On d-ary trees with restricted colorings

June 27, 2019

joint work with *Helmut Prodinger*

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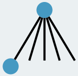
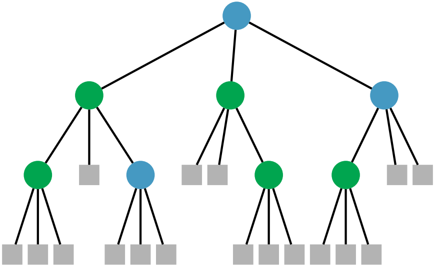
Hong & Park 2014: Hybrid *d*-ary Trees

Hybrid *d*-ary trees

- colored inner nodes:
  - ● (blue – “bad”)
  - ● (green – “good”)
- Forbidden:





A hybrid ternary tree of size 8.

- Hybrid (binary) trees: J. Pallo, ’94

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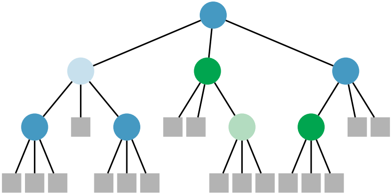
Their Generalization and A Potpourri of Results

(p, q)-hybrid d-ary tree

- p bad, q good colors
- Forbidden: "consecutive" same bad color.



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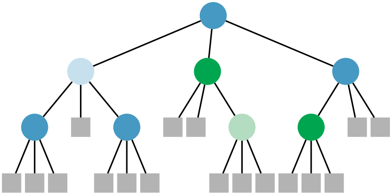

Some Results [Hong/Park '14]

Fibonacci Words

Blocks of (p, q)-hybrid d-ary trees $\to$ *generalized Fibonacci words.*

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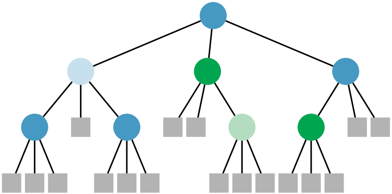

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Fibonacci Words

Blocks of (p, q)-hybrid d-ary trees $\to$ *generalized Fibonacci words*.

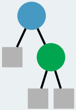
Number of Trees of size n

$$\frac{1}{(d-1)n+1}\sum_{k=0}^{n}\sum_{i=\lceil \frac{k}{2}\rceil}^{k}\binom{(d-1)n+1}{n-k}\binom{(d-1)n+i}{i}\binom{i}{k-i}(p+q-1)^{2i-k}q^{k-i}$$

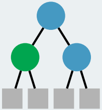
Our Take on the Problem

Question: What impact do different color restrictions have?

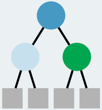
Bad → Leaf



Bad → Not Bad



Bad → Different ≙ Hybrid



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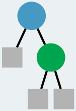

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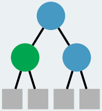
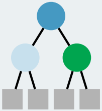


Bad $\rightarrow$ Different $\triangleq$ Hybrid



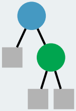
All colored d-ary trees

- Number of size n: $\frac{(p+q)^n}{n(d-1)+1}\binom{dn}{n}$
- Exponential growth: $(p+q)\frac{d^d}{(d-1)^{d-1}}$

Our Take on the Problem

Question: What impact do different color restrictions have?

Bad $\to$ Leaf



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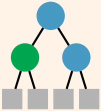

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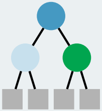

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- In this talk: focus on "Bad $\to$ Not Bad"

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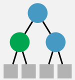
Generating Function Model



- z ... inner nodes, t ... "good" nodes
- $\alpha = p/q$ ... color ratio
- $\mathcal{H}$ ... d-ary trees, first child of bad node is not bad
  - $\mathcal{P}$ ... trees in $\mathcal{H}$ with bad root
  - $\mathcal{Q}$ ... trees in $\mathcal{H}$ with good root

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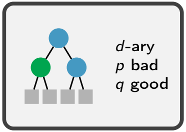

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Proposition

Generating functions $P = P(z,t)$, $Q = Q(z,t)$ satisfy

$$P = \frac{-(1+Q)t + \sqrt{t(1+Q)(t+(t+4\alpha)Q)}}{2t}$$

$$Q = qzt\left(\frac{(1+Q)t + \sqrt{t(1+Q)(t+(t+4\alpha)Q)}}{2t}\right)^d$$

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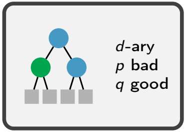
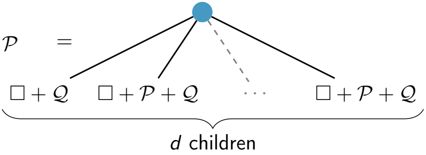
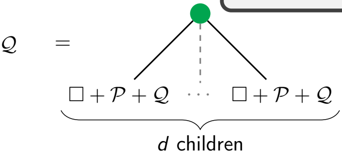
Generating Function Model – Proof







$$P = pz(1 + Q)(1 + P + Q)^{d-1}$$

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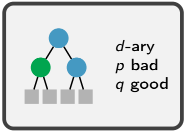

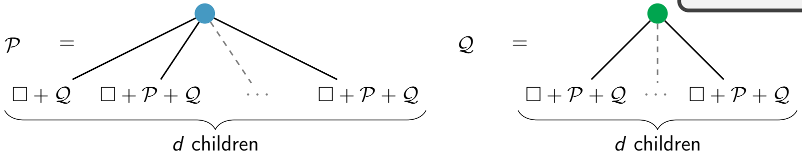

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$$P(1 + P + Q)t = \frac{p}{q}(1 + Q)Q = \alpha(1 + Q)Q.$$

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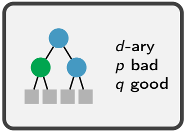

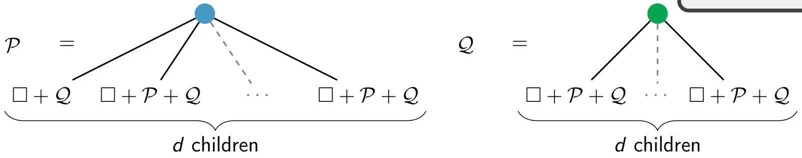

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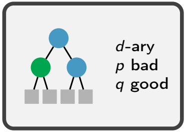
Solve for P:

$$P = \frac{-(1+Q)t \pm \sqrt{t(1+Q)(t+(t+4\alpha)Q)}}{2t}.$$

It's a Kind of Magic: A Special Substitution



Assume $v = v(z)$ satisfies $Q(z, 1) = \dfrac{v(1 + \alpha v)}{1 - v - \alpha v^2}$.

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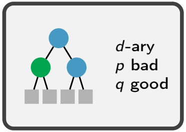

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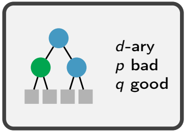

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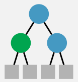

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- **Consequence:** $\sqrt{\dots}$ "disappears"!

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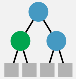

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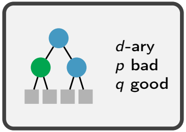
Proposition: Existence of $v(z)$

A function $v = v(z)$ that is analytic near $z = 0$ and satisfies $Q(z,1) = \frac{v(1+\alpha v)}{1-v-\alpha v^2}$ exists and is unique.

Magic Substitution: Proof and Ramifications



- Plug relation $Q(z,1) = \frac{v(1+\alpha v)}{1-v-\alpha v^2}$ in functional equation of Q:

$$\frac{v(1+\alpha v)}{1-v-\alpha v^2} = qz\Big(\frac{1+\alpha v}{1-v-\alpha v^2}\Big)^d$$

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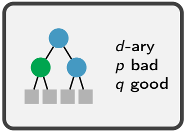

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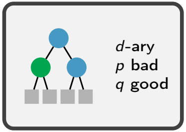

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- Φ is analytic at 0, $\Phi(0) \neq 0$, and has positive coefficients
  $\Rightarrow v(z)$ exists and is unique! □

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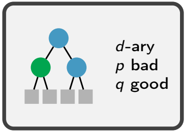

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- Φ is analytic at 0, $\Phi(0) \neq 0$, and has positive coefficients $\Rightarrow$ $v(z)$ exists and is unique! $\square$

New Representations:

$$Q(z,1) = \frac{v(1+\alpha v)}{1-v(1+\alpha v)}, \qquad P(z,1) = \frac{\alpha v}{1-v(1+\alpha v)}$$

$$H(z,1) = P(z,1) + Q(z,1) = -1 + \frac{1+\alpha v}{1-v(1+\alpha v)}$$

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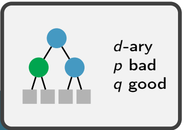
Result: Exact Growth



Theorem (H.–Prodinger '19+): Exact formulas

Number of "Bad $\to$ Not Bad"-trees with bad root:

$$\frac{q^n}{n}\sum_{k=0}^{n-1}\alpha^k\binom{dn-k}{n-k-1}\left(\alpha\binom{dn-k-1}{k}+\binom{dn-k-1}{k-2}\right),$$

with good root:

$$q^n d\sum_{k=0}^{n-1}\frac{\alpha^k}{dn-k}\binom{dn-k}{n-k-1}\binom{dn-k}{k},$$

and overall

$$\frac{q^n}{n}\sum_{k=0}^{n-1}\alpha^k\binom{dn-k}{n-k-1}\left(\binom{dn-k+1}{k}+\alpha\binom{dn-k-1}{k}\right).$$

Proof: Lagrange inversion.

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Playing with Counting Sequences: “Bad → Not Bad”

$d = 2$	$q = 1$	$q = 2$
$p = 1$	A007863	A216314
$p = 2$	A215661	

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- Sequence description for (d, p, q):
  - (2,1,1): hybrid binary trees
  - (2,1,2), (2,2,1), (3,1,1): generating function satisfies equation
  - (4,1,1): hybrid 4-ary trees

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Playing with Counting Sequences: “Bad → Leaf”

$d = 2$	$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q = 5$
$p = 1$	A006318	A103210	A103211	A133305	A133306
$p = 2$	A047891	A156017			
$p = 3$	A082298				
$p = 4$	A082301				

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$d=3$	$q=1$	$q=2$
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- Sequence description for (d, p, q):
  - (2,1,1): large Schröder numbers
  - (2,1,2–5): recurrence
  - (2,2,1): plane trees with tricolored leaves
  - (2,2,2): special Schroeder paths
  - (2,3–4,1) explicit generating function
  - (3,1,1): special lattice paths in first quadrant
  - (3,1,2), (3,2,1): generating function satisfies equation

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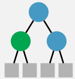
Analyzing Asymptotic Growth



- **Goal:** growth of $H(z, 1)$ … # of "Bad $\to$ Not Bad"-trees

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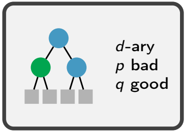

- **Goal:** growth of $H(z,1)$ … # of "Bad $\to$ Not Bad"-trees
- **Known:**

$$H = -1 + \frac{1+\alpha v}{1 - v - \alpha v^2}, \qquad v = z\Phi(v), \quad \Phi(u) = q\left(\frac{(1+\alpha u)}{1 - u - \alpha u^2}\right)^{d-1}$$

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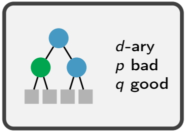

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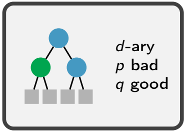
Strategy:

1. singular structure of $H(z,1)$ comes from $v(z)$

Analyzing Asymptotic Growth



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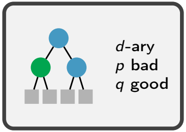

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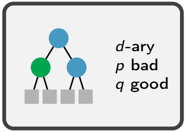

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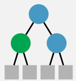

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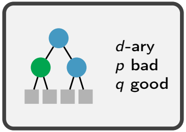

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If only there were software to carry this out for us…

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Intermission: SageMath to the Rescue!

Asymptotic Expansions in SageMath (H.–Heuberger–Krenn '15)

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This translates to

$$\tau + c_1\sqrt{1 - \frac{z}{\rho}} + c_2\Big(1 - \frac{z}{\rho}\Big) + c_3\Big(1 - \frac{z}{\rho}\Big)^{3/2} + O\Big(\Big(1 - \frac{z}{\rho}\Big)^2\Big)$$

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- String length of expansion of v up to $O((1-z/\rho)^2)$: 23225

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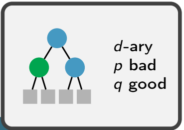
Result: Asymptotic Growth



Theorem: Asymptotic Growth (H.–Prodinger '19+)

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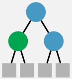

d-ary
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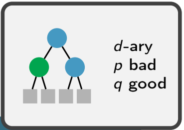

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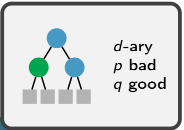

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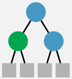

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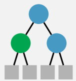

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- $\tau > 0$ satisfies $\alpha^2 d\tau^3 + 2\alpha d\tau^2 + \alpha d\tau - 2\alpha\tau + d\tau - 1 = 0$, and for $d \to \infty$: $\tau = \frac{1}{1+\alpha} d^{-1} + \frac{2\alpha^2}{(1+\alpha)^3} d^{-2} + O(d^{-3})$
- $\rho = \tau/\Phi(\tau)$, for $d \to \infty$: $\rho = \frac{1}{(p+q)e} d^{-1} + O(d^{-2})$

Proof. Expansion for τ via bootstrapping, for ρ as a consequence. Rest: Singularity Analysis. □

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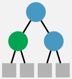
Result: Asymptotic Growth



Theorem: Asymptotic Growth (H.–Prodinger '19+)

Number of "Bad → Not Bad"-trees of size n grows like

$$C_1 \cdot \rho^{-n} n^{-3/2} + C_2 \cdot \rho^{-n} n^{-5/2} + O(\rho^{-n} n^{-7/2}).$$

The Constant C_1

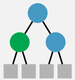
$$\frac{\sqrt{2}(\alpha^2\tau^2 + 2\,\alpha\tau + \alpha + 1)\,(1 + \alpha\tau)}{2\,\sqrt{\pi}(1 - \tau - \alpha\tau^2)}$$

$$\times \sqrt{\frac{-\tau}{(\alpha^4 d\tau^5 + 4\,\alpha^3 d\tau^4 + 2\,\alpha^3 d\tau^3 - 6\,\alpha^3\tau^3 + 6\,\alpha^2 d\tau^3 + 4\,\alpha^2 d\tau^2 + \alpha^2 d\tau - 12\,\alpha^2\tau^2 + 4\,\alpha d\tau^2 - 2\,\alpha^2\tau + 2\,\alpha d\tau - 8\,\alpha\tau + d\tau - 2\,\alpha - 2)(d-1)}}$$

- $\rho = \tau/\Phi(\tau)$, for $d \to \infty$: $\rho = \frac{1}{(p+q)e} d^{-1} + O(d^{-2})$

Proof. Expansion for τ via bootstrapping, for ρ as a consequence.
Rest: Singularity Analysis. □

Result: Asymptotic Growth



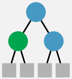
Theorem: Asymptotic Growth (H.–Prodinger '19+)

The Constant C_2

$$\sqrt{2}\big(2\,\alpha^{14}d^4\tau^{16} + 19\,\alpha^{14}d^3\tau^{16} + 28\,\alpha^{13}d^4\tau^{15} + 26\,\alpha^{14}d^2\tau^{16} + 14\,\alpha^{13}d^4\tau^{14} + 266\,\alpha^{13}d^3\tau^{15} + 115\,\alpha^{13}d^3\tau^{14} + 182\,\alpha^{12}d^4\tau^{14} + 364\,\alpha^{13}d^2\tau^{15} + 168$$
$$\alpha^{12}d^4\tau^{13} + 8\,\alpha^{13}d^2\tau^{14} + 1729\,\alpha^{12}d^3\tau^{14} + 42\,\alpha^{12}d^4\tau^{12} + 1362\,\alpha^{12}d^3\tau^{13} + 728\,\alpha^{11}d^4\tau^{13} - 12\,\alpha^{13}d\tau^{14} + 2366\,\alpha^{12}d^2\tau^{14} + 279\,\alpha^{12}d^3\tau^{12} + 924$$
$$\alpha^{11}d^4\tau^{12} - 60\,\alpha^{12}d^2\tau^{13} + 6916\,\alpha^{11}d^3\tau^{13} + 420\,\alpha^{11}d^4\tau^{11} - 762\,\alpha^{12}d^2\tau^{12} + 7380\,\alpha^{11}d^3\tau^{12} + 2002\,\alpha^{10}d^4\tau^{12} - 552\,\alpha^{12}d\tau^{13} + 9464\,\alpha^{11}d^2\tau^{13} + 7$$
$$\alpha^{11}d^4\tau^{10} + 2688\,\alpha^{11}d^3\tau^{11} + 3080\,\alpha^{10}d^4\tau^{11} - 1296\,\alpha^{12}d\tau^{12} - 1218\,\alpha^{11}d^2\tau^{12} + 19019\,\alpha^{10}d^3\tau^{12} + 335\,\alpha^{11}d^3\tau^{10} + 1890\,\alpha^{10}d^4\tau^{10} - 7878$$
$$\alpha^{11}d^2\tau^{11} + 24178\,\alpha^{10}d^3\tau^{11} + 4004\,\alpha^{9}d^4\tau^{11} - 144\,\alpha^{12}\tau^{12} - 5028\,\alpha^{11}d\tau^{12} + 26026\,\alpha^{10}d^2\tau^{12} + 560\,\alpha^{10}d^4\tau^{9} - 2108\,\alpha^{11}d^2\tau^{10} + 11565\,\alpha^{10}d^3\tau^{10}$$
$$+ 6930\,\alpha^{9}d^4\tau^{10} - 11460\,\alpha^{11}d\tau^{11} - 7210\,\alpha^{10}d^2\tau^{11} + 38038\,\alpha^{9}d^3\tau^{11} + 70\,\alpha^{10}d^4\tau^{8} + 2440\,\alpha^{10}d^3\tau^{9} + 5040\,\alpha^{9}d^4\tau^{9} + 1020\,\alpha^{11}d\tau^{10} - 36927$$
$$\alpha^{10}d^2\tau^{10} + 53295\,\alpha^{9}d^3\tau^{10} + 6006\,\alpha^{8}d^4\tau^{10} - 432\,\alpha^{11}\tau^{11} - 22692\,\alpha^{10}d\tau^{11} + 52052\,\alpha^{9}d^2\tau^{11} + 185\,\alpha^{10}d^3\tau^{8} + 1960\,\alpha^{9}d^4\tau^{8} - 16774\,\alpha^{10}d^2\tau^{9}$$
$$+ 29160\,\alpha^{9}d^3\tau^{9} + 11088\,\alpha^{8}d^4\tau^{9} + 3744\,\alpha^{11}\tau^{10} - 45396\,\alpha^{10}d\tau^{10} - 24030\,\alpha^{9}d^2\tau^{10} + 57057\,\alpha^{8}d^3\tau^{10} + 420\,\alpha^{9}d^4\tau^{7} - 2042\,\alpha^{10}d^2\tau^{8} + 7520$$
$$\alpha^{9}d^3\tau^{8} + 8820\,\alpha^{8}d^4\tau^{8} + 9156\,\alpha^{10}d\tau^{9} - 103662\,\alpha^{9}d^2\tau^{9} + 83160\,\alpha^{8}d^3\tau^{9} + 6864\,\alpha^{7}d^4\tau^{9} + 2160\,\alpha^{10}\tau^{10} - 63168\,\alpha^{9}d\tau^{10} + 78078\,\alpha^{8}d^2\tau^{10} + 42$$
$$\alpha^{9}d^4\tau^{6} + 810\,\alpha^{9}d^3\tau^{7} + 3920\,\alpha^{8}d^4\tau^{7} + 2928\,\alpha^{10}d\tau^{8} - 59171\,\alpha^{9}d^2\tau^{8} + 47430\,\alpha^{8}d^3\tau^{8} + 12936\,\alpha^{7}d^4\tau^{8} + 26928\,\alpha^{10}\tau^{9} - 105840\,\alpha^{9}d\tau^{9} - 52794$$
$$\alpha^{8}d^2\tau^{9} + 65208\,\alpha^{7}d^3\tau^{9} + 9\,\alpha^{9}d^3\tau^{6} + 1050\,\alpha^{8}d^4\tau^{6} - 11952\,\alpha^{9}d^2\tau^{7} + 12460\,\alpha^{8}d^3\tau^{7} + 10584\,\alpha^{7}d^4\tau^{7} + 36672\,\alpha^{9}d\tau^{8} - 193803\,\alpha^{8}d^2\tau^{8} + 93984$$
$$\alpha^{7}d^3\tau^{8} + 6006\,\alpha^{6}d^4\tau^{8} + 15840\,\alpha^{9}\tau^{9} - 119196\,\alpha^{8}d\tau^{9} + 89232\,\alpha^{7}d^2\tau^{9} + 168\,\alpha^{8}d^4\tau^{5} - 672\,\alpha^{9}d^2\tau^{6} + 1035\,\alpha^{8}d^3\tau^{6} + 4900\,\alpha^{7}d^4\tau^{6} + 17808\,\alpha^{9}d\tau$$
$$- 121492\,\alpha^{8}d^2\tau^{7} + 51408\,\alpha^{7}d^3\tau^{7} + 11088\,\alpha^{6}d^4\tau^{7} + 86112\,\alpha^{9}\tau^{8} - 160044\,\alpha^{8}d\tau^{8} - 81780\,\alpha^{7}d^2\tau^{8} + 57057\,\alpha^{6}d^3\tau^{8} + 14\,\alpha^{8}d^4\tau^{4} - 174\,\alpha^{8}d^3\tau^{5}$$
$$+ 1400\,\alpha^{7}d^4\tau^{5} + 540\,\alpha^{9}d\tau^{6} - 29808\,\alpha^{8}d^2\tau^{6} + 11270\,\alpha^{7}d^3\tau^{6} + 8820\,\alpha^{6}d^4\tau^{6} + 1008\,\alpha^{9}\tau^{7} + 86688\,\alpha^{8}d\tau^{7} - 253596\,\alpha^{7}d^2\tau^{7} + 77220\,\alpha^{6}d^3\tau^{7}$$
$$+ 4004\,\alpha^{5}d^4\tau^{7} + 44640\,\alpha^{8}\tau^{8} - 159912\,\alpha^{7}d\tau^{8} + 78078\,\alpha^{6}d^2\tau^{8} - 35\,\alpha^{8}d^3\tau^{4} + 252\,\alpha^{7}d^4\tau^{4} - 2568\,\alpha^{8}d^2\tau^{5} - 500\,\alpha^{7}d^3\tau^{5} + 3920\,\alpha^{6}d^4\tau^{5} + 864$$
$$\alpha^{9}\tau^{6} + 48192\,\alpha^{8}d\tau^{6} - 159992\,\alpha^{7}d^2\tau^{6} + 36666\,\alpha^{6}d^3\tau^{6} + 6930\,\alpha^{5}d^4\tau^{6} + 161712\,\alpha^{8}\tau^{7} - 162120\,\alpha^{7}d\tau^{7} - 92196\,\alpha^{6}d^2\tau^{7} + 38038\,\alpha^{5}d^3\tau^{7} + 28$$
$$\alpha^{7}d^4\tau^{3} + 26\,\alpha^{8}d^2\tau^{4} - 756\,\alpha^{7}d^3\tau^{4} + 1050\,\alpha^{6}d^4\tau^{4} + 3336\,\alpha^{8}d\tau^{5} - 40900\,\alpha^{7}d^2\tau^{5} + 4060\,\alpha^{6}d^3\tau^{5} + 5040\,\alpha^{5}d^4\tau^{5} + 6192\,\alpha^{8}\tau^{6} + 134508\,\alpha^{7}d\tau^{6}$$
$$- 237174\,\alpha^{6}d^2\tau^{6} + 45441\,\alpha^{5}d^3\tau^{6} + 2002\,\alpha^{4}d^4\tau^{6} + 74592\,\alpha^{7}\tau^{7} - 156168\,\alpha^{6}d\tau^{7} + 52052\,\alpha^{5}d^2\tau^{7} + 2\,\alpha^{7}d^4\tau^{2} - 148\,\alpha^{7}d^3\tau^{3} + 168\,\alpha^{6}d^4\tau^{3} - 96$$
$$\alpha^{8}d\tau^{4} - 3744\,\alpha^{7}d^2\tau^{4} - 2625\,\alpha^{6}d^3\tau^{4} + 1960\,\alpha^{5}d^4\tau^{4} + 4176\,\alpha^{8}\tau^{5} + 76200\,\alpha^{7}d\tau^{5} - 139888\,\alpha^{6}d^2\tau^{5} + 15840\,\alpha^{5}d^3\tau^{5} + 3080\,\alpha^{4}d^4\tau^{5} + 198576$$
$$\alpha^{7}\tau^{6} - 108024\,\alpha^{6}d\tau^{6} - 76500\,\alpha^{5}d^2\tau^{6} + 19019\,\alpha^{4}d^3\tau^{6} - 11\,\alpha^{7}d^3\tau^{2} + 14\,\alpha^{6}d^4\tau^{2} + 106\,\alpha^{7}d^2\tau^{3} - 1134\,\alpha^{6}d^3\tau^{3} + 420\,\alpha^{5}d^4\tau^{3} + 144\,\alpha^{8}\tau^{4} + 8028$$
$$\alpha^{7}d\tau^{4} - 32880\,\alpha^{6}d^2\tau^{4} - 1960\,\alpha^{5}d^3\tau^{4} + 1890\,\alpha^{4}d^4\tau^{4} + 15984\,\alpha^{7}\tau^{5} + 143760\,\alpha^{6}d\tau^{5} - 158556\,\alpha^{5}d^2\tau^{5} + 18370\,\alpha^{4}d^3\tau^{5} + 728\,\alpha^{3}d^4\tau^{5} + 82692$$
$$\alpha^{6}\tau^{6} - 111852\,\alpha^{5}d\tau^{6} + 26026\,\alpha^{4}d^2\tau^{6} + 20\,\alpha^{7}d^2\tau^{2} - 185\,\alpha^{6}d^3\tau^{2} + 42\,\alpha^{5}d^4\tau^{2} - 204\,\alpha^{7}d\tau^{3} - 2274\,\alpha^{6}d^2\tau^{3} - 2790\,\alpha^{5}d^3\tau^{3} + 560\,\alpha^{4}d^4\tau^{3} + 835$$

Rest: Singularity Analysis. □

Result: Asymptotic Growth



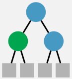
d-ary
p bad
q good

Theorem: Asymptotic Growth (H.–Prodinger '19+)

(A part of) The Constant C_2

$$
\begin{aligned}
&\sqrt{2}\big(2\,\alpha^{14}d^4\tau^{16} + 19\,\alpha^{14}d^3\tau^{16} + 28\,\alpha^{13}d^4\tau^{15} + 26\,\alpha^{14}d^2\tau^{16} + 14\,\alpha^{13}d^4\tau^{14} + 266\,\alpha^{13}d^3\tau^{15} + 115\,\alpha^{13}d^3\tau^{14} + 182\,\alpha^{12}d^4\tau^{14} + 364\,\alpha^{13}d^2\tau^{15} + 168\\
&\alpha^{12}d^4\tau^{13} + 8\,\alpha^{13}d^2\tau^{14} + 1729\,\alpha^{12}d^3\tau^{14} + 42\,\alpha^{12}d^4\tau^{12} + 1362\,\alpha^{12}d^3\tau^{13} + 728\,\alpha^{11}d^4\tau^{13} - 12\,\alpha^{13}d\tau^{14} + 2366\,\alpha^{12}d^2\tau^{14} + 279\,\alpha^{12}d^3\tau^{12} + 924\\
&\alpha^{11}d^4\tau^{12} - 60\,\alpha^{12}d^2\tau^{13} + 6916\,\alpha^{11}d^3\tau^{13} + 420\,\alpha^{11}d^4\tau^{11} - 762\,\alpha^{12}d^2\tau^{12} + 7380\,\alpha^{11}d^3\tau^{12} + 2002\,\alpha^{10}d^4\tau^{12} - 552\,\alpha^{12}d\tau^{13} + 9464\,\alpha^{11}d^2\tau^{13} + 7\\
&\alpha^{11}d^4\tau^{10} + 2688\,\alpha^{11}d^3\tau^{11} + 3080\,\alpha^{10}d^4\tau^{11} - 1296\,\alpha^{12}d\tau^{12} - 1218\,\alpha^{11}d^2\tau^{12} + 19019\,\alpha^{10}d^3\tau^{12} + 335\,\alpha^{11}d^3\tau^{10} + 1890\,\alpha^{10}d^4\tau^{10} - 7878\\
&\alpha^{11}d^2\tau^{11} + 24178\,\alpha^{10}d^3\tau^{11} + 4004\,\alpha^{9}d^4\tau^{11} - 144\,\alpha^{12}\tau^{12} - 5028\,\alpha^{11}d\tau^{12} + 26026\,\alpha^{10}d^2\tau^{12} + 560\,\alpha^{10}d^4\tau^{9} - 2108\,\alpha^{11}d^2\tau^{10} + 11565\,\alpha^{10}d^3\tau^{10}\\
&+ 6930\,\alpha^{9}d^4\tau^{10} - 11460\,\alpha^{11}d\tau^{11} - 7210\,\alpha^{10}d^2\tau^{11} + 38038\,\alpha^{9}d^3\tau^{11} + 70\,\alpha^{10}d^4\tau^{8} + 2440\,\alpha^{10}d^3\tau^{9} + 5040\,\alpha^{9}d^4\tau^{9} + 1020\,\alpha^{11}d\tau^{10} - 36927\\
&\alpha^{10}d^2\tau^{10} + 53295\,\alpha^{9}d^3\tau^{10} + 6006\,\alpha^{8}d^4\tau^{10} - 432\,\alpha^{11}\tau^{11} - 22692\,\alpha^{10}d\tau^{11} + 52052\,\alpha^{9}d^2\tau^{11} + 185\,\alpha^{10}d^3\tau^{8} + 1960\,\alpha^{9}d^4\tau^{8} - 16774\,\alpha^{10}d^2\tau^{9}\\
&+ 29160\,\alpha^{9}d^3\tau^{9} + 11088\,\alpha^{8}d^4\tau^{9} + 3744\,\alpha^{11}\tau^{10} - 45396\,\alpha^{10}d\tau^{10} - 24030\,\alpha^{9}d^2\tau^{10} + 57057\,\alpha^{8}d^3\tau^{10} + 420\,\alpha^{9}d^4\tau^{7} - 2042\,\alpha^{10}d^2\tau^{8} + 7520\\
&\alpha^{9}d^3\tau^{8} + 8820\,\alpha^{8}d^4\tau^{8} + 9156\,\alpha^{10}d\tau^{9} - 103662\,\alpha^{9}d^2\tau^{9} + 83160\,\alpha^{8}d^3\tau^{9} + 6864\,\alpha^{7}d^4\tau^{9} + 2160\,\alpha^{10}\tau^{10} - 63168\,\alpha^{9}d\tau^{10} + 78078\,\alpha^{8}d^2\tau^{10} + 42\\
&\alpha^{9}d^4\tau^{6} + 810\,\alpha^{9}d^3\tau^{7} + 3920\,\alpha^{8}d^4\tau^{7} + 2928\,\alpha^{10}d\tau^{8} - 59171\,\alpha^{9}d^2\tau^{8} + 47430\,\alpha^{8}d^3\tau^{8} + 12936\,\alpha^{7}d^4\tau^{8} + 26928\,\alpha^{10}\tau^{9} - 105840\,\alpha^{9}d\tau^{9} - 52794\\
&\alpha^{8}d^2\tau^{9} + 65208\,\alpha^{7}d^3\tau^{9} + 9\,\alpha^{9}d^3\tau^{6} + 1050\,\alpha^{8}d^4\tau^{6} - 11952\,\alpha^{9}d^2\tau^{7} + 12460\,\alpha^{8}d^3\tau^{7} + 10584\,\alpha^{7}d^4\tau^{7} + 36672\,\alpha^{9}d\tau^{8} - 193803\,\alpha^{8}d^2\tau^{8} + 93984\\
&\alpha^{7}d^3\tau^{8} + 6006\,\alpha^{6}d^4\tau^{8} + 15840\,\alpha^{9}\tau^{9} - 119196\,\alpha^{8}d\tau^{9} + 89232\,\alpha^{7}d^2\tau^{9} + 168\,\alpha^{8}d^4\tau^{5} - 672\,\alpha^{9}d^2\tau^{6} + 1035\,\alpha^{8}d^3\tau^{6} + 4900\,\alpha^{7}d^4\tau^{6} + 17808\,\alpha^{9}d\tau\\
&- 121492\,\alpha^{8}d^2\tau^{7} + 51408\,\alpha^{7}d^3\tau^{7} + 11088\,\alpha^{6}d^4\tau^{7} + 86112\,\alpha^{9}\tau^{8} - 160044\,\alpha^{8}d\tau^{8} - 81780\,\alpha^{7}d^2\tau^{8} + 57057\,\alpha^{6}d^3\tau^{8} + 14\,\alpha^{8}d^4\tau^{4} - 174\,\alpha^{8}d^3\tau^{5}\\
&+ 1400\,\alpha^{7}d^4\tau^{5} + 540\,\alpha^{9}d\tau^{6} - 29808\,\alpha^{8}d^2\tau^{6} + 11270\,\alpha^{7}d^3\tau^{6} + 8820\,\alpha^{6}d^4\tau^{6} + 1008\,\alpha^{9}\tau^{7} + 86688\,\alpha^{8}d\tau^{7} - 253596\,\alpha^{7}d^2\tau^{7} + 77220\,\alpha^{6}d^3\tau^{7}\\
&+ 4004\,\alpha^{5}d^4\tau^{7} + 44640\,\alpha^{8}\tau^{8} - 159912\,\alpha^{7}d\tau^{8} + 78078\,\alpha^{6}d^2\tau^{8} - 35\,\alpha^{8}d^3\tau^{4} + 252\,\alpha^{7}d^4\tau^{4} - 2568\,\alpha^{8}d^2\tau^{5} - 500\,\alpha^{7}d^3\tau^{5} + 3920\,\alpha^{6}d^4\tau^{5} + 864\\
&\alpha^{9}\tau^{6} + 48192\,\alpha^{8}d\tau^{6} - 159992\,\alpha^{7}d^2\tau^{6} + 36666\,\alpha^{6}d^3\tau^{6} + 6930\,\alpha^{5}d^4\tau^{6} + 161712\,\alpha^{8}\tau^{7} - 162120\,\alpha^{7}d\tau^{7} - 92196\,\alpha^{6}d^2\tau^{7} + 38038\,\alpha^{5}d^3\tau^{7} + 28\\
&\alpha^{7}d^4\tau^{3} + 26\,\alpha^{8}d^2\tau^{4} - 756\,\alpha^{7}d^3\tau^{4} + 1050\,\alpha^{6}d^4\tau^{4} + 3336\,\alpha^{8}d\tau^{5} - 40900\,\alpha^{7}d^2\tau^{5} + 4060\,\alpha^{6}d^3\tau^{5} + 5040\,\alpha^{5}d^4\tau^{5} + 6192\,\alpha^{8}\tau^{6} + 134508\,\alpha^{7}d\tau^{6}\\
&- 237174\,\alpha^{6}d^2\tau^{6} + 45441\,\alpha^{5}d^3\tau^{6} + 2002\,\alpha^{4}d^4\tau^{6} + 74592\,\alpha^{7}\tau^{7} - 156168\,\alpha^{6}d\tau^{7} + 52052\,\alpha^{5}d^2\tau^{7} + 2\,\alpha^{7}d^4\tau^{2} - 148\,\alpha^{7}d^3\tau^{3} + 168\,\alpha^{6}d^4\tau^{3} - 96\\
&\alpha^{8}d\tau^{4} - 3744\,\alpha^{7}d^2\tau^{4} - 2625\,\alpha^{6}d^3\tau^{4} + 1960\,\alpha^{5}d^4\tau^{4} + 4176\,\alpha^{8}\tau^{5} + 76200\,\alpha^{7}d\tau^{5} - 139888\,\alpha^{6}d^2\tau^{5} + 15840\,\alpha^{5}d^3\tau^{5} + 3080\,\alpha^{4}d^4\tau^{5} + 198576\\
&\alpha^{7}\tau^{6} - 108024\,\alpha^{6}d\tau^{6} - 76500\,\alpha^{5}d^2\tau^{6} + 19019\,\alpha^{4}d^3\tau^{6} - 11\,\alpha^{7}d^3\tau^{2} + 14\,\alpha^{6}d^4\tau^{2} + 106\,\alpha^{7}d^2\tau^{3} - 1134\,\alpha^{6}d^3\tau^{3} + 420\,\alpha^{5}d^4\tau^{3} + 144\,\alpha^{8}\tau^{4} + 802\\
&\alpha^{7}d\tau^{4} - 32880\,\alpha^{6}d^2\tau^{4} - 1960\,\alpha^{5}d^3\tau^{4} + 1890\,\alpha^{4}d^4\tau^{4} + 15984\,\alpha^{7}\tau^{5} + 143760\,\alpha^{6}d\tau^{5} - 158556\,\alpha^{5}d^2\tau^{5} + 18370\,\alpha^{4}d^3\tau^{5} + 728\,\alpha^{3}d^4\tau^{5} + 82692\\
&\alpha^{6}\tau^{6} - 111852\,\alpha^{5}d\tau^{6} + 26026\,\alpha^{4}d^2\tau^{6} + 20\,\alpha^{7}d^2\tau^{2} - 185\,\alpha^{6}d^3\tau^{2} + 42\,\alpha^{5}d^4\tau^{2} - 204\,\alpha^{7}d\tau^{3} - 2274\,\alpha^{6}d^2\tau^{3} - 2790\,\alpha^{5}d^3\tau^{3} + 560\,\alpha^{4}d^4\tau^{3} + 835
\end{aligned}
$$

Result: Asymptotic Growth



Theorem: Asymptotic Growth (H.–Prodinger '19+)

(A part of (the numerator of)) The Constant C_2

$$\sqrt{2}(2\,\alpha^{14}d^4\tau^{16} + 19\,\alpha^{14}d^3\tau^{16} + 28\,\alpha^{13}d^4\tau^{15} + 26\,\alpha^{14}d^2\tau^{16} + 14\,\alpha^{13}d^4\tau^{14} + 266\,\alpha^{13}d^3\tau^{15} + 115\,\alpha^{13}d^3\tau^{14} + 182\,\alpha^{12}d^4\tau^{14} + 364\,\alpha^{13}d^2\tau^{15} + 168$$

$$\alpha^{12}d^4\tau^{13} + 8\,\alpha^{13}d^2\tau^{14} + 1729\,\alpha^{12}d^3\tau^{14} + 42\,\alpha^{12}d^4\tau^{12} + 1362\,\alpha^{12}d^3\tau^{13} + 728\,\alpha^{11}d^4\tau^{13} - 12\,\alpha^{13}d\tau^{14} + 2366\,\alpha^{12}d^2\tau^{14} + 279\,\alpha^{12}d^3\tau^{12} + 924$$

$$\alpha^{11}d^4\tau^{12} - 60\,\alpha^{12}d^2\tau^{13} + 6916\,\alpha^{11}d^3\tau^{13} + 420\,\alpha^{11}d^4\tau^{11} - 762\,\alpha^{12}d^2\tau^{12} + 7380\,\alpha^{11}d^3\tau^{12} + 2002\,\alpha^{10}d^4\tau^{12} - 552\,\alpha^{12}d\tau^{13} + 9464\,\alpha^{11}d^2\tau^{13} + 7$$

$$\alpha^{11}d^4\tau^{10} + 2688\,\alpha^{11}d^3\tau^{11} + 3080\,\alpha^{10}d^4\tau^{11} - 1296\,\alpha^{12}d\tau^{12} - 1218\,\alpha^{11}d^2\tau^{12} + 19019\,\alpha^{10}d^3\tau^{12} + 335\,\alpha^{11}d^3\tau^{10} + 1890\,\alpha^{10}d^4\tau^{10} - 7878$$

$$\alpha^{11}d^2\tau^{11} + 24178\,\alpha^{10}d^3\tau^{11} + 4004\,\alpha^{9}d^4\tau^{11} - 144\,\alpha^{12}\tau^{12} - 5028\,\alpha^{11}d\tau^{12} + 26026\,\alpha^{10}d^2\tau^{12} + 560\,\alpha^{10}d^4\tau^{9} - 2108\,\alpha^{11}d^2\tau^{10} + 11565\,\alpha^{10}d^3\tau^{10}$$

$$+ 6930\,\alpha^{9}d^4\tau^{10} - 11460\,\alpha^{11}d\tau^{11} - 7210\,\alpha^{10}d^2\tau^{11} + 38038\,\alpha^{9}d^3\tau^{11} + 70\,\alpha^{10}d^4\tau^{8} + 2440\,\alpha^{10}d^3\tau^{9} + 5040\,\alpha^{9}d^4\tau^{9} + 1020\,\alpha^{11}d\tau^{10} - 36927$$

$$\alpha^{10}d^2\tau^{10} + 53295\,\alpha^{9}d^3\tau^{10} + 6006\,\alpha^{8}d^4\tau^{10} - 432\,\alpha^{11}\tau^{11} - 22692\,\alpha^{10}d\tau^{11} + 52052\,\alpha^{9}d^2\tau^{11} + 185\,\alpha^{10}d^3\tau^{8} + 1960\,\alpha^{9}d^4\tau^{8} - 16774\,\alpha^{10}d^2\tau^{9}$$

$$+ 29160\,\alpha^{9}d^3\tau^{9} + 11088\,\alpha^{8}d^4\tau^{9} + 3744\,\alpha^{11}\tau^{10} - 45396\,\alpha^{10}d\tau^{10} - 24030\,\alpha^{9}d^2\tau^{10} + 57057\,\alpha^{8}d^3\tau^{10} + 420\,\alpha^{9}d^4\tau^{7} - 2042\,\alpha^{10}d^2\tau^{8} + 7520$$

$$\alpha^{9}d^3\tau^{8} + 8820\,\alpha^{8}d^4\tau^{8} + 9156\,\alpha^{10}d\tau^{9} - 103662\,\alpha^{9}d^2\tau^{9} + 83160\,\alpha^{8}d^3\tau^{9} + 6864\,\alpha^{7}d^4\tau^{9} + 2160\,\alpha^{10}\tau^{10} - 63168\,\alpha^{9}d\tau^{10} + 78078\,\alpha^{8}d^2\tau^{10} + 42$$

$$\alpha^{9}d^4\tau^{6} + 810\,\alpha^{9}d^3\tau^{7} + 3920\,\alpha^{8}d^4\tau^{7} + 2928\,\alpha^{10}d\tau^{8} - 59171\,\alpha^{9}d^2\tau^{8} + 47430\,\alpha^{8}d^3\tau^{8} + 12936\,\alpha^{7}d^4\tau^{8} + 26928\,\alpha^{10}\tau^{9} - 105840\,\alpha^{9}d\tau^{9} - 52794$$

$$\alpha^{8}d^2\tau^{9} + 65208\,\alpha^{7}d^3\tau^{9} + 9\,\alpha^{9}d^3\tau^{6} + 1050\,\alpha^{8}d^4\tau^{6} - 11952\,\alpha^{9}d^2\tau^{7} + 12460\,\alpha^{8}d^3\tau^{7} + 10584\,\alpha^{7}d^4\tau^{7} + 36672\,\alpha^{9}d\tau^{8} - 193803\,\alpha^{8}d^2\tau^{8} + 93984$$

$$\alpha^{7}d^3\tau^{8} + 6006\,\alpha^{6}d^4\tau^{8} + 15840\,\alpha^{9}\tau^{9} - 119196\,\alpha^{8}d\tau^{9} + 89232\,\alpha^{7}d^2\tau^{9} + 168\,\alpha^{8}d^4\tau^{5} - 672\,\alpha^{9}d^2\tau^{6} + 1035\,\alpha^{8}d^3\tau^{6} + 4900\,\alpha^{7}d^4\tau^{6} + 17808\,\alpha^{9}d\tau$$

$$- 121492\,\alpha^{8}d^2\tau^{7} + 51408\,\alpha^{7}d^3\tau^{7} + 11088\,\alpha^{6}d^4\tau^{7} + 86112\,\alpha^{9}\tau^{8} - 160044\,\alpha^{8}d\tau^{8} - 81780\,\alpha^{7}d^2\tau^{8} + 57057\,\alpha^{6}d^3\tau^{8} + 14\,\alpha^{8}d^4\tau^{4} - 174\,\alpha^{8}d^3\tau^{5}$$

$$+ 1400\,\alpha^{7}d^4\tau^{5} + 540\,\alpha^{9}d\tau^{6} - 29808\,\alpha^{8}d^2\tau^{6} + 11270\,\alpha^{7}d^3\tau^{6} + 8820\,\alpha^{6}d^4\tau^{6} + 1008\,\alpha^{9}\tau^{7} + 86688\,\alpha^{8}d\tau^{7} - 253596\,\alpha^{7}d^2\tau^{7} + 77220\,\alpha^{6}d^3\tau^{7}$$

$$+ 4004\,\alpha^{5}d^4\tau^{7} + 44640\,\alpha^{8}\tau^{8} - 159912\,\alpha^{7}d\tau^{8} + 78078\,\alpha^{6}d^2\tau^{8} - 35\,\alpha^{8}d^3\tau^{4} + 252\,\alpha^{7}d^4\tau^{4} - 2568\,\alpha^{8}d^2\tau^{5} - 500\,\alpha^{7}d^3\tau^{5} + 3920\,\alpha^{6}d^4\tau^{5} + 864$$

$$\alpha^{9}\tau^{6} + 48192\,\alpha^{8}d\tau^{6} - 159992\,\alpha^{7}d^2\tau^{6} + 36666\,\alpha^{6}d^3\tau^{6} + 6930\,\alpha^{5}d^4\tau^{6} + 161712\,\alpha^{8}\tau^{7} - 162120\,\alpha^{7}d\tau^{7} - 92196\,\alpha^{6}d^2\tau^{7} + 38038\,\alpha^{5}d^3\tau^{7} + 28$$

$$\alpha^{7}d^4\tau^{3} + 26\,\alpha^{8}d^2\tau^{4} - 756\,\alpha^{7}d^3\tau^{4} + 1050\,\alpha^{6}d^4\tau^{4} + 3336\,\alpha^{8}d\tau^{5} - 40900\,\alpha^{7}d^2\tau^{5} + 4060\,\alpha^{6}d^3\tau^{5} + 5040\,\alpha^{5}d^4\tau^{5} + 6192\,\alpha^{8}\tau^{6} + 134508\,\alpha^{7}d\tau^{6}$$

$$- 237174\,\alpha^{6}d^2\tau^{6} + 45441\,\alpha^{5}d^3\tau^{6} + 2002\,\alpha^{4}d^4\tau^{6} + 74592\,\alpha^{7}\tau^{7} - 156168\,\alpha^{6}d\tau^{7} + 52052\,\alpha^{5}d^2\tau^{7} + 2\,\alpha^{7}d^4\tau^{2} - 148\,\alpha^{7}d^3\tau^{3} + 168\,\alpha^{6}d^4\tau^{3} - 96$$

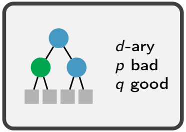
$$\alpha^{8}d\tau^{4} - 3744\,\alpha^{7}d^2\tau^{4} - 2625\,\alpha^{6}d^3\tau^{4} + 1960\,\alpha^{5}d^4\tau^{4} + 4176\,\alpha^{8}\tau^{5} + 76200\,\alpha^{7}d\tau^{5} - 139888\,\alpha^{6}d^2\tau^{5} + 15840\,\alpha^{5}d^3\tau^{5} + 3080\,\alpha^{4}d^4\tau^{5} + 198576$$

$$\alpha^{7}\tau^{6} - 108024\,\alpha^{6}d\tau^{6} - 76500\,\alpha^{5}d^2\tau^{6} + 19019\,\alpha^{4}d^3\tau^{6} - 11\,\alpha^{7}d^3\tau^{2} + 14\,\alpha^{6}d^4\tau^{2} + 106\,\alpha^{7}d^2\tau^{3} - 1134\,\alpha^{6}d^3\tau^{3} + 420\,\alpha^{5}d^4\tau^{3} + 144\,\alpha^{8}\tau^{4} + 802$$

$$\alpha^{7}d\tau^{4} - 32880\,\alpha^{6}d^2\tau^{4} - 1960\,\alpha^{5}d^3\tau^{4} + 1890\,\alpha^{4}d^4\tau^{4} + 15984\,\alpha^{7}\tau^{5} + 143760\,\alpha^{6}d\tau^{5} - 158556\,\alpha^{5}d^2\tau^{5} + 18370\,\alpha^{4}d^3\tau^{5} + 728\,\alpha^{3}d^4\tau^{5} + 82692$$

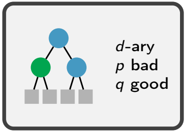
$$\alpha^{6}\tau^{6} - 111852\,\alpha^{5}d\tau^{6} + 26026\,\alpha^{4}d^2\tau^{6} + 20\,\alpha^{7}d^2\tau^{2} - 185\,\alpha^{6}d^3\tau^{2} + 42\,\alpha^{5}d^4\tau^{2} - 204\,\alpha^{7}d\tau^{3} - 2274\,\alpha^{6}d^2\tau^{3} - 2790\,\alpha^{5}d^3\tau^{3} + 560\,\alpha^{4}d^4\tau^{3} + 835$$

Counting Good Vertices



- **Goal:** study G_n … # of good nodes in tree of size n

Counting Good Vertices

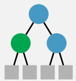


- **Goal:** study $G_n \ldots$ # of good nodes in tree of size n

Strategy:

1. Take functional equations for $P(z,t)$, $Q(z,t)$, implicit differentiation w.r.t. t; set $t = 1$

Counting Good Vertices

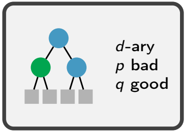


- **Goal:** study G_n … # of good nodes in tree of size n

Strategy:

1. Take functional equations for $P(z,t)$, $Q(z,t)$, implicit differentiation w.r.t. t; set $t = 1$
2. Solve for $\partial_t P(z,1)$, $\partial_t Q(z,1)$,

Counting Good Vertices



- **Goal:** study G_n … # of good nodes in tree of size n

Strategy:

1. Take functional equations for $P(z,t)$, $Q(z,t)$, implicit differentiation w.r.t. t; set $t = 1$
2. Solve for $\partial_t P(z,1)$, $\partial_t Q(z,1)$,
3. use substitution, add: $\partial_t H(z,1) = \partial_t P(z,1) + \partial_t Q(z,1)$, find

$$\partial_t H = \frac{\alpha^3 v^4 + 3\,\alpha^2 v^3 + 3\,\alpha v^2 + v}{\alpha^3 d v^5 + 3\,\alpha^2 d v^4 - (2\,\alpha^2 - 3\,\alpha d)v^3 - ((\alpha-1)d + 3\,\alpha)v^2 - ((\alpha+1)d - 2\,\alpha + 1)v + 1}$$

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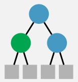
Counting Good Vertices



- **Goal:** study G_n ... # of good nodes in tree of size n

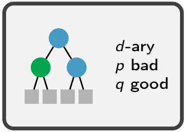
Strategy:

1. Take functional equations for $P(z,t)$, $Q(z,t)$, implicit differentiation w.r.t. t; set $t=1$
2. Solve for $\partial_t P(z,1)$, $\partial_t Q(z,1)$,
3. use substitution, add: $\partial_t H(z,1) = \partial_t P(z,1) + \partial_t Q(z,1)$, find

$$\partial_t H = \frac{\alpha^3 v^4 + 3\,\alpha^2 v^3 + 3\,\alpha v^2 + v}{\alpha^3 d v^5 + 3\,\alpha^2 d v^4 - (2\,\alpha^2 - 3\,\alpha d)v^3 - ((\alpha-1)d + 3\,\alpha)v^2 - ((\alpha+1)d - 2\,\alpha + 1)v + 1}$$

4. Plug in expansion for $v \to$ expansion for $\partial_t H$

Counting Good Vertices



- **Goal:** study G_n ... # of good nodes in tree of size n

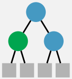
Strategy:

1. Take functional equations for $P(z,t)$, $Q(z,t)$, implicit differentiation w.r.t. t; set $t=1$
2. Solve for $\partial_t P(z,1)$, $\partial_t Q(z,1)$,
3. use substitution, add: $\partial_t H(z,1) = \partial_t P(z,1) + \partial_t Q(z,1)$, find

$$\partial_t H = \frac{\alpha^3 v^4 + 3\,\alpha^2 v^3 + 3\,\alpha v^2 + v}{\alpha^3 d v^5 + 3\,\alpha^2 d v^4 - (2\,\alpha^2 - 3\,\alpha d) v^3 - ((\alpha-1)d + 3\,\alpha) v^2 - ((\alpha+1)d - 2\,\alpha + 1) v + 1}$$

4. Plug in expansion for $v \to$ expansion for $\partial_t H$
   string length: 223391

Counting Good Vertices



- **Goal:** study G_n ... # of good nodes in tree of size n

Strategy:

1. Take functional equations for $P(z,t)$, $Q(z,t)$, implicit differentiation w.r.t. t; set $t = 1$
2. Solve for $\partial_t P(z,1)$, $\partial_t Q(z,1)$,
3. use substitution, add: $\partial_t H(z,1) = \partial_t P(z,1) + \partial_t Q(z,1)$, find

$$\partial_t H = \frac{\alpha^3 v^4 + 3\,\alpha^2 v^3 + 3\,\alpha v^2 + v}{\alpha^3 d v^5 + 3\,\alpha^2 d v^4 - (2\,\alpha^2 - 3\,\alpha d) v^3 - ((\alpha - 1)d + 3\,\alpha) v^2 - ((\alpha + 1)d - 2\,\alpha + 1) v + 1}$$

4. Plug in expansion for $v \to$ expansion for $\partial_t H$
   string length: 223391
5. Singularity Analysis; normalize by # of trees $\to \mathbb{E} G_n$

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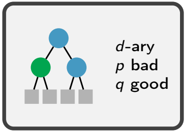
Counting Good Vertices



- **Goal:** study G_n … # of good nodes in tree of size n

Strategy:

1. Take functional equations for $P(z,t)$, $Q(z,t)$, implicit differentiation w.r.t. t; set $t = 1$
2. Solve for $\partial_t P(z,1)$, $\partial_t Q(z,1)$,
3. use substitution, add: $\partial_t H(z,1) = \partial_t P(z,1) + \partial_t Q(z,1)$, find

$$\partial_t H = \frac{\alpha^3 v^4 + 3\,\alpha^2 v^3 + 3\,\alpha v^2 + v}{\alpha^3 d v^5 + 3\,\alpha^2 d v^4 - (2\,\alpha^2 - 3\,\alpha d)v^3 - ((\alpha-1)d + 3\,\alpha)v^2 - ((\alpha+1)d - 2\,\alpha + 1)v + 1}$$

4. Plug in expansion for $v \to$ expansion for $\partial_t H$
   string length: 223391
5. Singularity Analysis; normalize by # of trees $\to \mathbb{E} G_n$

Repeat for $\partial_t^2 H$ (string length 60809) $\rightsquigarrow \mathbb{E} G_n(G_n - 1)$

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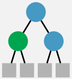
Counting Good Nodes: Results



d-ary
p bad
q good

Theorem: Good Nodes (H.–Prodinger '19+)

In [155]:
```
good_nodes_expectation = (H_d1_coef_asy / H_coef_asy).map_coefficients(lambda ex: simplify_by_tau(ex.simplify_rational
good_nodes_expectation
```

Out[155]:

$$\left(\frac{2\,\alpha^2 d\tau^2 + 3\,\alpha d\tau + 1}{4\,\alpha + 1}\right) n + \frac{\left(8\,\alpha^3 d^3\tau^2 - 32\,\alpha^3 d^2\tau^2 + 29\,\alpha^2 d^3\tau^2 + 6\,\alpha^2 d^3\tau + 32\,\alpha^3 d\tau^2 - 19\,\alpha^2 d^2\tau^2 + 9\,\alpha d^3\tau^2 - 18\,\alpha^2 d^2\tau + 44\,\alpha d^3\tau - 7\,\alpha d^2\tau^2 + 12\,\alpha^2 d\tau - 27\,\alpha d^2\tau + 14\,\alpha d^3\tau - 38\,\alpha d^2 - 9\,\alpha d\tau - 12\,d^2\tau + 14\,\alpha d - 14\,d^2 + 16\,\alpha + 12\,d\right)}{\left(4\,\alpha^2 d^3 - 24\,\alpha^2 d^2 + 8\,\alpha d^3 + 48\,\alpha^2 d + 4\,\alpha d^2 + 4\,d^3 - 32\,\alpha^2 - 13\,\alpha d - 4\,d^2\right)(4\,\alpha + 1)} + O\left(n^{-\frac{1}{2}}\right)$$

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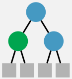
Counting Good Nodes: Results



Theorem: Good Nodes (H.–Prodinger '19+)

$$\mathbb{E}G_n = \frac{2\alpha^2 d\tau^2 + 3\alpha d\tau + 1}{4\alpha + 1} n + O(1),$$

In [155]:
```
good_nodes_expectation = (H_d1_coef_asy / H_coef_asy).map_coefficients(lambda ex: simplify_by_tau(ex.simplify_rational
good_nodes_expectation
```

Out[155]:

$$\left(\frac{2\,\alpha^2 d\tau^2 + 3\,\alpha d\tau + 1}{4\,\alpha + 1}\right) n + \frac{\left(8\,\alpha^3 d^3\tau^2 - 32\,\alpha^3 d^2\tau^2 + 29\,\alpha^2 d^3\tau^2 + 6\,\alpha^2 d^3\tau + 32\,\alpha^3 d\tau^2 - 19\,\alpha^2 d^2\tau^2 + 9\,\alpha d^3\tau^2 - 18\,\alpha^2 d^2\tau + 44\,\alpha d^3\tau - 7\,\alpha d^2\tau^2 + 12\,\alpha^2 d\tau - 27\,\alpha d^2\tau + 14\,\alpha d^3\tau - 38\,\alpha d^2 - 9\,\alpha d\tau - 12\,d^2\tau + 14\,\alpha d - 14\,d^2 + 16\,\alpha + 12\,d\right)}{\left(4\,\alpha^2 d^3 - 24\,\alpha^2 d^2 + 8\,\alpha d^3 + 48\,\alpha^2 d + 4\,\alpha d^2 + 4\,d^3 - 32\,\alpha^2 - 13\,\alpha d - 4\,d^2\right)(4\,\alpha + 1)} + O\left(n^{-\frac{1}{2}}\right)$$

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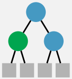
Counting Good Nodes: Results



Theorem: Good Nodes (H.–Prodinger '19+)

$$\mathbb{E}G_n = \frac{2\alpha^2 d\tau^2 + 3\alpha d\tau + 1}{4\alpha + 1} n + O(1),$$

and for $d \to \infty$,

$$\frac{2\alpha^2 d\tau^2 + 3\alpha d\tau + 1}{4\alpha + 1} = \frac{q}{p+q} + \frac{2\alpha^2}{(\alpha+1)^3} d^{-1} + O(d^{-2})$$

In [155]:
```
good_nodes_expectation = (H_d1_coef_asy / H_coef_asy).map_coefficients(lambda ex: simplify_by_tau(ex.simplify_rational
good_nodes_expectation
```

Out[155]:

$$\left(\frac{2\alpha^2 d\tau^2 + 3\alpha d\tau + 1}{4\alpha + 1}\right) n + \frac{\left(8\alpha^3 d^3\tau^2 - 32\alpha^3 d^2\tau^2 + 29\alpha^2 d^3\tau^2 + 6\alpha^2 d^3\tau + 32\alpha^3 d\tau^2 - 19\alpha^2 d^2\tau^2 + 9\alpha d^3\tau^2 - 18\alpha^2 d^2\tau + 44\alpha d^3\tau - 7\alpha d^2\tau^2 + 12\alpha^2 d\tau - 27\alpha d^2\tau + 14\alpha d^3\tau - 38\alpha d^2 - 9\alpha d\tau - 12 d^2\tau + 14\alpha d - 14 d^2 + 16\alpha + 12 d\right)}{\left(4\alpha^2 d^3 - 24\alpha^2 d^2 + 8\alpha d^3 + 48\alpha^2 d + 4\alpha d^2 + 4 d^3 - 32\alpha^2 - 13\alpha d - 4 d^2\right)(4\alpha + 1)} + O\left(n^{-\frac{1}{2}}\right)$$

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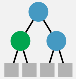
Counting Good Nodes: Results



Theorem: Good Nodes (H.–Prodinger '19+)

$$\mathbb{E}G_n = \frac{2\alpha^2 d\tau^2 + 3\alpha d\tau + 1}{4\alpha + 1} n + O(1),$$

and for $d \to \infty$,

$$\frac{2\alpha^2 d\tau^2 + 3\alpha d\tau + 1}{4\alpha + 1} = \frac{q}{p+q} + \frac{2\alpha^2}{(\alpha+1)^3} d^{-1} + O(d^{-2})$$

In [155]:
```
good_nodes_expectation = (H_d1_coef_asy / H_coef_asy).map_coefficients(lambda ex: simplify_by_tau(ex.simplify_rational
good_nodes_expectation
```

Out[155]:

$$\left(\frac{2\,\alpha^2 d\tau^2 + 3\,\alpha d\tau + 1}{4\,\alpha + 1}\right) n + \frac{\left(8\,\alpha^3 d^3\tau^2 - 32\,\alpha^3 d^2\tau^2 + 29\,\alpha^2 d^3\tau^2 + 6\,\alpha^2 d^3\tau + 32\,\alpha^3 d\tau^2 - 19\,\alpha^2 d^2\tau^2 + 9\,\alpha d^3\tau^2 - 18\,\alpha^2 d^2\tau + 44\,\alpha d^3\tau - 7\,\alpha d^2\tau^2 + 12\,\alpha^2 d\tau - 27\,\alpha d^2\tau + 14\,\alpha d^3\tau - 38\,\alpha d^2 - 9\,\alpha d\tau - 12\,d^2\tau + 14\,\alpha d - 14\,d^2 + 16\,\alpha + 12\,d\right)}{\left(4\,\alpha^2 d^3 - 24\,\alpha^2 d^2 + 8\,\alpha d^3 + 48\,\alpha^2 d + 4\,\alpha d^2 + 4\,d^3 - 32\,\alpha^2 - 13\,\alpha d - 4\,d^2\right)(4\,\alpha + 1)} + O\left(n^{-\frac{1}{2}}\right)$$

In [187]:
```
(factorial_moment_2_asy
 - good_nodes_expectation^2
 + good_nodes_expectation).map_coefficients(lambda ex: simplify_by_tau(ex.simplify_rational()))
```

Out[187]: $O\left(n^{\frac{3}{2}}\right)$

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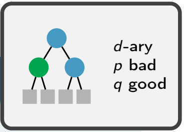
Counting Good Nodes: Results



Theorem: Good Nodes (H.–Prodinger '19+)

$$\mathbb{E}G_n = \frac{2\alpha^2 d\tau^2 + 3\alpha d\tau + 1}{4\alpha + 1} n + O(1),$$

and for $d \to \infty$,

$$\frac{2\alpha^2 d\tau^2 + 3\alpha d\tau + 1}{4\alpha + 1} = \frac{q}{p+q} + \frac{2\alpha^2}{(\alpha+1)^3} d^{-1} + O(d^{-2})$$

Furthermore, $\mathbb{V}G_n = O(n^{3/2})$. ☹ – higher precision needed!

In [155]:
```
good_nodes_expectation = (H_d1_coef_asy / H_coef_asy).map_coefficients(lambda ex: simplify_by_tau(ex.simplify_rational
good_nodes_expectation
```

Out[155]:

$$\left(\frac{2\alpha^2 d\tau^2 + 3\alpha d\tau + 1}{4\alpha + 1}\right) n + \frac{(8\alpha^3 d^3\tau^2 - 32\alpha^3 d^2\tau^2 + 29\alpha^2 d^3\tau^2 + 6\alpha^2 d^3\tau + 32\alpha^3 d\tau^2 - 19\alpha^2 d^2\tau^2 + 9\alpha d^3\tau^2 - 18\alpha^2 d^2\tau + 44\alpha d^3\tau - 7\alpha d^2\tau^2 + 12\alpha^2 d\tau - 27\alpha d^2\tau + 14\alpha d^3\tau - 38\alpha d^2 - 9\alpha d\tau - 12 d^2\tau + 14\alpha d - 14 d^2 + 16\alpha + 12 d)}{(4\alpha^2 d^3 - 24\alpha^2 d^2 + 8\alpha d^3 + 48\alpha^2 d + 4\alpha d^2 + 4 d^3 - 32\alpha^2 - 13\alpha d - 4 d^2)(4\alpha + 1)} + O\left(n^{-\frac{1}{2}}\right)$$

In [187]:
```
(factorial_moment_2_asy
 - good_nodes_expectation^2
 + good_nodes_expectation).map_coefficients(lambda ex: simplify_by_tau(ex.simplify_rational()))
```

Out[187]: $O\left(n^{\frac{3}{2}}\right)$

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Asymptotics: Numerical Example

- Today: **27-06-2019** $\rightsquigarrow p = 27,\ q = 6,\ d = 2019$

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$\tau = 0.0000901\ldots$

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  - in comparison: $(p+q)de \approx 181110.96$
- $\mathbb{E}G_n = 0.181938\ldots \cdot n + O(1)$
  - in comparison: $\frac{q}{p+q} = \frac{6}{33} = \frac{2}{11} \approx 0.1818181818$

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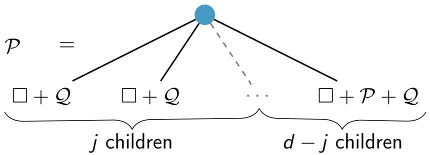
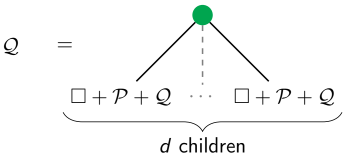
Outlook: Why only constrain the first child?





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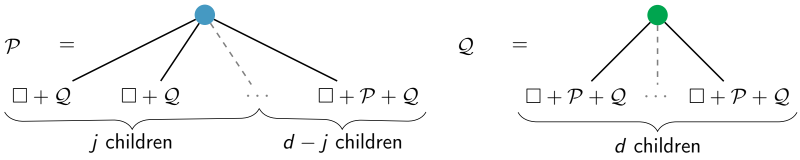

$$P = pz(1 + Q)^j (1 + P + Q)^{d-j}$$

$$Q = qzt(1 + P + Q)^d$$

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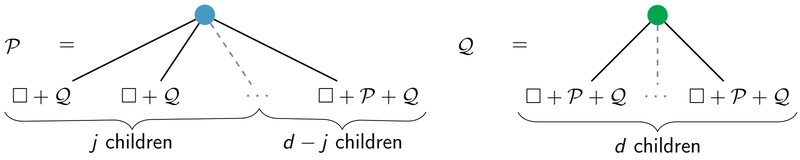

$$P = pz(1 + Q)^j (1 + P + Q)^{d-j}$$
$$Q = qzt(1 + P + Q)^d$$

P cannot be solved explicitly any more:

$$P(1 + P + Q)^j t = \alpha Q(1 + Q)^j$$

Substitution that simplifies this case? Other Approaches?

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Related Problem: More Systematic Restrictions

Credit: Stephan Wagner

- Given colors $\{1, \ldots, k\}$, matrix $(a_{ij})_{1 \le i,j \le k} \in \{0,1\}^{k \times k}$
- Color all nodes of plane trees (alternative: just rooted)

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$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Counting sequence: $2 \cdot C_{n-1}$

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Counting sequence: $2 \cdot C_{n-1}$

Another Example

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Counting seq.: $3 \cdot 2^{n-1} \cdot C_{n-1}$

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Related Problem: More Systematic Restrictions (ctd.)

- $T_i(z) \ldots$ GF of trees with root of color i
- $T(z) = \sum_{i=1}^{k} T_i(z) \ldots$ overall GF

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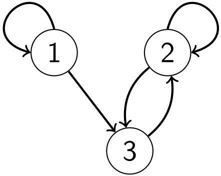

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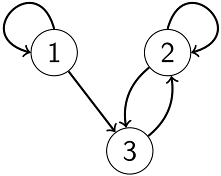

- **Proposition:** Isomorphic graphs $\rightsquigarrow$ same counting sequence

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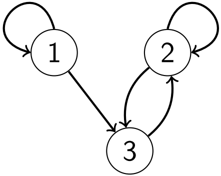

- **Proposition:** Isomorphic graphs $\rightsquigarrow$ same counting sequence
- Which matrices lead to the same counting sequence?

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