Counting Ascents in Generalized Dyck Paths

Benjamin Hackl

joint work with *Clemens Heuberger* and *Helmut Prodinger*

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT | WIEN GRAZ

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mso DK modeling simulation optimization

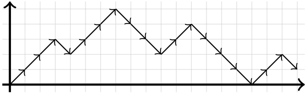
KARL POPPER KOLLEG

FWF Der Wissenschaftsfonds.

Łukasiewicz Paths

Dyck Meanders:

- Sequences of $\{-1, 1\} \triangleq \{\searrow, \nearrow\}$,
- Never below axis.



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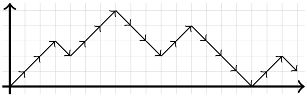

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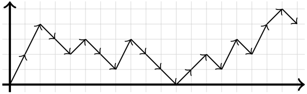

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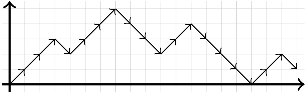

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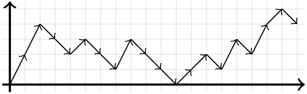
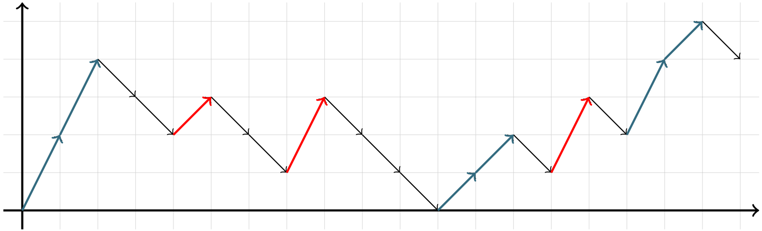


Notation. $S(u)$ ... GF of $\mathcal{S}$, $S_+(u) = S(u) - u^{-1}$.

$$\mathcal{S} = \{-1, 0, 2\} \leftrightsquigarrow S(u) = u^{-1} + 1 + u^2,\; S_+(u) = 1 + u^2$$

Ascents



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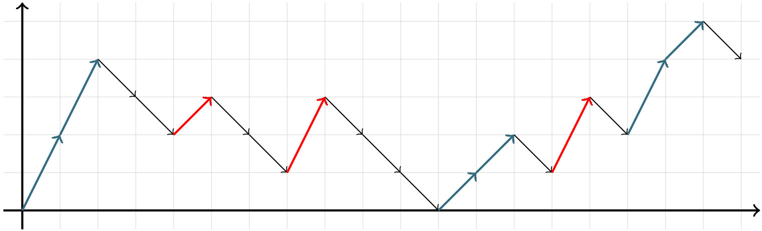

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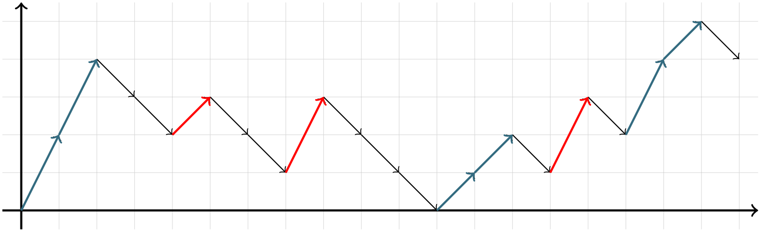
Ascents



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- ***r*-Ascent:** ascent of length r.

Path Classes of Interest

Background: Banderier–Flajolet, Basic AC of dir. lattice paths

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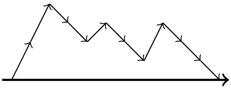

Excursion

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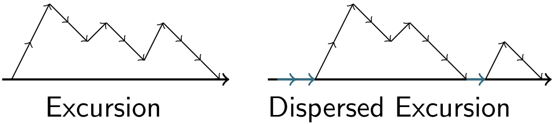

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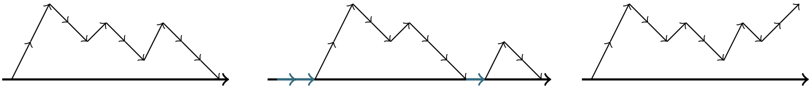
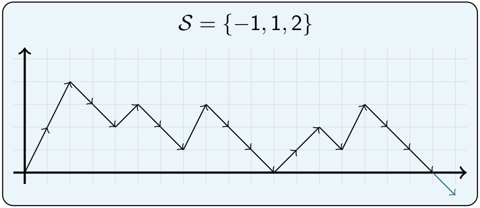
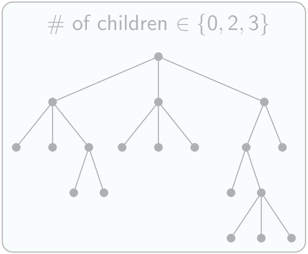
- **Meanders**: unrestricted



Excursion Dispersed Excursion Meander

Bijection: Excursions $\longleftrightarrow$ Trees





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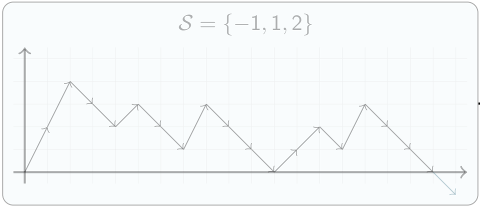

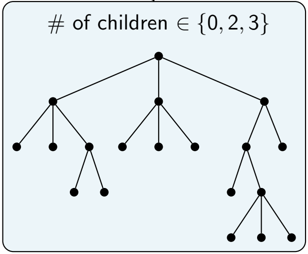

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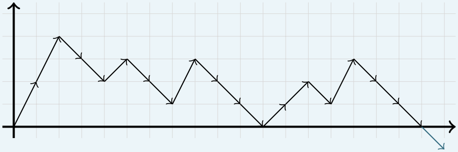
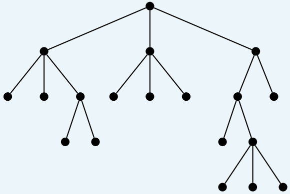


\# of children $\in \{0, 2, 3\}$



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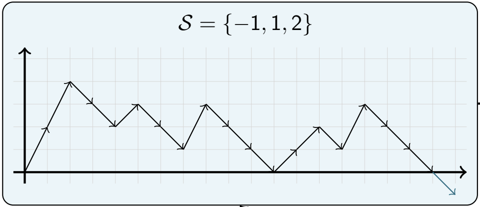

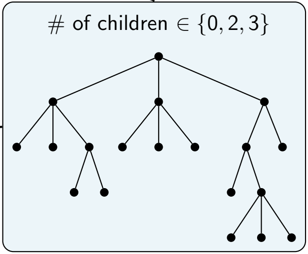

Observation

Excursions($\mathcal{S}$) $\longleftrightarrow$ Plane Trees($\mathcal{S} + 1$)

Exploiting the Tree Structure

Proposition

- $V(z,t)$ … *OGF of* $\mathcal{V}$*: plane trees,* $\#$ *of children* $\in \mathcal{S} + 1$

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which enumerates seq. of non-negative steps. v ... height

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1. Consequence of bijection.

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1. Consequence of bijection.
2. Decompose w.r.t. leftmost path:

$$\mathcal{V}=\circ\times\mathrm{SEQ}(\circ\times\sum\nolimits_{s\in\mathcal{S},\ s\geq 0}\mathcal{V}^s)$$

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- $v = 0 \rightsquigarrow$ excursions, $v = 1 \rightsquigarrow$ meanders

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- analytic approach via **singular inversion**!

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- $V(z, 1)$ has p square root singularities on its radius of convergence

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2. *dominant singularities: square-root singularities at $\zeta\rho$*
   - *ζ . . . pth root of unity*
3. *Singular expansion $z \to \zeta\rho$:*

$$V(z,1) = \zeta\tau - \zeta\sqrt{\frac{2S(\tau)}{S''(\tau)}}\Bigl(1 - \frac{z}{\zeta\rho}\Bigr)^{1/2} + O\Bigl(1 - \frac{z}{\zeta\rho}\Bigr).$$

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Excursions – Example 1/2

Example (r-Ascents in Dyck paths)

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Example (June 29, 2018)

- $\mathcal{S} = \{-1, 6, 29, 2018\}$, $S(u) = u^{-1} + u^{6} + u^{29} + u^{2018}$, $p = 1$,

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r-Ascents in Dispersed Excursions, $\tau \neq 1$

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- $r = 1 \rightsquigarrow$ known result recovered

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$$\mathbb{E}M_{n,r} = \mu n + c_{\mathcal{S}} + O\Bigl(\Bigl(\frac{S(\tau)}{S(1)}\Bigr)^{n} n^{5/2}\Bigr), \quad \mathbb{V}M_{n,r} = \sigma^2 n + O(1),$$

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Also, $M_{n,r}$ is asymptotically normally distributed for $n \to \infty$.

Meanders – Special Cases

Proposition (Dyck Meanders)

- $\mathcal{S} = \{-1, 1\}$, $p = 2$, $\tau = 1$.

$$\mathbb{E}M_{n,r} = \frac{n}{2^{r+2}} + \frac{\sqrt{2\pi}(r-2)}{2^{r+3}} n^{1/2} - \frac{r^2 - r - 8}{2^{r+3}} + O(n^{-1/2}),$$

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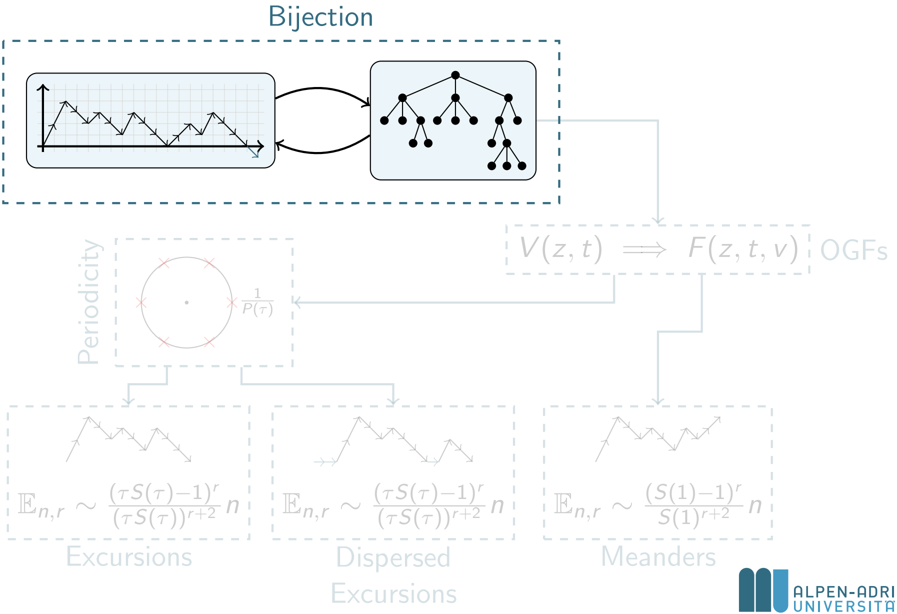
Proposition (Motzkin Meanders)

- $\mathcal{S} = \{-1, 0, 1\}$, $p = 1$, $\tau = 1$.

$$\mathbb{E}M_{n,r} = \frac{2^r}{3^{r+2}} n + \frac{\sqrt{3\pi}(r-4)2^{r-2}}{3^{r+2}} n^{1/2} + O(1),$$

$$\mathbb{V}M_{n,r} = \frac{3^{r+2}2^{r+4} - 2^{2r}(3r^2(\pi - 2) - 8r(3\pi - 10) + 48\pi - 144)}{16 \cdot 3^{2r+4}} n + O(n^{1/2}).$$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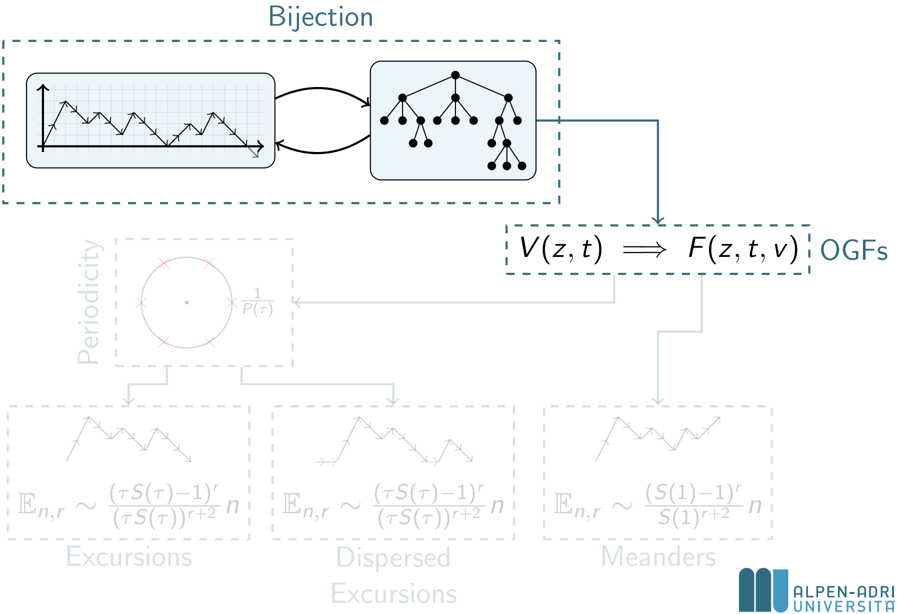
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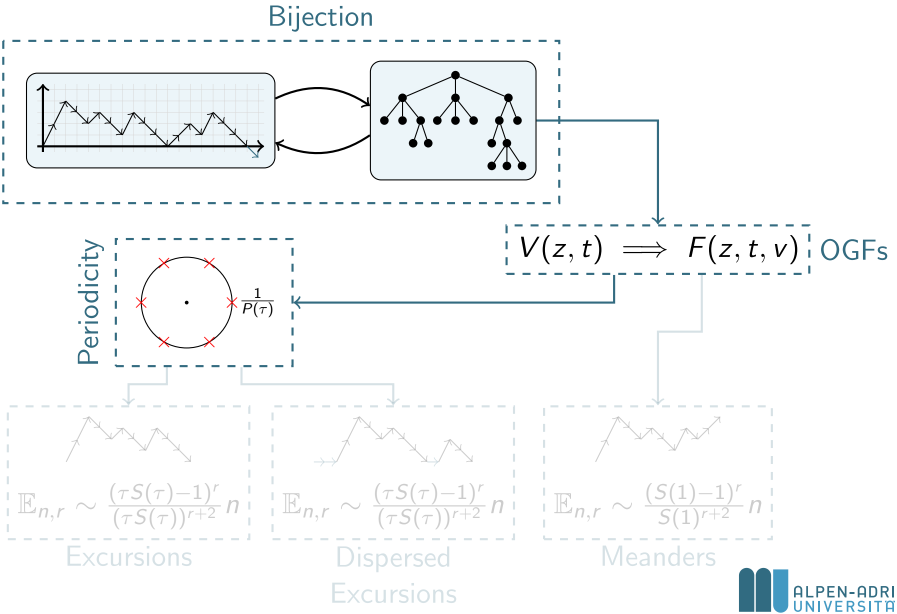
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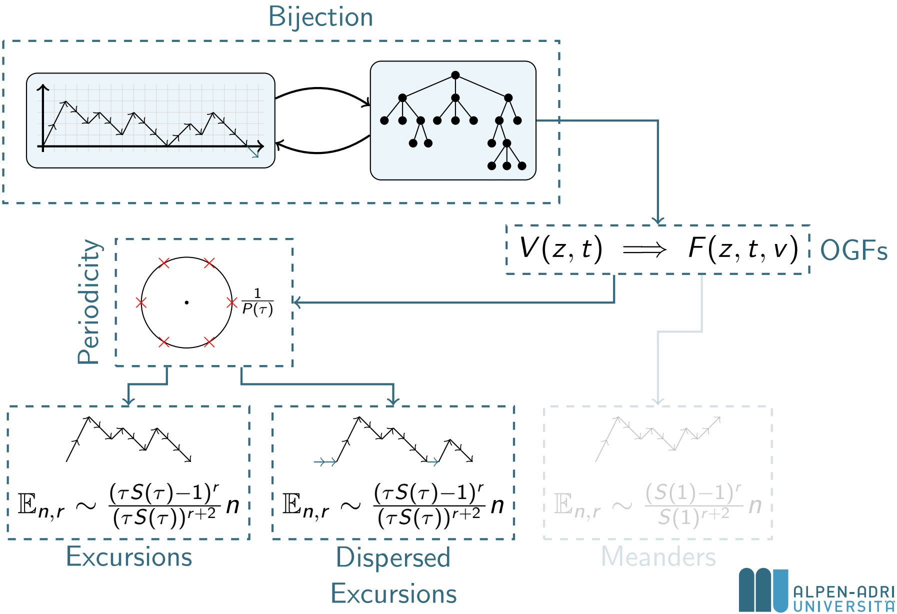
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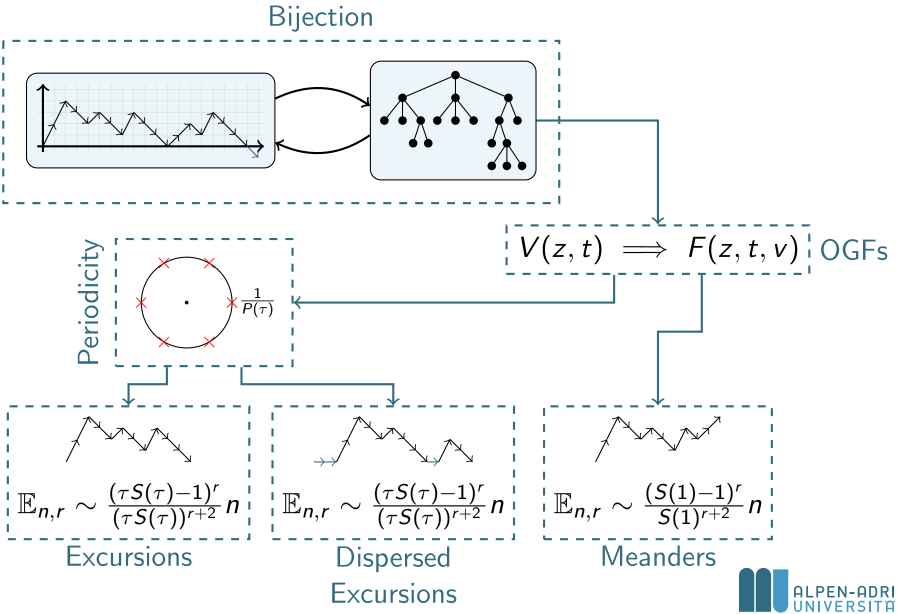
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