Introductory Example

- Given: \(2 \times n\) checkerboard, \(n\) identical \(1 \times 2\) domino pieces

- \# of possibilities to cover the board?

\[
\begin{align*}
\text{n = 1:} & \quad \begin{array}{c}
\square
\end{array} & \text{1} \\
\text{n = 2:} & \quad \begin{array}{c}
\square, \square
\end{array} & \text{2} \\
\text{n = 3:} & \quad \begin{array}{c}
\square, \square, \square
\end{array} & \text{3} \\
\text{n = 4:} & \quad \begin{array}{c}
\square, \square, \square, \square
\end{array} & \text{5}
\end{align*}
\]

In general, \(\exists \ldots \) all tilings \(\Rightarrow \exists = \emptyset + 0 \times \exists + \square \times \exists \rightarrow f_n = f_{n-1} + f_{n-2} \rightarrow \text{Fibonacci!}\)
Overview: Some Algebraic Classes for Sequences

- Algebraic sequences
- C-finite sequences
- Holonomic sequences
- Polynomial sequences
- Hypergeometric sequences

All sequences
Polynomial Sequences

- \((a_n)_{n \geq 0}\) is polynomial :\(\iff\) 
  \[ a_n = p(n) \text{ for polynomial } p \]
Polynomial Sequences

- \((a_n)_{n \geq 0}\) is polynomial \(\iff a_n = p(n)\) for polynomial \(p\)

Generating Functions:

\[
\sum_{n \geq 0} a_n z^n = \frac{q(z)}{(1 - z)^{\delta+1}}, \quad \text{with } \delta = \deg p, \, \deg q \leq \delta
\]

- \(\frac{1}{1-z} = 1 + z + z^2 + \ldots = \sum_{n \geq 0} z^n \) (geometric series)
- \(\frac{z}{(1-z)} = \sum_{n \geq 0} n z^{n-1} = \left(\frac{z}{(1-z)^2}\right) - \frac{z}{1-z} = \sum_{n \geq 0} n z^n\)
- \(\frac{2z^2 + z (1-z)}{(1-z)^3} = \sum_{n \geq 0} n^2 z^n\)
C-finite Sequences

\[ (a_n)_{n \geq 0} \text{ is C-finite } \iff a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1a_{n+1} + c_0a_n = 0 \]

for constants \( c_0, \ldots, c_{r-1} \) and all \( n \geq 0 \)
C-finite Sequences

- $(a_n)_{n \geq 0}$ is C-finite if and only if:
  
  \[ a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1a_{n+1} + c_0a_n = 0 \]
  
  for constants $c_0, \ldots, c_{r-1}$ and all $n \geq 0$

- $\leadsto$ linear recurrence with Constant coefficients
C-finite Sequences

- \((a_n)_{n \geq 0}\) is \textbf{C-finite} if

\[
a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1a_{n+1} + c_0a_n = 0
\]

for constants \(c_0, \ldots, c_{r-1}\) and all \(n \geq 0\).

- \(\leadsto\) linear recurrence with \textbf{Constant} coefficients

\[\textbf{Solution Space}\]

Assume

\[
z^r + c_{r-1}z^{r-1} + \cdots + c_1z + c_0 = (z - \alpha_1)^{e_1} \cdots (z - \alpha_m)^{e_m}.
\]

Basis of vector space containing solutions:

\[
(n^j \alpha_k^n)_{n \geq 0} \quad (1 \leq k \leq m, \ 0 \leq j < e_k)
\]
C-finite Sequences

- \((a_n)_{n \geq 0}\) is C-finite if
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\]

Generating Functions: Rational Functions!

\[
\sum_{n \geq 0} a_n z^n = \frac{p(z)}{1 + c_{r-1}z + \cdots + c_0z^r}, \quad \text{with } \deg p < r
\]
C-finite Example: Fibonacci

\[ f_{n+2} = f_{n+1} + f_n, \quad n \geq 0 \quad (f_0 = 1, f_1 = 1) \]

From symbolic Equation:

\[ \mathcal{Z} = \emptyset \cdot \emptyset + \emptyset \cdot 2 \cdot \emptyset = \emptyset \]

\[ \mathcal{Z} (\emptyset - \emptyset - \emptyset) = \emptyset \quad \therefore \quad \mathcal{Z} (\emptyset - (0 + \emptyset)) \]

\[ \implies \mathcal{Z} = \emptyset - (0 + \emptyset) \]

Spezialisation: \( \mathcal{A} \equiv z^0, \quad \mathcal{D} \equiv z_1, \quad \mathcal{E} \equiv z^2 \)

\[ T(z) = \frac{2^0}{2^0 - (z + z^2)} = \frac{1}{1 - (z + z^2)} \quad \rightarrow \quad [z^n] T(z) = \frac{1}{1 - \left( \frac{1 + \sqrt{5}}{2} \right)^n} - \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

(Binet’s formula.)
Algebraic Sequences

\[(a_n)_{n \geq 0} \text{ is algebraic } \iff \]

\[p_0(z) + p_1(z)A(z) + \cdots + p_d(z)A(z)^d = 0\]

for GF \(A(z) = \sum_{n \geq 0} a_n z^n\) and polynomials \(p_0, \ldots, p_d\)
Algebraic Sequences

- \((a_n)_{n \geq 0}\) is algebraic if
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- \(A(z)\) is algebraic over \(\mathbb{K}[z]\)
Algebraic Sequences

- $\langle a_n \rangle_{n \geq 0}$ is algebraic if
  \[ p_0(z) + p_1(z)A(z) + \cdots + p_d(z)A(z)^d = 0 \]
  for GF $A(z) = \sum_{n \geq 0} a_n z^n$ and polynomials $p_0, \ldots, p_d$

- $A(z)$ is algebraic over $\mathbb{K}[z]$

Counting Sequence

$\langle a_n \rangle_{n \geq 0}$ satisfies linear recurrence with polynomial coefficients
Algebraic Sequences

- \((a_n)_{n \geq 0}\) is algebraic if
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- \(\sim\) \(A(z)\) is algebraic over \(\mathbb{K}[z]\)

Analytic structure

- Puiseux series!
  \[
  A(z) \overset{z \rightarrow \zeta}{\longrightarrow} \sum_{k \geq k_0} \tilde{a}_k (1 - z/\zeta)^{k/r}
  \]
- “Singularity Analysis”

Counting Sequence

\((a_n)_{n \geq 0}\) satisfies linear recurrence with polynomial coefficients

\[
[z^n] \left(1 - \frac{z}{\zeta}\right)^{k/r} = \frac{n^{-k/r}}{\Gamma(-k/r)} \sum_{\mu=0}^{\nu} \binom{\frac{n}{r} - k/r}{\mu} \nu^\mu (1 + O(1/n))
\]

NB: Rational Exponent!
Algebraic Example: Catalan

Count Dyck Paths:

Generating func. via Decomposition:

\[ C(z) = \frac{1}{1 - 2z - z^2} \]

Singly daily analysis:

\[ \frac{-2}{\pi} \frac{n^{-3/2}}{\Gamma(-3/2)} \left( \frac{1}{4} \right)^n \left( 1 + O(1/n) \right) \]

\[ C(z) = \frac{1 - \sqrt{1 - 4z}}{2z} \]
Holonomic Sequences

- \((a_n)_{n \geq 0}\) is holonomic :

\[ p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0 \]

for polynomials \(p_0, \ldots, p_d\) and all \(n \geq 0\)
Holonomic Sequences

- $(a_n)_{n \geq 0}$ is holonomic $:\iff$
  $$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0$$
  for polynomials $p_0, \ldots, p_d$ and all $n \geq 0$

Generating Functions:

- $A(z) = \sum_{n \geq 0} a_n z^n \rightsquigarrow$ linear differential equation with polynomial coefficients
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**Generating Functions:**

- $A(z) = \sum_{n \geq 0} a_n z^n \rightsquigarrow$ linear differential equation with polynomial coefficients

- Solutions: “generalized series”, shape

$$z^\alpha \exp(q(z^{-1/s}))(a_0(z) + \log(z)a_1(z) + \cdots + \log(z)^m a_m(z))$$

**Strategy:**

1. Identify Singularity closest to $0$.
2. Use ODE, extract gen. series
3. Singularity analysis!
Holonomic Example

– Harmonic Numbers: \( H_n = \sum_{1 \leq k \leq n} \frac{1}{k} \)

\[
H_{n+1} = H_n + \frac{1}{n+1} \quad | \cdot (n+1)
\]

\[
(n+1) \cdot H_{n+1} = (n+1) \cdot H_n + 1 \quad | \cdot (-1)
\]

\[
(n+2) \cdot H_{n+2} = (n+2) \cdot H_{n+1} + 1 \quad \Rightarrow
\]

\[
(n+2) \cdot H_{n+2} - (2n+3) \cdot H_{n+1} + (n+1) \cdot H_n = 0
\]

\[
\log (1 + z) = \sum_{n \geq 1} \frac{(-1)^{n+1} z^n}{n} \Rightarrow \log (1 - z) = \sum_{n \geq 1} \frac{z^n}{n^2} \cdot \left| \cdot \frac{1}{1 - z} \right|
\]

\[
H = \frac{1}{1 - z} \cdot \log \left( \frac{1}{1 - z} \right) = \sum_{n \geq 1} H_n \cdot z^n \quad \rightarrow \quad H' = \frac{A}{(1 - z)^2} \cdot \log \left( \frac{1}{1 - z} \right) + \frac{A}{1 - z} \cdot (1 - z) \cdot \frac{1}{(1 - z)^2}
\]

\[
= \frac{A}{1 - z} \cdot H + \frac{A}{(1 - z)^2} \quad | \cdot (1 - z)^2
\]

\[
\Rightarrow \quad (1 - z)^2 \cdot H' = (1 - z) \cdot H + 1.
\]
A Bit About Permutations

- **Permutations**: bijective maps $\sigma : [n] \rightarrow [n]$
- **Run / Fall**: consecutive elements getting larger / smaller

1342756

134|27|56 (runs)

1|3|42|75|6 (falls)
Flip-Sorting

Repeat until sorted:

- Partition $\sigma$ into falls
- Reverse (“flip”) all falls

Theorem (Ungar, 1982)

Any permutation of $[n]$ is sorted after at most $n - 1$ flip-rounds.
Permutations are **pop-stacked** if they result from a flip-round.

Theorem (Asinowski–Banderier–Billey–H.–Linusson, 2019+)

1. $\pi$ is pop-stacked $\iff$ consecutive runs overlap
2. number of pop-stacked permutations with $k$ runs $\rightarrow$ C-finite
Pop-Stacked with $k$ Runs – Proof

Bijection: permutations with $k$ runs $\leftrightarrow \mathcal{L}_k \subset \text{words over } [k]$

261453 $\leftrightarrow$ 213221
Pop-Stacked with $k$ Runs – Proof

**Bijection:** permutations with $k$ runs $\leftrightarrow \mathcal{L}_k$ ⊂ words over $[k]$

261453 $\leftrightarrow$ 213221

**Strategy:** construct DFA recognizing $\mathcal{L}_k \Rightarrow$ sequence C-finite
Pop-Stacked: Experimental Observations

- Counting sequence is C-finite $\Rightarrow$ generating function is rational
Pop-Stacked: Experimental Observations

- Counting sequence is C-finite $\Rightarrow$ generating function is rational
- Thus: GF can be guessed from data!

\[ P_1(z) = z \]
\[ P_2(z) = 2z^3 \]
\[ P_3(z) = 2z^4 \]
\[ P_4(z) = 2z^6 \]

Conjecture:
\[ P_k(z) = q_k(z) Q_{\lfloor j \rfloor} \left(1 - jz\right)^k \]

Benjamin Hackl
Flip-Sorting / Lattice Walks
Counting sequence is C-finite $\implies$ generating function is rational

Thus: GF can be guessed from data!

\[
P_1(z) = \frac{z}{1 - z}
\]

\[
P_2(z) = \frac{2z^3}{(1 - z)^2(1 - 2z)}
\]

\[
P_3(z) = \frac{2z^4(1 + 3z - 6z^2)}{(1 - z)^3(1 - 2z)^2(1 - 3z)}
\]

\[
P_4(z) = \frac{2z^6(21 - 74z + 5z^2 + 180z^3 - 144z^4)}{(1 - z)^4(1 - 2z)^3(1 - 3z)^2(1 - 4z)}
\]
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\]

Conjecture:

\[
P_k(z) = \frac{q_k(z)}{\prod_{1 \leq j \leq k}(1 - jz)^{k-j+1}}
\]
A Bit About (2D) Lattice Walks

- **Step set:** $S \subseteq \mathbb{Z}^2$
- **Lattice walk:** sequence over $S$
A Bit About (2D) Lattice Walks

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- **Lattice walk:** sequence over $S$

**Typical Questions:** find \# of walks with $n$ steps that
- end in given point
- avoid given region

Relevant Work by Bostan, Bousquet-Mélou, Mishna, Kauers:
- classification of lattice walks
  - with “small steps”
  - remaining in the quarter plane
A Bit About (2D) Lattice Walks

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Step-Changing Lattice Walks  joint work with Manuel Kauers

- **Idea:** step set depends on current position

- **Unrestricted walks:** functional equation $\rightsquigarrow$ **algebraic**

- **Goal:** systematic analysis of walks *avoiding left half plane*

  - $2\times$ small steps $\rightarrow (2^8)^2 = 65536$ different families!
Experimental Results

- Total number of “interesting” families: 38963
Experimental Results

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- Guessing recurrences from 2500 sequence terms each
Experimental Results

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