Iterative Cutting and Pruning of Planar Trees

Benjamin Hackl

joint work in progress with
Sara Kropf and *Helmut Prodinger*

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KLAGENFURT | WIEN GRAZ

Analco17

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mso DK modeling simulation optimization

KARL POPPER KOLLEG

FWF Der Wissenschaftsfonds.

Procedural Tree Generation

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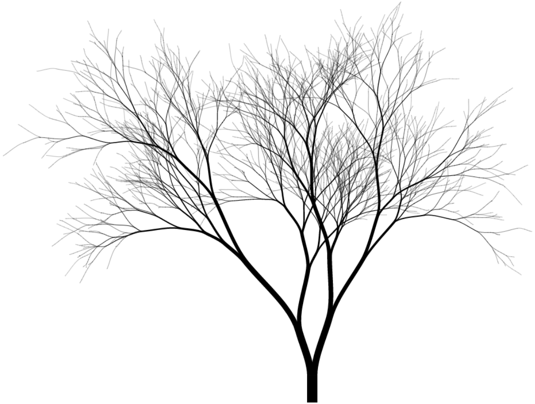

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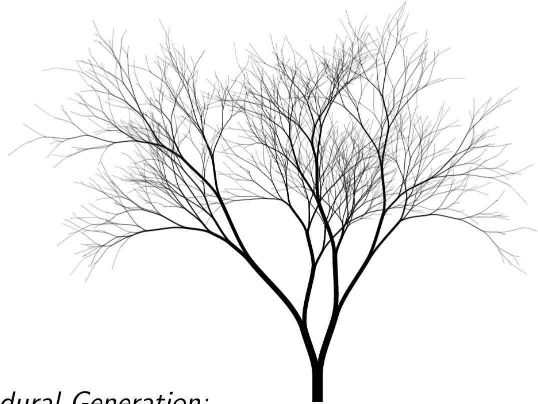

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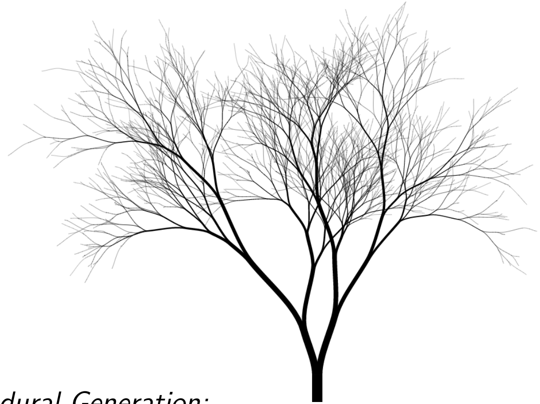

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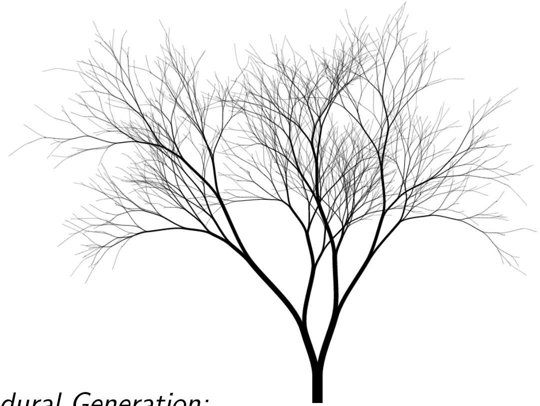

- *Procedural Generation:*
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Growing rooted plane trees

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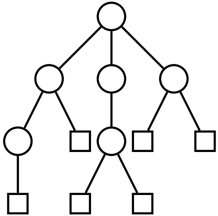

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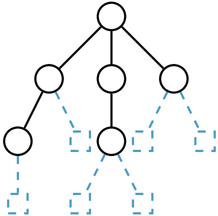

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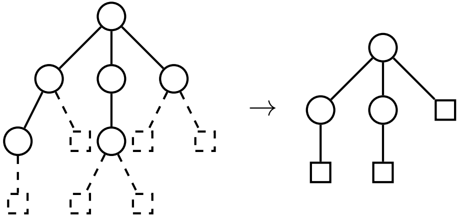

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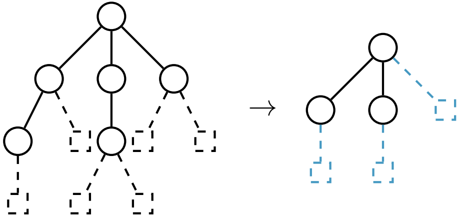

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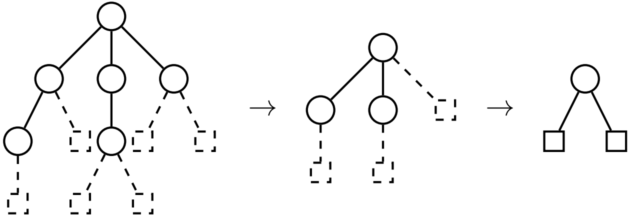

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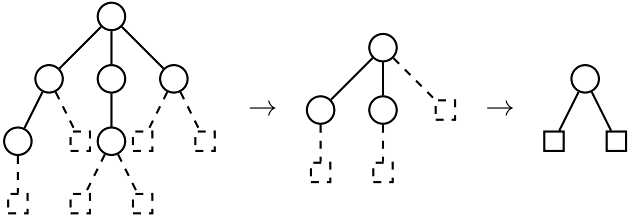

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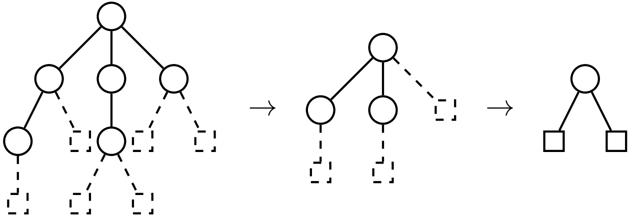

- Growing trees:
  - grow new leaves out of current leaves and inner nodes

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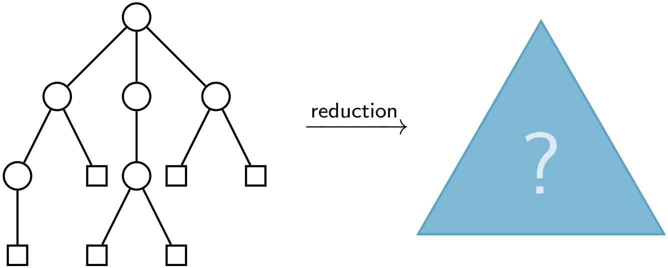

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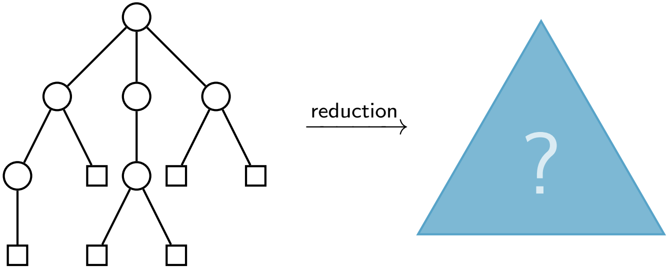

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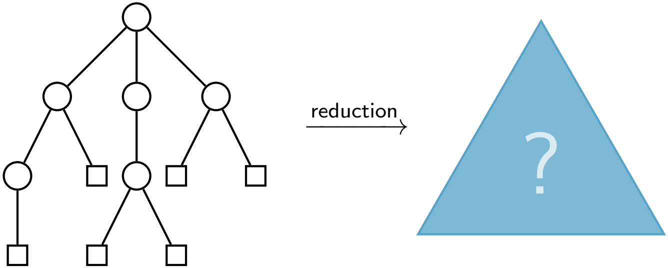

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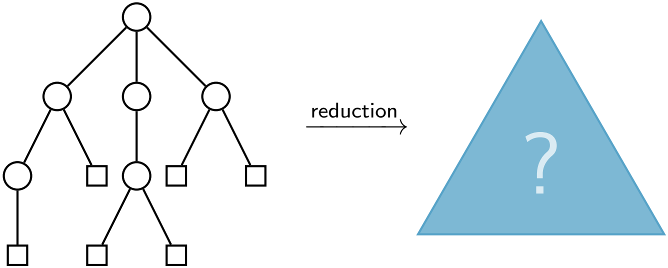

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- Coefficient extraction; Parameter distribution

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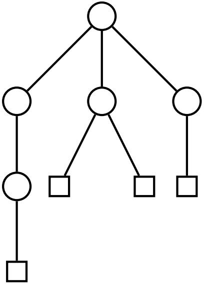

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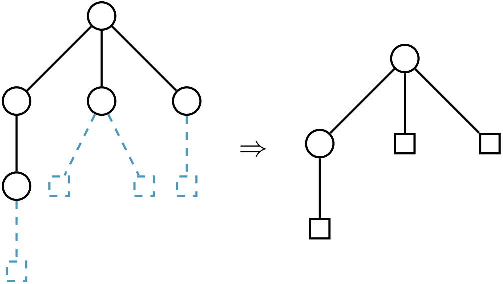

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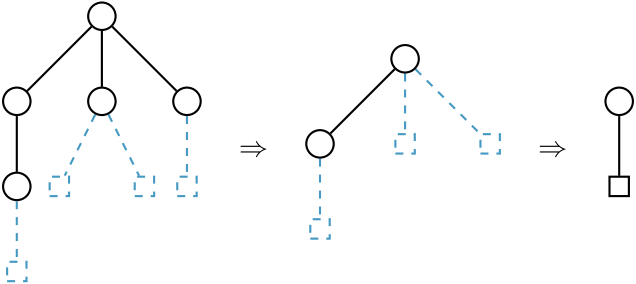

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Proof. Symbolic equation

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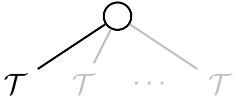

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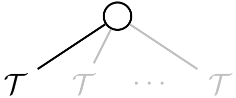

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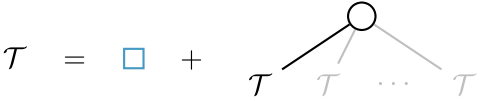

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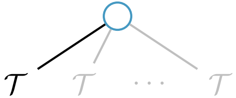

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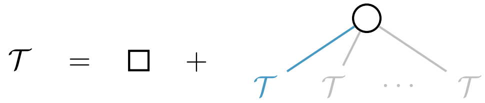

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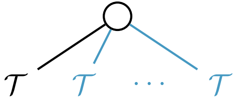

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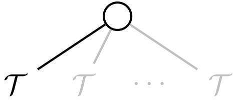
$\mathcal{T} \quad = \quad \square \quad +$ 

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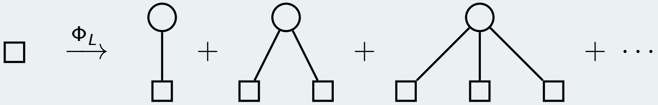
- inverse operation to leaf reduction
  - attach leaves to all current leaves (necessary)
  - attach leaves to inner nodes (optional)



$$\Rightarrow \quad \Phi_L(t) = zt + zt^2 + zt^3 + \cdots$$

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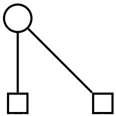

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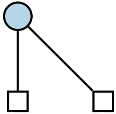

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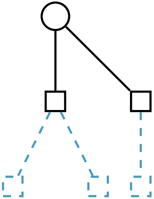

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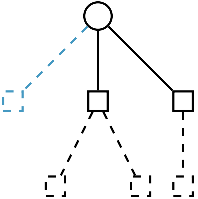

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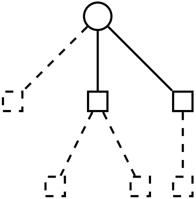

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- Linear extension of Φ_L proves the proposition.

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- Asymptotic normality: $X_{n,r}$ is a tree parameter with small toll function, limit law by Wagner (2015)

How do we cut our trees? (2)

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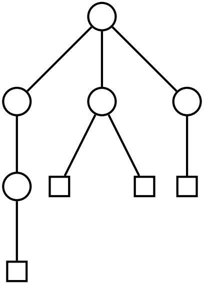

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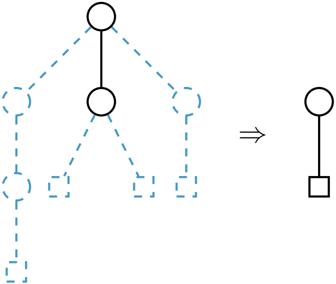

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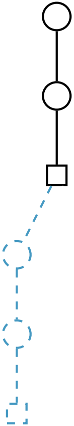

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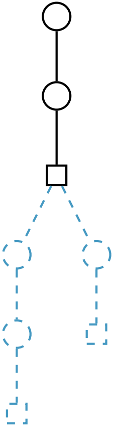

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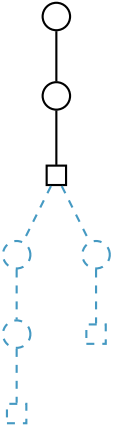

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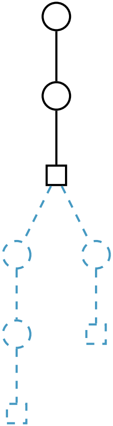

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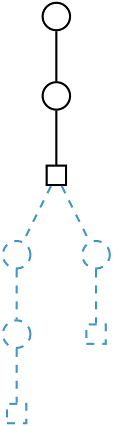

Proposition

The linear operator given by

$$\Phi_P(f(z,t)) = (1-p) f\left(\frac{z}{(1-p)^2}, \frac{zp^2}{(1-p)^2}\right)$$

is the path expansion operator.

Generating function for path reductions

Proposition

BGF for size comparison ($z \rightsquigarrow$ original size, $v \rightsquigarrow$ r-fold path reduced size) is

$$\frac{1-u^{2^{r+1}}}{(1-u^{2^{r+1}-1})(1+u)}T\Big(\frac{u(1-u^{2^{r+1}-1})^2}{(1-u^{2^{r+1}})^2}v, \frac{u^{2^{r+1}-1}(1-u)^2}{(1-u^{2^{r+1}})^2}v\Big),$$

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Observation. This is the BGF for leaf reductions

$$\frac{1-u^{r+2}}{(1-u^{r+1})(1+u)} T\Big(\frac{u(1-u^{r+1})^2}{(1-u^{r+2})^2} v, \frac{u^{r+1}(1-u)^2}{(1-u^{r+2})^2} v\Big)$$

with $r \mapsto 2^{r+1} - 2$.

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Theorem (H.–Kropf–Prodinger, 2016)

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- Proof: subsequence of RV's from cutting leaves

Counting total number of paths

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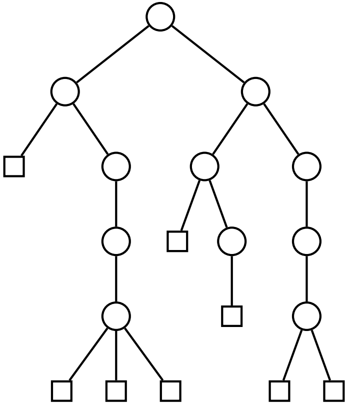

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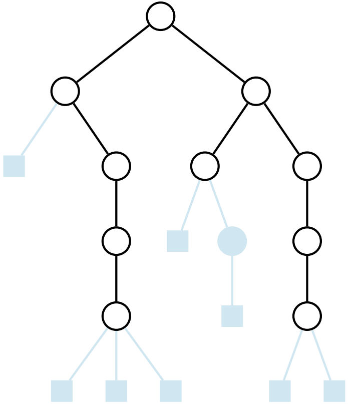

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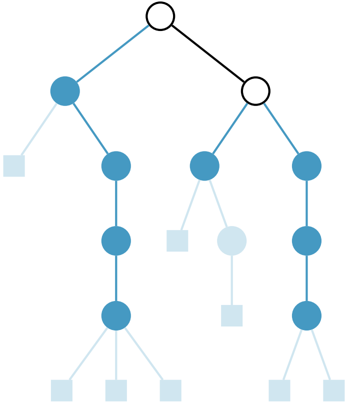

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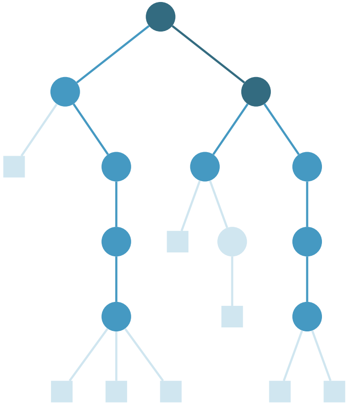

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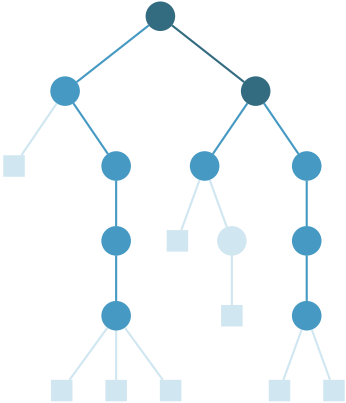

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- **Proof:** Sum of leaves in all reductions, Mellin-transform, singularity analysis.

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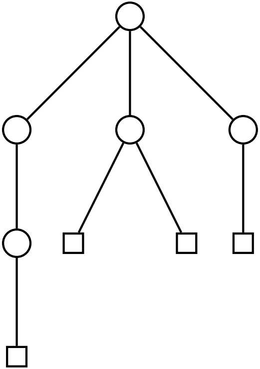

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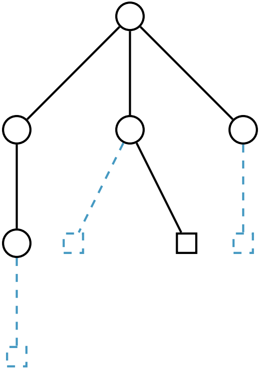

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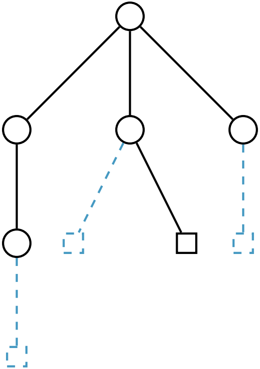

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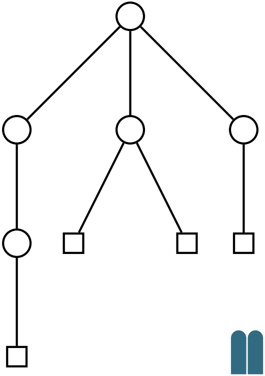

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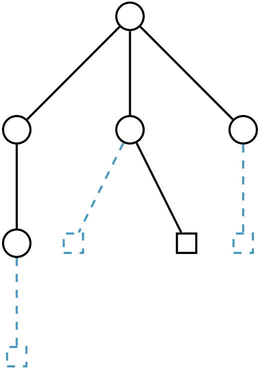

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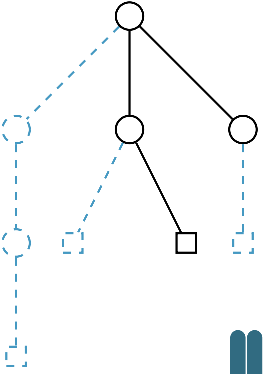

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Note. Via Flajolet, Odlyzko (1982):

$$B_r(1/4) = 2 - \frac{4}{r} - \frac{4\log r}{r^2} + O(r^{-3}), \quad r \to \infty$$

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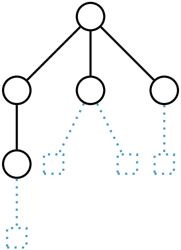

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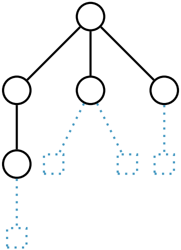

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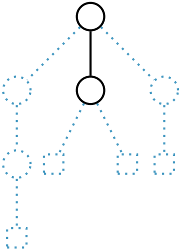

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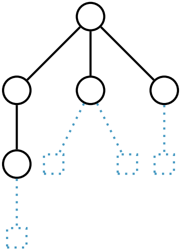

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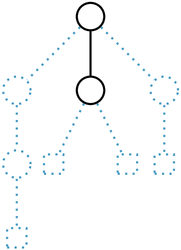

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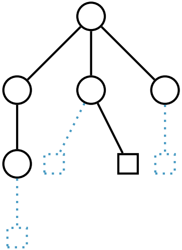

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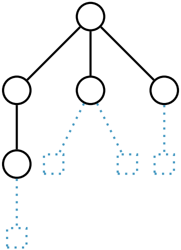

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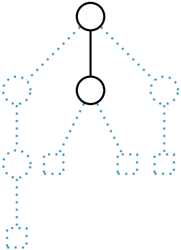

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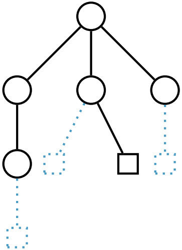
$\mathbb{V} = \Theta(n)$

limit law: ???

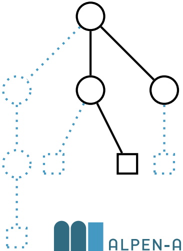


Old paths

$\mathbb{E} \sim \frac{2n}{r+2}$

$\mathbb{V} \sim \frac{2r(r+1)}{3(r+2)^2} n$

limit law: ???



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