Growing and Destroying Classes of Plane Trees

Benjamin Hackl

joint work with *Helmut Prodinger*

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT | WIEN GRAZ

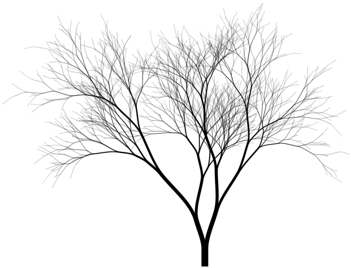
June 23, 2017

mso DK modeling simulation optimization

KARL POPPER KOLLEG

FWF Der Wissenschaftsfonds.

(Rooted) Plane trees



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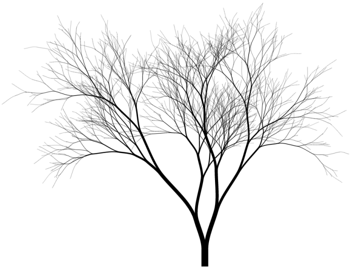

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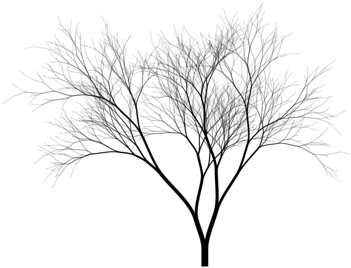

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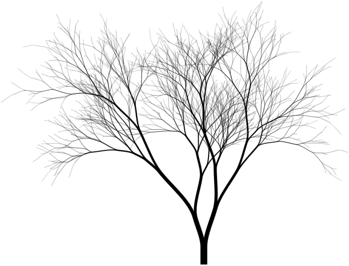

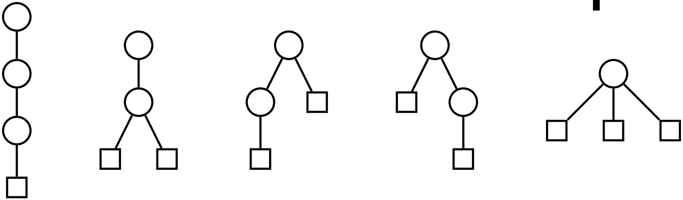

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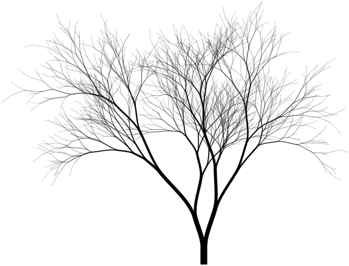

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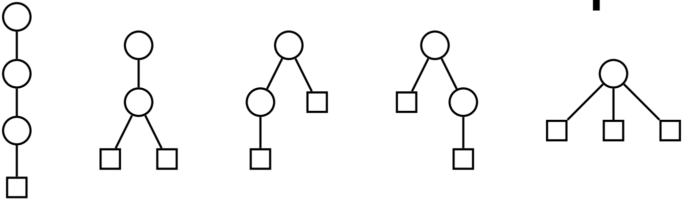

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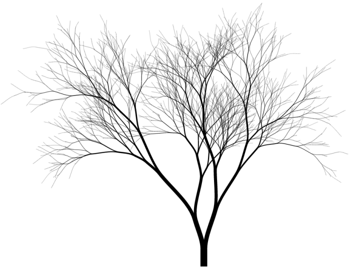

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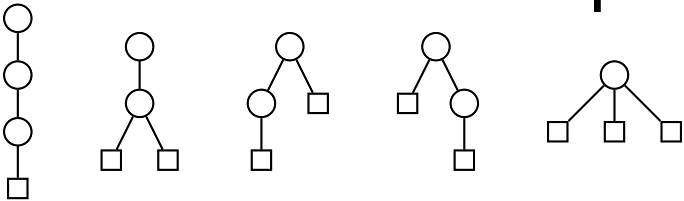

- $C_n = \frac{1}{n+1}\binom{2n}{n}$ plane trees of size $n+1$
- combinatorial class $\mathcal{T}$, g.f. $T(z) = \frac{1-\sqrt{1-4z}}{2}$

Growing plane trees

- How can we grow trees?

~~Growing~~ Trimming plane trees

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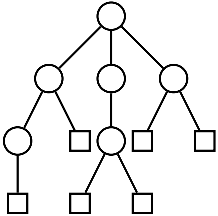

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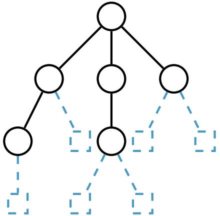

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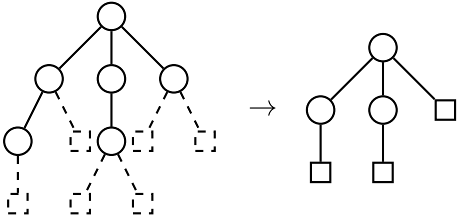

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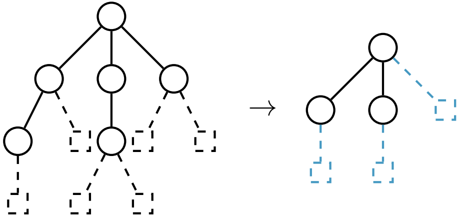

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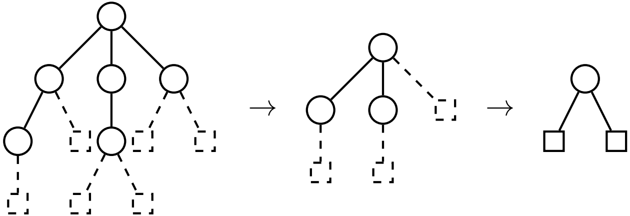

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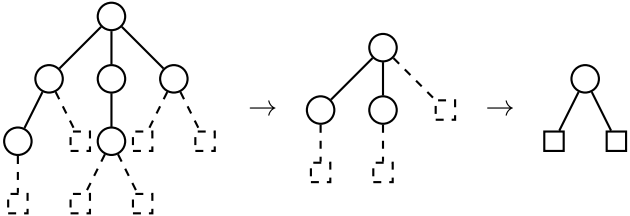

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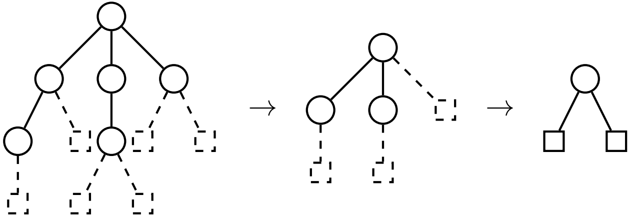

- Growing trees:
  - grow new leaves out of current leaves and inner nodes

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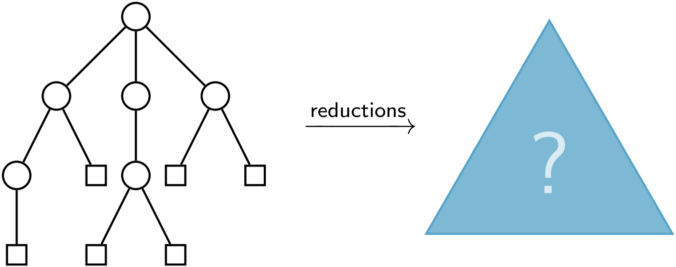

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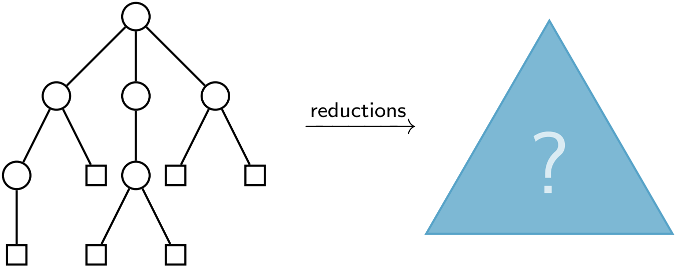

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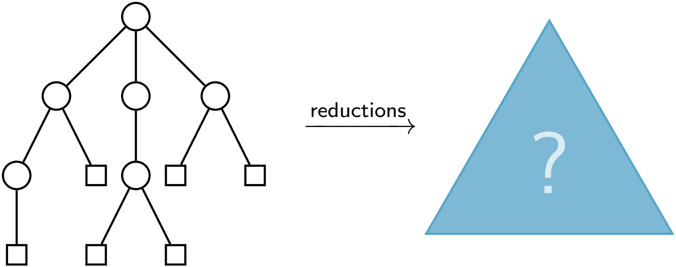

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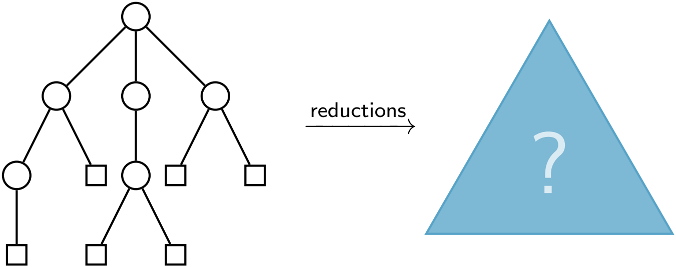

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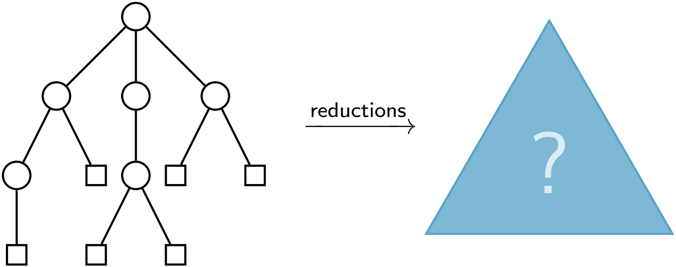

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- Parameters: **Age** and **Ancestor size**

Summary: Age

Definition

- $\tau \ldots$ *some plane tree*

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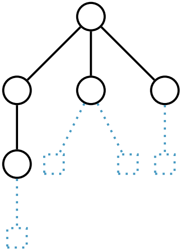
Leaves

⇝ height (Knuth, de Brujin, Rice)
$\mathbb{E} \sim \sqrt{\pi n}$



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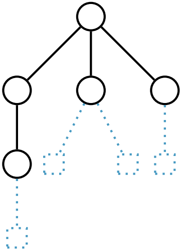
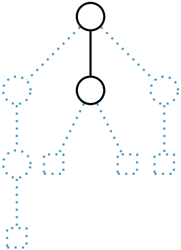


Paths

$\rightsquigarrow$ Pruning number (Zeilberger)
$\mathbb{E} \sim \log_4 n$



Summary: Size of rth Ancestor

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

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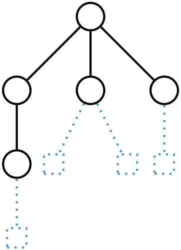
limit law: ✓



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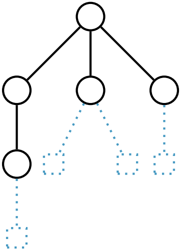

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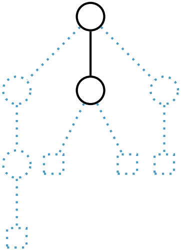

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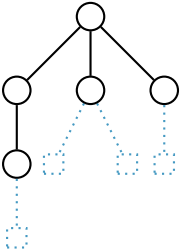

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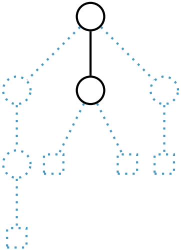

Old leaves

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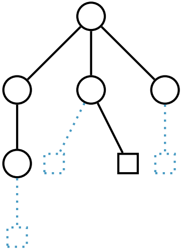

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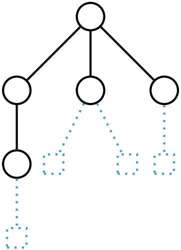

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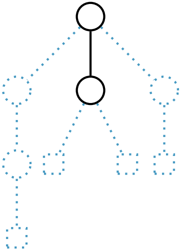

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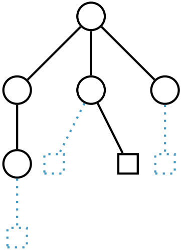

Old paths

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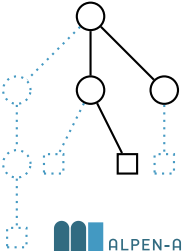

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Stanley, Catalan bijection #26

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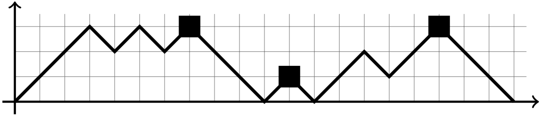

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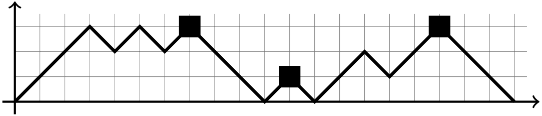

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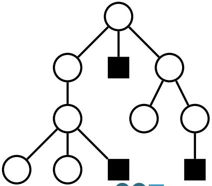
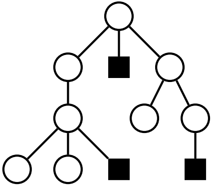


Catalan–Stanley Trees



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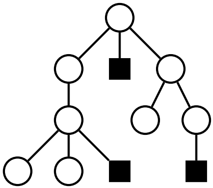

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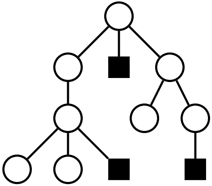

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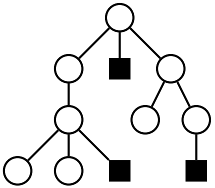

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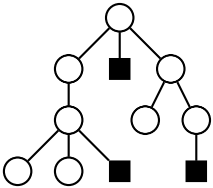

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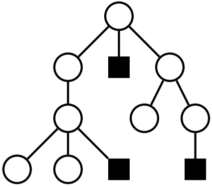

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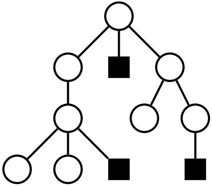

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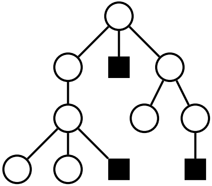

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and for $n \geq 2$ there are C_{n-2} Catalan–Stanley trees with n nodes.

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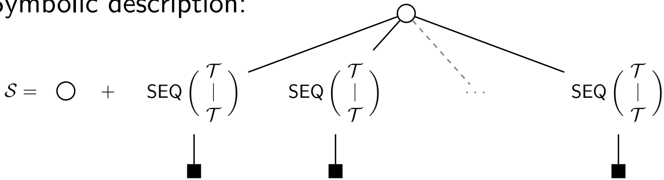

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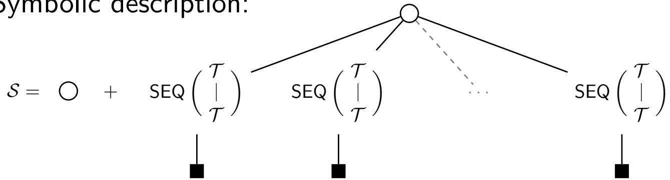

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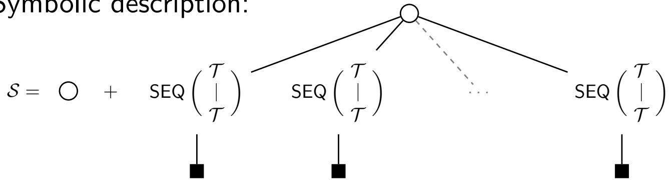

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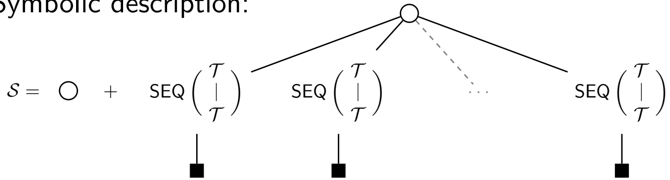

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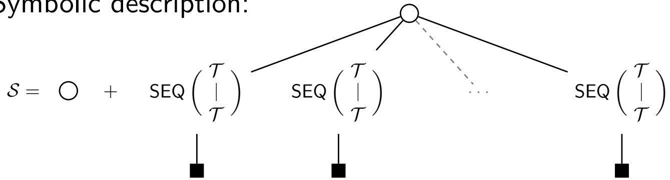

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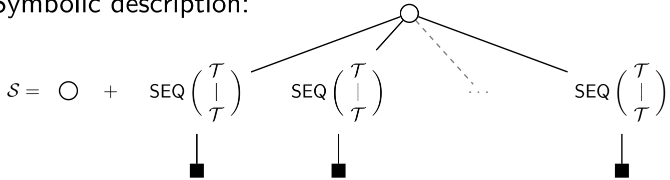

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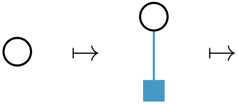

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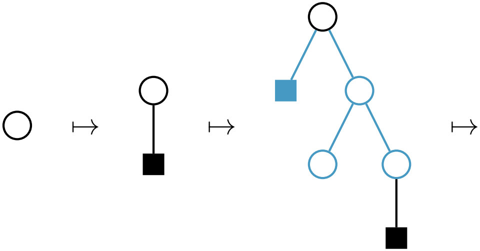

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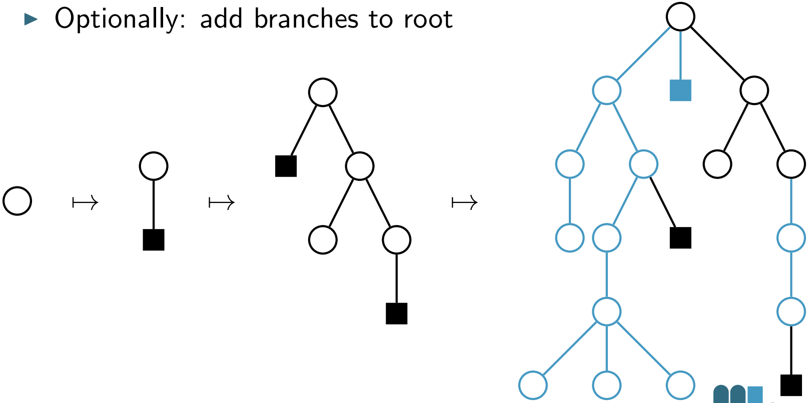

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Generating Function for Ancestors

Corollary

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Proof: mark original tree size with v, expand r times. □

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Proof: mark original tree size with v, expand r times. □

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In particular, we have

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Summary: Age and Ancestors of Catalan–Stanley trees

- "nice", not too artificial growth process with different parameter behavior ✓

Age

Size of rth Ancestor

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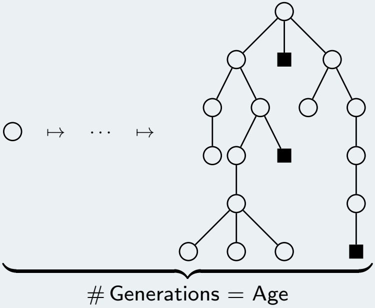

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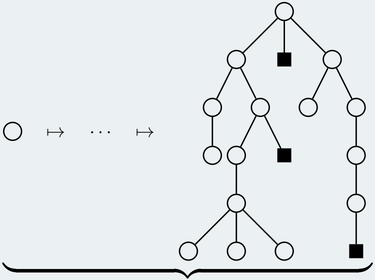

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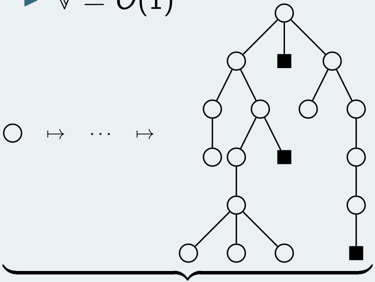

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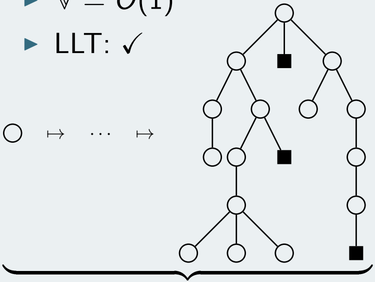

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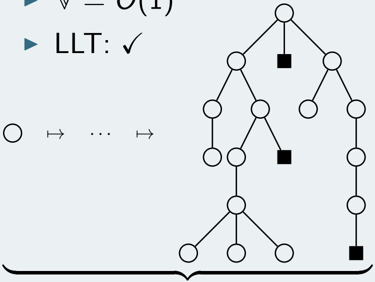

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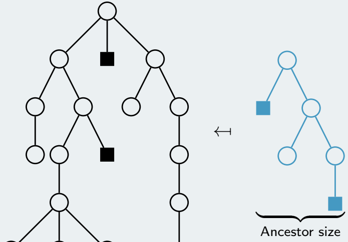

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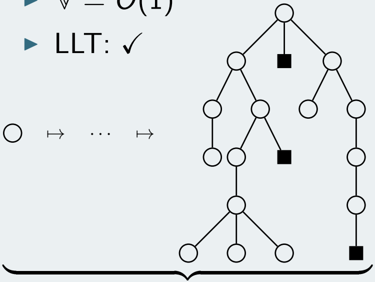

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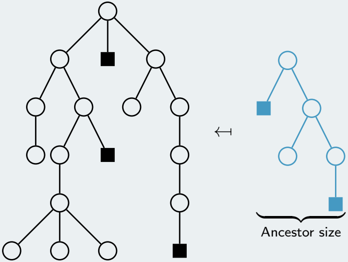

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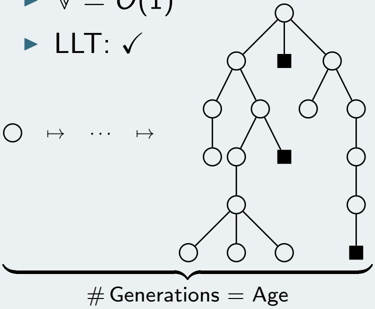

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