The Register Function and Reductions of Binary Trees and Lattice Paths

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joint work with

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Trimming binary trees

Binary trees can be “trimmed” by the following strategy:

- Remove all leaves
- Merge nodes with only one descendant
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“Surviving” nodes

Label all nodes in the tree by the following rules:

- Leaves $\rightarrow 0$ (they do not survive a single reduction)
- $\text{val(left child)} = \text{val(right child)} \rightarrow$ increase by 1
- Otherwise: take the maximum
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The register function

The register function

Number in the root of the tree: Register function, a.k.a. Horton-Strahler number.

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  - Required stack size for evaluating an expression
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- Applications:
  - Required stack size for evaluating an expression
  - Branching complexity of river networks (e.g. Danube: 9)
The register function – selection of known results

- Flajolet, Raoult, Vuillemin (1979): asymptotic expansion of expected value
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- Louchard, Prodinger (2008): register function for directed lattice paths
Reduction of lattice paths

Reduction of a simple, two-dimensional lattice path (i.e. a sequence of \{↑, →, ↓, ←\}):

- If the path starts with ↑ or ↓: rotate it
- If the path ends with → or ←: rotate the last step
- Consider the pairs of horizontal-vertical segments:
  - Replace → ... ↑ ... by ↗,
  - → ... ↓ ... by ↘,
  - ← ... ↓ ... by↙,
  - ← ... ↑ ... by ↖.
- Rotate the entire path again
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Reduction – Example
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Compactification degree and functional equation

- **Compactification degree**: number of reductions until a path is compactified to an atomic step \{↑, →, ↓, ←\}
Compactification degree and functional equation

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**Proposition**

The generating function of simple two-dimensional lattice paths of length $\geq 1$, $L(z) = \frac{4z}{1-4z}$, fulfills the functional equation

$$L(z) = 4z + 4L\left(\frac{z^2}{(1-2z)^2}\right).$$
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Can be checked directly—or proven combinatorially!
Functional equation (combinatorial proof)

Read the reduction *backwards*:

- Replace $\rightarrow$ by $\rightarrow \ldots \uparrow \ldots$ and so on...
- Optionally rotate the entire path and/or the last step.
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⇒ Replacement corresponds to \( z \mapsto \frac{z^2}{(1-2z)^2} \).
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Optional rotations: factor 4.
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$$4L\left(\frac{z^2}{(1-2z)^2}\right)$$

counts all reducible paths.
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Adding $4z$ (for $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$) then counts all paths. □
Compactification degree – Recursion

- $L_r(z)$ ... OGF for paths with compactification degree $r$
Compactification degree – Recursion

- $L_r^\equiv(z) \ldots$ OGF for paths with compactification degree $r$
- Only $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$ have comp. deg. 0 $\Rightarrow L_0^\equiv(z) = 4z$
Compactification degree – Recursion

- \( L^r_r(z) \) ... OGF for paths with compactification degree \( r \)
- Only \( \{\uparrow, \rightarrow, \downarrow, \leftarrow\} \) have comp. deg. 0 ⇒ \( L^0_0(z) = 4z \)
- Recursion:

\[
L^r_r(z) = 4L^r_{r-1}\left(\frac{z^2}{(1 - 2z)^2}\right), \quad r \geq 1
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Compactification degree – Recursion

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- “Magic substitution” $z = \frac{u}{(1 + u)^2}$: $z \mapsto \frac{z^2}{(1 - 2z)^2}$ becomes $u \mapsto u^2$
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  \( u \mapsto u^2 \)
- Overall:
  
  \[
  L_r^\equiv(z) = 4^{r+1}\left.\frac{u}{(1 + u)^2}\right|_{u \mapsto u^2^r} = 4^{r+1}\frac{u^{2^r}}{(1 + u^{2^r})^2}
  \]
Compactification degree – Random variables

- $X_n \ldots$ compactification degree of a (uniformly) random lattice path of length $n$

$\Rightarrow \mathbb{P}(X_n = r) = \frac{[z^n] L_r^=(z)}{4^n}$
Compactification degree – Random variables

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$$ \Rightarrow P(X_n = r) = \frac{[z^n]L_r(z)}{4^n} $$

- Probability densities of $X_1$ up to $X_{512}$:
Analysis of $\mathbb{E}X_n$ (1)

As we have $\mathbb{E}X_n = 4^{-n}[z^n] \sum_{r \geq 0} rL_r^\infty(z)$, we analyze

$$G(z) = \sum_{r \geq 0} rL_r^\infty(z)$$
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- With $z = \frac{u}{(1+u)^2}$ and $u = e^{-t}$, we have

$$G(z) = \sum_{r, \lambda \geq 0} r 4^{r+1} (-1)^{\lambda-1} \lambda e^{-t \lambda 2^r}$$
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$\rightsquigarrow$ Local expansion for $t \to 0$ ($z \to \frac{1}{4}$) via Mellin transform
Analysis of $\mathbb{E}X_n (2)$

- By basic properties of the Mellin transform we find

$$G^*(s) = \Gamma(s)\zeta(s - 1) \frac{2^{2-s}}{1 - 2^{2-s}}$$
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- Double pole at $s = 2$, simple poles at $s = 2 + \frac{2\pi i}{\log 2} k = 2 + \chi_k$ for $k \in \mathbb{Z} \setminus \{0\}$
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- Mellin inversion:

$$G(z) = \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} \Gamma(s)\zeta(s - 1)\frac{2^{2-s}}{1 - 2^{2-s}} t^{-s} \, ds$$
Analysis of $\mathbb{E}X_n(2)$

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- Obtain contribution by shifting line of integration
Analysis of $\mathbb{E}X_n$ (3)

- Residue at $s = 2$:

\[-\frac{4}{\log 2} t^{-2} \log t + \left(\frac{4}{\log 2} - 2\right) t^{-2}\]
Analysis of $\mathbb{E}X_n(3)$

- Residue at $s = 2$:

$$-\frac{4}{\log 2} t^{-2} \log t + \left(\frac{4}{\log 2} - 2\right) t^{-2}$$

- Substituting $z$ back for $t$ and expanding locally for $z \to \frac{1}{4}$ yields

$$-\frac{\log(1 - 4z)}{\log 2 (1 - 4z)} + \frac{2 - 3 \log 2}{\log 2 (1 - 4z)} + \frac{\log 2 - 1}{\log 2} + \frac{\log(1 - 4z)}{3 \log 2} + O(1 - 4z)$$
Analysis of $E X_n (4)$

After division by $4^n$, the local expansion translates into

$$\log_4 n + \frac{\gamma + 2 - 3 \log 2}{2 \log 2} + O(n^{-2}).$$
Analysis of $\mathbb{E}X_n(4)$

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- Plot against exact values (left: comparison, right: difference):
Analysis of $\mathbb{E}X_n$ (5)

Collecting the contributions at $s = 2 + \chi_k$ yields:

**Theorem (H.–Heuberger–Prodinger, 2016)**

The expected compactification degree among all simple 2D lattice paths of length $n$ admits the asymptotic expansion

$$
\mathbb{E}X_n = \log_4 n + \frac{\gamma + 2 - 3 \log 2}{2 \log 2} + \delta_1(\log_4 n) + O(n^{-1}),
$$

where $\delta_1(\cdot)$ is a small 1-periodic fluctuation.
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where

$$
\delta_1(x) = \frac{1}{\log 2} \sum_{k \neq 0} \frac{\Gamma(2 + \chi_k) \zeta(1 + \chi_k)}{\Gamma(1 + \chi_k/2)} e^{2k\pi i x}
$$

*is a small 1-periodic fluctuation.*
Analysis of $\nabla X_n$

Similarly: variance $\nabla X_n$ can be determined.

**Theorem (H.–Heuberger–Prodinger, 2016)**

The corresponding variance is given by

$$\nabla X_n = \frac{\pi^2 - 24 \log^2 \pi - 48 \zeta''(0) - 24}{24 \log^2 2} - \frac{2 \log \pi}{\log 2} - \frac{11}{12}$$

$$+ \delta_2(\log_4 n) + \frac{\gamma + 2 - 3 \log 2}{\log 2} \delta_1(\log_4 n)$$

$$+ \delta_1^2(\log_4 n) + O\left(\frac{1}{\log n}\right),$$

where $\delta_1(x)$ is defined as above and $\delta_2(x)$ is a small 1-periodic fluctuation as well.
Expectation and Variance: exact vs. asymptotic
Expectation and Variance: exact vs. asymptotic

Computations $\leadsto$ Asymptotic Expansions in SageMath!
Fringe Analysis

- *Fringe*: lattice path together with all reductions
Fringe Analysis

- **Fringe**: lattice path together with all reductions
- Size of $r$th fringe... length of $r$th lattice path reduction

![Diagram of lattice paths and reductions]
Fringe Analysis

- **Fringe**: lattice path together with all reductions
- Size of $r$th fringe... length of $r$th lattice path reduction

- How large is the $r$th fringe and the entire fringe on average?
Bivariate generating function

- $H_r(z, v) \ldots$ BGF counting path length (with $z$) and $r$th fringe size (with $v$)
Bivariate generating function

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- Recursion:

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H_0(z, v) = \frac{4zv}{1 - 4zv}, \quad H_r(z, v) = 4H_{r-1}\left(\left(\frac{z}{1 - 2z}\right)^2, v\right)
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- Intuition: $v$ “remembers” original size; path expansion in $z$
Bivariate generating function

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- Intuition: $v$ “remembers” original size; path expansion in $z$
- Explicit solution with $z = \frac{u}{(1+u)^2}$:

$$H_r(z, v) = \frac{4^{r+1}u^{2^r}v}{(1 + u^{2^r})^2 - 4u^{2^r}v}$$
Size of $r$th fringe

**Theorem (H.–Heuberger–Prodinger, 2016)**

The expectation $E_{n;r}^L$ and variance $V_{n;r}^L$ of the $r$th fringe size of a random path of length $n$ have the asymptotic expansions

$$E_{n;r}^L = \frac{n}{4r} + \frac{1 - 4^{-r}}{3} + O(n^3 \theta_r^{-n}),$$

where $\theta_r = 4^{2+2 \cos(\frac{2\pi}{2}r)} > 1$. For $r > 0$, the random variables modeling the $r$th fringe size of lattice paths of length $n$ are asymptotically normally distributed.
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V_{n;r}^L = \frac{4^r - 1}{3 \cdot 16^r} n + \frac{-2 \cdot 16^r - 5 \cdot 4^r + 7}{45 \cdot 16^r} + O(n^5 \theta_r^{-n}),
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For $r > 0$, the random variables modeling the $r$th fringe size of lattice paths of length $n$ are asymptotically normally distributed.
Overall fringe size

Strategy: sum over $H_r(z, v)$, expansion via Mellin transform, singularity analysis.

Theorem (H.–Heuberger–Prodinger, 2016)

The expected fringe size $E_L(n)$ for a random path of length $n$ admits the asymptotic expansion

$$E_L(n) = \frac{4}{3} n + \frac{1}{3} \log 4 n + 5 + \frac{3}{\gamma} - \frac{11 \log 2}{18 \log 2} + \delta(n) \log 4 n + O(n^{-1} \log n),$$

where $\delta(n)$ is a $1$-periodic fluctuation of mean zero with

$$\delta(n) = \frac{2}{3} \sqrt{\pi \log 2} \sum_{k \neq 0} \Gamma(3 + \chi k^2) \left( \frac{2}{\zeta(\chi k - 1)} + \frac{\zeta(\chi k + 1)}{2} \right) e^{2k \pi i n}.$$
Overall fringe size

Strategy: sum over $H_r(z, v)$, expansion via Mellin transform, singularity analysis.

**Theorem (H.–Heuberger–Prodinger, 2016)**

The expected fringe size $E_n^L$ for a random path of length $n$ admits the asymptotic expansion

$$E_n^L = \frac{4}{3} n + \frac{1}{3} \log_4 n + \frac{5 + 3\gamma - 11 \log 2}{18 \log 2} + \delta(\log_4 n) + O(n^{-1} \log n),$$

where $\delta(x)$ is a 1-periodic fluctuation of mean zero with

$$\delta(x) = \frac{2}{3\sqrt{\pi} \log 2} \sum_{k \neq 0} \Gamma\left(\frac{3 + \chi k}{2}\right) \left(2\zeta(\chi k - 1) + \zeta(\chi k + 1)\right) e^{2k\pi i x}.$$
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