Growing and Destroying Classes of Plane Trees

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joint work with Helmut Prodinger

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(Rooted) Plane trees

Characterization:
- unlabeled
(Rooted) Plane trees

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- special node: root
(Rooted) Plane trees

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- order of children matters
(Rooted) Plane trees

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Characterization:

- unlabeled
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\[ C_n = \frac{1}{n+1} \binom{2n}{n} \text{ plane trees of size } n + 1 \]
(Rooted) Plane trees

Characterization:

- unlabeled
- special node: root
- order of children matters

\[ C_n = \frac{1}{n+1} \binom{2n}{n} \text{ plane trees of size } n + 1 \]

- combinatorial class \( \mathcal{T} \), g.f. \( T(z) = \frac{1 - \sqrt{1 - 4z}}{2} \)
Growing plane trees

- How can we grow trees?
Growing Trimming plane trees

- How can we grow trees?
- Easier question: what could be the inverse operation?
Growing Trimming plane trees

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  - Most straightforward: cut away all leaves!
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Growing trees:
Growing Trimming plane trees

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Growing trees:
- grow new leaves out of current leaves and inner nodes
“What?” and “How?”

▶ **Aim:** analysis of tree structure under iterated reduction
“What?” and “How?”

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“What?” and “How?”

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  ![Tree Structure Diagram]

- Algorithmic description
“What?” and “How?”

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- Algorithmic description
- Investigation of “tree expansion” $\leadsto$ g.f.
“What?” and “How?”

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![Tree structure diagram]

- Algorithmic description
- Investigation of “tree expansion” $\leadsto$ g.f.
- Coefficient extraction; Parameter distribution
“What?” and “How?”

- **Aim**: analysis of tree structure under iterated reduction

- Algorithmic description
- Investigation of “tree expansion” $\rightsquigarrow$ g.f.
- Coefficient extraction; Parameter distribution
- Parameters: **Age** and **Ancestor size**
Summary: Age

Definition

- $\tau \ldots$ some plane tree
Summary: Age

Definition

- $\tau \ldots$ some plane tree
- **Age** of $\tau$: # of generations required to grow $\tau$ from $\bigcirc$
Summary: Age

Definition

- \( \tau \ldots \) some plane tree
- **Age** of \( \tau \): \# of generations required to grow \( \tau \) from \( \bigcirc \)

Leaves

\( \leadsto \) height (Knuth, de Bruijn, Rice)

\[ \mathbb{E} \sim \sqrt{\pi n} \]
Summary: Age

Definition

- $\tau$... *some plane tree*
- **Age of $\tau$: # of generations required to grow $\tau$ from $\circ$**

Leaves

$\sim$ height (Knuth, de Bruijn, Rice)

$E \sim \sqrt{\pi n}$

Paths

$\sim$ Pruning number (Zeilberger)

$E \sim \log_4 n$
Summary: Size of \( r \)th Ancestor

Leaves

\[
E \sim \frac{n}{r+1} \\
V \sim \frac{r(r+2)}{6(r+1)^2} n
\]

limit law: ✓
Summary: Size of $r$th Ancestor

**Leaves**

\[ E \sim \frac{n}{r+1} \]
\[ V \sim \frac{r(r+2)}{6(r+1)^2} n \]

Limit law: ✓

**Paths**

\[ E \sim \frac{n}{2^{r+1} - 1} \]
\[ V \sim \frac{2^{r+1} (2^r - 1)}{3(2^{r+1} - 1)^2} n \]

Limit law: ✓
Summary: Size of $r$th Ancestor

Leaves

$E \sim \frac{n}{r+1}$

$V \sim \frac{r(r+2)}{6(r+1)^2} n$

limit law: ✓

Paths

$E \sim \frac{n}{2^{r+1}-1}$

$V \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n$

limit law: ✓

Old leaves

$E \sim (2 - B_r(1/4)) n$

$V = \Theta(n)$

limit law: ✓
## Summary: Size of rth Ancestor

### Leaves

- Expected size of leaves: $E \sim \frac{n}{r+1}$
- Variance of leaves: $\mathbb{V} \sim \frac{r(r+2)}{6(r+1)^2}n$
- Limit law: ✓

### Paths

- Expected size of paths: $E \sim \frac{n}{2^{r+1}-1}$
- Variance of paths: $\mathbb{V} \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2}n$
- Limit law: ✓

### Old leaves

- Expected size of old leaves: $E \sim (2 - B_r(1/4))n$
- Variance of old leaves: $\mathbb{V} = \Theta(n)$
- Limit law: ✓

### Old paths

- Expected size of old paths: $E \sim \frac{2n}{r+2}$
- Variance of old paths: $\mathbb{V} \sim \frac{2r(r+1)}{3(r+2)^2}n$
- Limit law: ???

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Something New

- **Want:** not too artificial reduction with different parameter behavior
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**Stanley, Catalan bijection #26**

Dyck paths from \((0, 0)\) to \((2n + 2, 0)\) such that every maximal sequence of consecutive steps \((1, -1)\) ending on the \(x\)-axis has odd length.
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Stanley, Catalan bijection #26

Dyck paths from \((0, 0)\) to \((2n + 2, 0)\) such that every maximal sequence of consecutive steps \((1, -1)\) ending on the \(x\)-axis has odd length.
Catalan–Stanley Trees

Catalan–Stanley Tree:

A Catalan–Stanley tree is a binary tree where the rightmost leaves in all branches of the root have odd distance. The generating function for the class of Catalan–Stanley trees, denoted by $S(z, t)$, is given by:

$$S(z, t) = z + zt(1 - t - T_2),$$

and for $n \geq 2$, there are $C_{n-2}$ Catalan–Stanley trees with $n$ nodes.
Catalan–Stanley Trees

Catalan–Stanley Tree:  
- ... plane tree
Catalan–Stanley Trees

Catalan–Stanley Tree:
- ... plane tree
- ... rightmost leaves in all branches of root have odd distance
Catalan–Stanley Trees

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Proposition

- $S$ … class of Catalan–Stanley trees, g.f. $S(z, t)$
Catalan–Stanley Trees

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- ... plane tree
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- $S$... class of Catalan–Stanley trees, g.f. $S(z, t)$
- $z \equiv \bigcirc, \ t \equiv \blacksquare$
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- $S$ ... class of Catalan–Stanley trees, g.f. $S(z, t)$
- $z \triangleq \bigcirc$, $t \triangleq \blacksquare$
- $T = T(z)$ ... g.f. of plane trees
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\[
S(z, t) = z + \frac{zt}{1 - t - T^2},
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S(z, t) = z + \frac{zt}{1 - t - T^2},
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Catalan–Stanley Trees (Proof)

- $T$... class of plane trees
Catalan–Stanley Trees (Proof)

- $\mathcal{T}$… class of plane trees
- Symbolic description:

$$S = \bigcirc + \text{SEQ}\left(\frac{\mathcal{T}}{\mathcal{T}}\right) \quad \text{SEQ}\left(\frac{\mathcal{T}}{\mathcal{T}}\right) \quad \ldots \quad \text{SEQ}\left(\frac{\mathcal{T}}{\mathcal{T}}\right)$$
Catalan–Stanley Trees (Proof)

- $\mathcal{T}$... class of plane trees
- Symbolic description:

$$S = \bigcirc + \text{SEQ} \left( \frac{T}{T} \right) \quad \text{SEQ} \left( \frac{T}{T} \right) \quad \ldots \quad \text{SEQ} \left( \frac{T}{T} \right)$$

$$\Rightarrow S(z, t) = z + \frac{zt}{1 - t - T^2} = z + \frac{zt}{1 - T^2}$$
Catalan–Stanley Trees (Proof)

- 🌳 class of plane trees
- Symbolic description:

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S = \bigcirc + \text{SEQ}
\left(\frac{T}{T}\right)
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- Count w.r.t. size: set \( t = z \), use \( T = \frac{z}{1-T} \)
Catalan–Stanley Trees (Proof)

- $\mathcal{T}$... class of plane trees
- Symbolic description:

\[
S = \bigcirc + \text{SEQ} \left( \frac{\mathcal{T}}{\mathcal{T}} \right) \text{SEQ} \left( \frac{\mathcal{T}}{\mathcal{T}} \right) \cdots \text{SEQ} \left( \frac{\mathcal{T}}{\mathcal{T}} \right)
\]

\[
\Rightarrow S(z, t) = z + \frac{zt}{1 - t - T^2}
\]

- Count w.r.t. size: set $t = z$, use $T = \frac{z}{1 - T}$

\[
\Rightarrow S(z, z) = z + \frac{z^2}{1 - (z + T^2)} = z + \frac{z^2}{1 - T} = z + zT
\]
Catalan–Stanley Trees (Proof)

- \( \mathcal{T} \) ... class of plane trees
- Symbolic description:

\[
S(\mathcal{T}) = \bigcirc + \text{SEQ} \left( \frac{\mathcal{T}}{T} \right) + \text{SEQ} \left( \frac{\mathcal{T}}{T} \right) + \ldots
\]

\[
\Rightarrow S(z, t) = z + \frac{zt}{1 - \frac{t}{1 - T^2}} = z + \frac{zt}{1 - t - T^2}
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- Count w.r.t. size: set \( t = z \), use \( T = \frac{z}{1 - T} \)

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- \( T(z) = \sum_{n \geq 1} C_{n-1} z^n \)
Catalan–Stanley Trees (Proof)

- Class of plane trees
- Symbolic description:

\[ S = \bigcirc + \text{SEQ} \left( \frac{T}{T} \right) \quad \text{SEQ} \left( \frac{T}{T} \right) \quad \ldots \quad \text{SEQ} \left( \frac{T}{T} \right) \]

\[ \Rightarrow S(z, t) = z + \frac{zt}{1 - t - T^2} \]

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\[ \Rightarrow S(z, z) = z + \frac{z^2}{1 - (z + T^2)} = z + \frac{z^2}{1 - T} = z + zT \]

- \( T(z) = \sum_{n \geq 1} C_{n-1} z^n \Rightarrow S(z, z) = z + \sum_{n \geq 2} C_{n-2} z^n \)
Growing Catalan–Stanley Trees

- **Idea:** grow tree at ■ and ensure that odd-distance property is satisfied
Growing Catalan–Stanley Trees

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- **Strategy:** insert a sequence of two plane trees before every □
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A Generating Function Approach

Q: Describe tree growth via linear operator Φ:

\[
\Phi(f(z,t)) = 1 - t f(z, 1 - t T_2)
\]
A Generating Function Approach

- **Q:** Describe tree growth via linear operator $\Phi$:
  - $\mathcal{F}$...some subclass of Catalan–Stanley trees, $f(z, t)$ g.f.
A Generating Function Approach

Q: Describe tree growth via linear operator $\Phi$:

- $\mathcal{F}$...some subclass of Catalan–Stanley trees, $f(z, t)$ g.f.
- $\Phi(f(z, t))$...counts trees grown from those in $\mathcal{F}$
A Generating Function Approach

- **Q:** Describe tree growth via linear operator $\Phi$:
  - $\mathcal{F} \ldots$ some subclass of Catalan–Stanley trees, $f(z, t)$ g.f.
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- **Facts:** $\Phi$ is linear, Image of $z^n t^k$:

\[ \Phi(z^n t^k) = \]
A Generating Function Approach

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Facts: $\Phi$ is linear, Image of $z^nt^k$:
   - $\bigcirc$ stay as they are: $z \mapsto z$

$$\Phi(z^nt^k) = z^n$$
A Generating Function Approach

**Q:** Describe tree growth via linear operator Φ:
- \( \mathcal{F} \) ... some subclass of Catalan–Stanley trees, \( f(z, t) \) g.f.
- \( \Phi(f(z, t)) \) ... counts trees grown from those in \( \mathcal{F} \)

**Facts:** Φ is linear, Image of \( z^n t^k \):
- \( \bigcirc \) stay as they are: \( z \mapsto z \)
- \( \blacksquare \) get two trees attached: \( t \mapsto t T^2 \)

\[
\Phi(z^n t^k) = z^n (t T^2)^k
\]
A Generating Function Approach

**Q:** Describe tree growth via linear operator $\Phi$:

- $\mathcal{F}$... some subclass of Catalan–Stanley trees, $f(z, t)$ g.f.
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**Facts:** $\Phi$ is linear, Image of $z^nt^k$:

- stay as they are: $z \mapsto z$
- get two trees attached: $t \mapsto tT^2$
- add sequences of $\blacksquare$ to the root ($k + 1$ positions): $(\frac{1}{1-t})^{k+1}$

$$\Phi(z^nt^k) = z^n(tT^2)^k(\frac{1}{1-t})^{k+1}$$
A Generating Function Approach

- **Q:** Describe tree growth via linear operator $\Phi$:
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\[
\Phi(z^n t^k) = z^n (t T^2)^k \left(\frac{1}{1-t}\right)^{k+1}
\]

- **A:** This proves

\[
\Phi(f(z,t)) = \frac{1}{1-t} f(z, \frac{t}{1-t} T^2)
\]
A Generating Function Approach

- **Q:** Describe tree growth via linear operator $\Phi$:
  - $\mathcal{F}$... some subclass of Catalan–Stanley trees, $f(z, t)$ g.f.
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\Phi(z^n t^k) = z^n (tT^2)^k \left(\frac{1}{1-t}\right)^{k+1}
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\Phi(f(z, t)) = \frac{1}{1-t} f\left(z, \frac{t}{1-t} T^2\right)
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- **Note:** $\Phi(z) = \frac{z}{1-t} = z + zt + zt^2 + \cdots$
A Generating Function Approach

Q: Describe tree growth via linear operator $\Phi$:
- $\mathcal{F}$...some subclass of Catalan–Stanley trees, $f(z, t)$ g.f.
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Facts: $\Phi$ is linear, Image of $z^n t^k$:
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$$\Phi(z^n t^k) = z^n (t T^2)^k \left(\frac{1}{1-t}\right)^{k+1}$$

A: This proves
$$\Phi(f(z, t)) = \frac{1}{1-t} f(z, \frac{t}{1-t} T^2)$$

Note: $\Phi(z) = \frac{z}{1-t} = z + zt + zt^2 + \cdots$
- $\bigcirc$ may not grow!
$r$-fold Iterated Growth

**Proposition**

- $\mathcal{F} \ldots$ *family of Catalan–Stanley trees*
Proposition

- \( \mathcal{F} \ldots \text{family of Catalan–Stanley trees} \)
- \( \text{Generating function } f(z, t) \)
Proposition

- \(\mathcal{F}\ldots\text{family of Catalan–Stanley trees}\)
- \(\text{Generating function } f(z, t)\)
- \(r \in \mathbb{Z}_{\geq 0}\)
Proposition

- $\mathcal{F}$...family of Catalan–Stanley trees
- Generating function $f(z, t)$
- $r \in \mathbb{Z}_{\geq 0}$

$$\Phi^r(f(z, t)) = \frac{1}{1 - t \frac{1 - T^{2r}}{1 - T^2}} f\left(z, \frac{t T^{2r}}{1 - t \frac{1 - T^{2r}}{1 - T^2}}\right)$$

counts trees grown from $\mathcal{F}$ after $r$ generations.
$r$-fold Iterated Growth

Proposition

- $\mathcal{F}$ ... family of Catalan–Stanley trees
- Generating function $f(z, t)$
- $r \in \mathbb{Z}_{\geq 0}$

\[
\Phi^r(f(z, t)) = \frac{1}{1 - t^{1-T^2} f(z, \frac{tT^{2r}}{1 - t^{1-T^2}})}
\]

counts trees grown from $\mathcal{F}$ after $r$ generations.

For some Catalan–Stanley tree $\tau$:

- **Age of $\tau$**: min. # of generations to grow $\tau$ from $\bigcirc$
Proposition

- \( \mathcal{F} \) ... family of Catalan–Stanley trees
- Generating function \( f(z, t) \)
- \( r \in \mathbb{Z}_{\geq 0} \)

\[
\Phi^r(f(z, t)) = \frac{1}{1 - t \frac{1 - T^{2r}}{1 - T^2}} f\left(z, \frac{t T^{2r}}{1 - t \frac{1 - T^{2r}}{1 - T^2}} \right)
\]

counts trees grown from \( \mathcal{F} \) after \( r \) generations.

For some Catalan–Stanley tree \( \tau \):

- **Age of** \( \tau \): min. \( \# \) of generations to grow \( \tau \) from \( \bigcirc \)
- **Size of** \( r \)-th ancestor: size of \( r \)-fold reduced \( \tau \)
Trees of Given Age

Corollary

\[ F_r^\leq(z, t) = \Phi^r(z) = \frac{z}{1 - t \frac{1 - T^{2r}}{1 - T^2}} \]

counts Catalan–Stanley trees of age \( \leq r \) w.r.t. \( z \equiv \bigcirc, \ t \equiv \blacksquare \).
Trees of Given Age

**Corollary**

\[
F_r^\leq(z, t) = \Phi^r(z) = \frac{z}{1 - t \frac{1-T^{2r}}{1-T^2}}
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counts Catalan–Stanley trees of age \( \leq r \) w.r.t. \( z \triangleq \bigcirc \), \( t \triangleq \blacksquare \).

**Proof:** \( \bigcirc \) may not grow \( \Rightarrow \) \( \Phi^r(z) \) counts trees of age \( \leq r \). \( \square \)
Trees of Given Age

Corollary

\[ F_{r}^{\leq}(z,t) = \Phi^{r}(z) = \frac{z}{1 - t \frac{1 - T^{2r}}{1 - T}} \]

counts Catalan–Stanley trees of age \( \leq r \) w.r.t. \( z \triangleq \bigcirc, t \triangleq \blacksquare \).

**Proof:** \( \bigcirc \) may not grow \( \Rightarrow \Phi^{r}(z) \) counts trees of age \( \leq r \).

- \( D_{n} \ldots \) age of random Catalan–Stanley tree of size \( n \)
Trees of Given Age

Corollary

\[ F_{r}^{\leq}(z, t) = \Phi^{r}(z) = \frac{z}{1 - t^{\frac{1-T^{2r}}{1-T^{2}}}} \]

counts Catalan–Stanley trees of age \( \leq r \) w.r.t. \( z \triangleq \bigcirc, t \triangleq \blacksquare \).

Proof: \( \bigcirc \) may not grow \( \Rightarrow \) \( \Phi^{r}(z) \) counts trees of age \( \leq r \). \( \square \)

- \( D_{n} \ldots \) age of random Catalan–Stanley tree of size \( n \)
- Facts: \( \mathbb{E}D_{n} = \sum_{r \geq 1} \mathbb{P}(D_{n} \geq r), \forall D_{n} = \mathbb{E}(D_{n}^{2}) - (\mathbb{E}D_{n})^{2}, \)
  \( \mathbb{E}(D_{n}^{2}) = \sum_{r \geq 1}(2r - 1)\mathbb{P}(D_{n} \geq r) \)
Trees of Given Age

**Corollary**

\[
F_r^\leq(z, t) = \Phi^r(z) = \frac{z}{1 - t \frac{1 - T^{2r}}{1 - T^2}}
\]

counts Catalan–Stanley trees of age \( \leq r \) w.r.t. \( z \triangleq \bigcirc, t \triangleq \blacksquare \).

**Proof:** \( \bigcirc \) may not grow \( \Rightarrow \) \( \Phi^r(z) \) counts trees of age \( \leq r \).  \( \square \)

- \( D_n \)... age of random Catalan–Stanley tree of size \( n \)
- **Facts:** \( \mathbb{E}D_n = \sum_{r \geq 1} \mathbb{P}(D_n \geq r) \), \( \forall D_n = \mathbb{E}(D_n^2) - (\mathbb{E}D_n)^2 \), \( \mathbb{E}(D_n^2) = \sum_{r \geq 1} (2r - 1) \mathbb{P}(D_n \geq r) \)
- **Want:** \( F_r^\geq(z) \)... g.f. for trees of age \( \geq r \)
Trees of Given Age

Corollary

\[ F_r^\leq (z, t) = \Phi^r (z) = \frac{z}{1 - t^{1 - T^r}} \]

counts Catalan–Stanley trees of age \( \leq r \) w.r.t. \( z \triangleq \bigcirc, t \triangleq \blacksquare \).

Proof:
\( \bigcirc \) may not grow \( \Rightarrow \) \( \Phi^r (z) \) counts trees of age \( \leq r \).

- \( D_n \ldots \) age of random Catalan–Stanley tree of size \( n \)
- Facts: \( \mathbb{E}D_n = \sum_{r \geq 1} \mathbb{P}(D_n \geq r) \), \( \forall D_n = \mathbb{E}(D_n^2) - (\mathbb{E}D_n)^2 \),
  \( \mathbb{E}(D_n^2) = \sum_{r \geq 1} (2r - 1)\mathbb{P}(D_n \geq r) \)
- Want: \( F_r^\geq (z) \ldots \) g.f. for trees of age \( \geq r \)

\[
F_r^\geq (z) = S(z, z) - F_{r-1}^\leq (z, z) = z(1 + T)^{\frac{T^{2r-1}}{1 + T^{2r-1}}}
\]

\[
= \sum_{n \geq 0} f_{n,r} z^n
\]
Singular Expansion

- Recall: \[ T = \frac{1 - \sqrt{1 - 4z}}{2} \]
Singular Expansion

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\]

\[
= \frac{1}{(1 + 2^{2r-1})(1 + \frac{2^{2r-1}}{1+2^{2r-1}} \sum_{j \geq 1} \binom{2r+j-2}{j} (1 - 4z)^{j/2})}
\]
Recall: $T = \frac{1 - \sqrt{1 - 4z}}{2}$

\[
\frac{T^{2r-1}}{1 + T^{2r-1}} = \frac{1}{1 + T^{1-2r}} = \frac{1}{1 + 2^{2r-1}(1 - \sqrt{1 - 4z})^{1-2r}}
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Inversion,
Singular Expansion

- Recall: \( T = \frac{1 - \sqrt{1 - 4z}}{2} \)

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\frac{T^{2r-1}}{1 + T^{2r-1}} = \frac{1}{1 + T^{1-2r}} = \frac{1}{1 + 2^{2r-1}(1 - \sqrt{1 - 4z})^{1-2r}}
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\]

- Inversion,
- Multiplication by expansion of \( z(1 + T) \),
Singular Expansion

- Recall: \( T = \frac{1 - \sqrt{1 - 4z}}{2} \)

\[
\begin{align*}
\frac{T^{2r-1}}{1 + T^{2r-1}} &= \frac{1}{1 + T^{1-2r}} = \frac{1}{1 + 2^{2r-1}(1 - \sqrt{1 - 4z})^{1-2r}} \\
&= \frac{1}{(1 + 2^{2r-1})(1 + \sum_{j \geq 1} \binom{2r + j - 2}{j} (1 - 4z)^{j/2})}
\end{align*}
\]

- Inversion,
- Multiplication by expansion of \( z(1 + T) \),
- Coefficient extraction
Result – Age

Theorem (H–Prodinger, 2017)

The age of a (uniformly random) Catalan–Stanley tree of size $n$ follows a discrete limiting distribution with
Result – Age

Theorem (H–Prodinger, 2017)

The age of a (uniformly random) Catalan–Stanley tree of size $n$ follows a discrete limiting distribution with

$$P(D_n = r) = \frac{1}{C_{n-2}}(f_{n,r} - f_{n,r+1}),$$

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Result – Age

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Result – Age

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Result – Age

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$$E_D n = 2.71825 \ldots - 4.22209 \ldots n^{-1} + O(n^{-2}),$$
Result – Age

**Theorem (H–Prodinger, 2017)**

*The age of a (uniformly random) Catalan–Stanley tree of size n follows a discrete limiting distribution with*

\[ P(D_n = r) = \frac{1}{C_{n-2}} (f_{n,r} - f_{n,r+1}), \]

where

\[ f_{n,r} = \frac{4(4^r (3r - 1) + 1)}{(4^r + 2)^2} - 6.64^r (2r^3 - 5r^2 + 4r - 1) - 6.16^r (16r^3 - 24r^2 + 10r - 1) + 24.4^r (2r^3 - r^2) n^{-1} \]

\[ + O(r^5 3^{-r} n^{-2}), \]

\[ \mathbb{E} D_n = 2.71825 \ldots - 4.22209 \ldots n^{-1} + O(n^{-2}), \]

\[ \nabla D_n = 0.91845 \ldots - 9.16217 \ldots n^{-1} + O(n^{-2}). \]
Generating Function for Ancestors

**Corollary**

\[
G_r(z, v) = \Phi^r(S(zv, tv))|_{t=z} = \frac{1}{1 - z \frac{1 - T^{2r}}{1 - T^2}} S \left( zv, \frac{zT^{2r}}{1 - z \frac{1 - T^{2r}}{1 - T^2}} v \right)
\]
Generating Function for Ancestors

Corollary

\[ G_r(z, v) = \Phi^r(S(zv, tv))|_{t=z} = \frac{1}{1 - z \frac{1-T^2 r}{1-T^2}} S\left(zv, \frac{zT^{2r}}{1 - z \frac{1-T^2 r}{1-T^2}} v\right) \]

is the bivariate generating function enumerating Catalan–Stanley trees w.r.t. size \((\triangleq z)\) and the size of the \(r\)th ancestor \((\triangleq v)\).
Generating Function for Ancestors

**Corollary**

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G_r(z, v) = \Phi^r(S(zv, tv))|_{t=z} = \frac{1}{1 - z \frac{1 - T^{2r}}{1 - T^2}} S\left( zv, \frac{zT^{2r}}{1 - z \frac{1 - T^{2r}}{1 - T^2}} v \right)
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is the bivariate generating function enumerating Catalan–Stanley trees w.r.t. size (\(\triangleq z\)) and the size of the rth ancestor (\(\triangleq v\)).

**Proof:** mark original tree size with \(v\), expand \(r\) times. \(\square\)
Generating Function for Ancestors

**Corollary**

\[
G_r(z, v) = \Phi^r(S(zv, tv))|_{t=z} = \frac{1}{1 - z} \frac{1 - T^{2r}}{1 - T^2} S\left(zv, \frac{zT^{2r}}{1 - z} \frac{1 - T^{2r}}{1 - T^2} \right)
\]

is the bivariate generating function enumerating Catalan–Stanley trees w.r.t. size (≜ z) and the size of the rth ancestor (≜ v).

**Proof:** mark original tree size with \( v \), expand \( r \) times. □

- \( X_{n,r} \ldots \) size of \( r \)th ancestor of (unif. random) Catalan–Stanley tree of size \( n \)
Newtonian Function for Ancestors

**Corollary**

\[ G_r(z, v) = \Phi^r(S(zv, tv))|_{t=z} = \frac{1}{1 - z \frac{1 - T^{2r}}{1 - T^2}} S\left(zv, \frac{z T^{2r}}{1 - z \frac{1 - T^{2r}}{1 - T^2}} v\right) \]

is the bivariate generating function enumerating Catalan–Stanley trees w.r.t. size (≜ z) and the size of the rth ancestor (≜ v).

**Proof:** mark original tree size with v, expand r times. □

- \( X_{n,r} \ldots \) size of rth ancestor of (unif. random) Catalan–Stanley tree of size n

\[ \mathbb{E} X_{n,r}^d = \frac{1}{C_{n-2}} \frac{\partial^d}{\partial v^d} G_r(z, v)|_{v=1} \]
Generating Function for Ancestors

Corollary

\[ G_r(z, v) = \Phi^r(S(zv, tv)) \bigg|_{t=z} = \frac{1}{1 - z \frac{1 - T^{2r}}{1 - T^2}} \left( zv, \frac{zT^{2r}}{1 - z \frac{1 - T^{2r}}{1 - T^2}} v \right) \]

is the bivariate generating function enumerating Catalan–Stanley trees w.r.t. size (\(\triangleq z\)) and the size of the \(r\)th ancestor (\(\triangleq v\)).

Proof: mark original tree size with \(v\), expand \(r\) times.

▶ \(X_{n,r} \ldots\) size of \(r\)th ancestor of (unif. random) Catalan–Stanley tree of size \(n\)

▶ \(\mathbb{E}X_{n,r}^d = \frac{1}{C_{n-2}} \frac{\partial^d}{\partial v^d} G_r(z, v) \big|_{v=1}\)

▶ Singular expansion \(z \rightarrow 1/4\), Singularity Analysis
Result – Size of \( r \)th Ancestor

**Theorem (H–Prodinger, 2017)**

\[ X_{n,r}, \text{ the size of the } r \text{th ancestor of a (uniformly random) Catalan–Stanley tree of size } n \text{ satisfies} \]

\[
E[X_{n,r}] = 1 + \frac{1}{C_n} - 2 \left( 2^n - 2r - 4n - 2 \right)
\]

\[
V[X_{n,r}] = \frac{(2^r + 1)(2^r - 1)}{16r^2} - \frac{\sqrt{\pi}}{16} \left( 4r(3r + 1) - 1 \right) - \frac{1}{3} \cdot \frac{1}{16r^3} + O\left( 4^{-r} r^2 n \right)
\]
Result – Size of rth Ancestor

Theorem (H–Prodinger, 2017)

\[ X_{n,r}, \text{ the size of the rth ancestor of a (uniformly random) Catalan–Stanley tree of size } n \text{ satisfies} \]

\[
\mathbb{E}X_{n,r} = \frac{1}{4r} n + \frac{2 \cdot 4^r - 2r^2 + r - 2}{2 \cdot 4^r} + O(n^{-1}),
\]
Result – Size of $r$th Ancestor

Theorem (H–Prodinger, 2017)

$X_{n,r}$, the size of the $r$th ancestor of a (uniformly random) Catalan–Stanley tree of size $n$ satisfies

$$
\mathbb{E}X_{n,r} = \frac{1}{4r} n + \frac{2 \cdot 4^r - 2r^2 + r - 2}{2 \cdot 4^r} + O(n^{-1}),
$$

$$
\mathbb{V}X_{n,r} = \frac{(2^r + 1)(2^r - 1)}{16^r} n^2 - \frac{\sqrt{\pi}(4^r(3r + 1) - 1)}{3 \cdot 16^r} n^{3/2} + O(4^{-r}r^2 n).
$$
Result – Size of $r$th Ancestor

**Theorem (H–Prodinger, 2017)**

Let $X_{n,r}$ be the size of the $r$th ancestor of a (uniformly random) Catalan–Stanley tree of size $n$. Then,

\[
\mathbb{E}X_{n,r} = \frac{1}{4r} n + \frac{2 \cdot 4^r - 2r^2 + r - 2}{2 \cdot 4^r} + O(n^{-1}),
\]

\[
\mathbb{V}X_{n,r} = \frac{(2^r + 1)(2^r - 1)}{16^r} n^2 - \frac{\sqrt{\pi}(4^r(3r + 1) - 1)}{3 \cdot 16^r} n^{3/2}
\]

\[+ O(4^{-r} r^2 n).\]

In particular, we have

\[
\mathbb{E}X_{n,r} = 1 + \frac{1}{C_n} \binom{2n - 2r - 4}{n - 2}.
\]
Summary: Age and Ancestors of Catalan–Stanley trees

- “nice”, not too artificial growth process with different parameter behavior ✓

<table>
<thead>
<tr>
<th>Age</th>
<th>Size of $r$th Ancestor</th>
</tr>
</thead>
</table>

$E \sim O(1)$

$V \sim (2^r + 1)(2^r - 1) / 16^r$
Summary: Age and Ancestors of Catalan–Stanley trees

- “nice”, not too artificial growth process with different parameter behavior ✓

Age

Size of $r$th Ancestor

$E = O(1)$

$V = O(1)$

LLT: ✓

$\#\text{Generations} = \text{Age}$
Summary: Age and Ancestors of Catalan–Stanley trees

▶ “nice”, not too artificial growth process with different parameter behavior ✓

Age

▶ \( E = O(1) \)

Size of \( r \)th Ancestor

\[
E \sim \frac{1}{4^r} \\
V \sim \frac{(2r+1)(2r-1)}{16^r}
\]

# Generations = Age
Summary: Age and Ancestors of Catalan–Stanley trees

- “nice”, not too artificial growth process with different parameter behavior ✓

**Age**

- $E = O(1)$
- $V = O(1)$

**Size of rth Ancestor**

# Generations = Age
Summary: Age and Ancestors of Catalan–Stanley trees

- “nice”, not too artificial growth process with different parameter behavior ✓

**Age**

- $E = O(1)$
- $V = O(1)$
- LLT: ✓

**Size of $r$th Ancestor**

# Generations = Age
Summary: Age and Ancestors of Catalan–Stanley trees

- “nice”, not too artificial growth process with different parameter behavior ✓

Age

- \( E = O(1) \)
- \( V = O(1) \)
- LLT: ✓

\[ \#	ext{Generations} = \text{Age} \]

Size of \( r \)th Ancestor

\[ E \sim 1^{4r^n} \]
\[ V \sim (2r+1)(2r-1)^{16r^n/2} \]

\[ \leftarrow \]

\( \text{Ancestor size} \)
Summary: Age and Ancestors of Catalan–Stanley trees

- “nice”, not too artificial growth process with different parameter behavior ✓

**Age**
- $E = O(1)$
- $V = O(1)$
- LLT: ✓

```
# Generations = Age
```

**Size of $r$th Ancestor**
- $E \sim \frac{1}{4^r} n$
Summary: Age and Ancestors of Catalan–Stanley trees

▶ “nice”, not too artificial growth process with different parameter behavior ✓

**Age**

- \( E = O(1) \)
- \( V = O(1) \)
- LLT: ✓

\[# Generations = Age\]

**Size of \( r \)th Ancestor**

- \( E \sim \frac{1}{4^r} n \)
- \( V \sim \frac{(2^r+1)(2^r-1)}{16^r} n^2 \)