Flip-Sorting and Step-Changing Lattice Walks

Asymptotic Counting and Analytic Properties via Algebraic Classification

September 11, 2019

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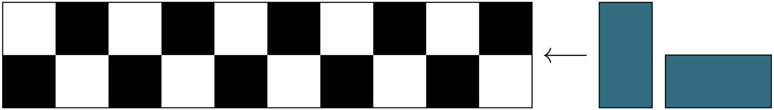
Introductory Example

- Given: $2 \times n$ checkerboard, n identical 1×2 domino pieces



- # of possibilities to cover the board?

$n = 1$: 1

$n = 2$: 2

$n = 3$: 3

$n = 4$: 5

⋮

size 1 size 2

In general τ ... all tilings $\Rightarrow \tau = \emptyset + \tau + \tau$

$\Rightarrow f_n = f_{n-1} + f_{n-2} \longrightarrow$ Fibonacci!

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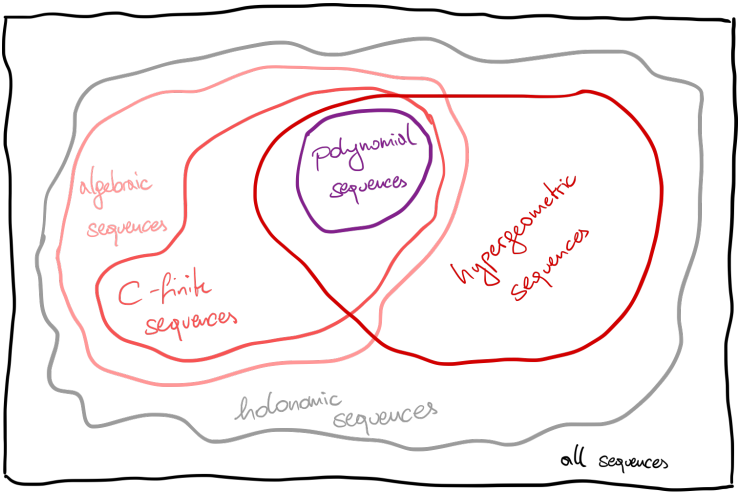
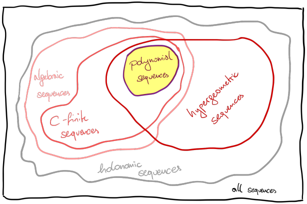
Overview: Some Algebraic Classes for Sequences



Polynomial Sequences



- $(a_n)_{n\geq 0}$ is **polynomial** $:\Longleftrightarrow$
  $a_n = p(n)$ for polynomial p

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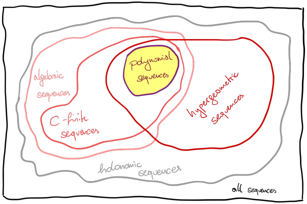

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Generating Functions:

$$\sum_{n\geq 0} a_n z^n = \frac{q(z)}{(1-z)^{\delta+1}}, \qquad \text{with } \delta = \deg p,\ \deg q \leq \delta$$

- $\frac{1}{1-z} = 1 + z + z^2 + \dots = \sum_{n\geq 0} z^n$ (geometric series)

↓ Diff.

$$\frac{1}{(1-z)^2} = \sum_{n\geq 0} n z^{n-1} \Rightarrow \frac{z}{(1-z)^2} = \sum_{n\geq 0} n z^n$$

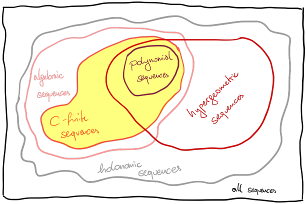
↓

$$\frac{2}{(1-z)^3} = \sum_{n\geq 0} n(n-1) z^{n-2} \Rightarrow \frac{2z^2}{(1-z)^3} = \sum_{n\geq 0} n(n-1) z^n.$$

$+$ ⟶

$$\frac{2z^2 + z(1-z)}{(1-z)^3} = \sum_{n\geq 0} n^2 z^n$$

C-finite Sequences

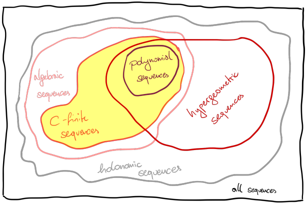


- $(a_n)_{n\geq 0}$ is **C-finite** $:\Longleftrightarrow$

$$a_{n+r} + c_{r-1}a_{n+r-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0$$

for constants $c_0, \ldots, c_{r-1}$ and all $n \geq 0$

C-finite Sequences



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- $\rightsquigarrow$ linear recurrence with **C**onstant coefficients

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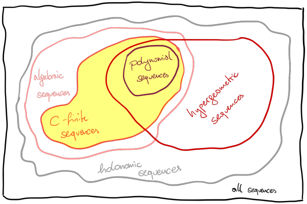

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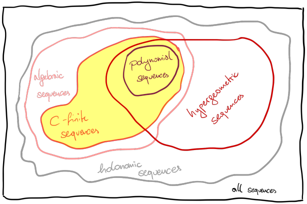
- $\leadsto$ linear recurrence with **C**onstant coefficients

Solution Space

Assume $z^r + c_{r-1}z^{r-1} + \cdots + c_1 z + c_0 = (z-\alpha_1)^{e_1} \ldots (z-\alpha_m)^{e_m}$.
Basis of vector space containing solutions:

$$(n^j \alpha_k^n)_{n\geq 0} \qquad (1 \leq k \leq m,\ 0 \leq j < e_k)$$

C-finite Sequences



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Generating Functions: Rational Functions!

$$\sum_{n\geq 0} a_n z^n = \frac{p(z)}{1 + c_{r-1}z + \cdots + c_0 z^r}, \qquad \text{with } \deg p < r$$

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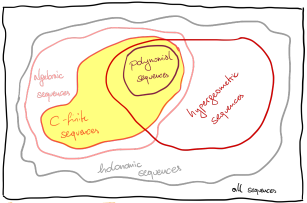
C-finite Example: Fibonacci



$f_{n+2} = f_{n+1} + f_n, \quad n \geq 0 \quad (f_0 = 1,\ f_1 = 1)$

From symbolic Equation:

$$\mathcal{T} = \emptyset + \square\,\mathcal{T} + \boxminus\,\mathcal{T} \quad \Big|\ - \square\,\mathcal{T} - \boxminus\,\mathcal{T}$$

$$\mathcal{T}\,(\emptyset - \square - \boxminus) = \emptyset \quad \Big|\ : (\emptyset - (\square + \boxminus))$$

$$\Rightarrow \mathcal{T} = \frac{\emptyset}{\emptyset - (\square + \boxminus)}.$$

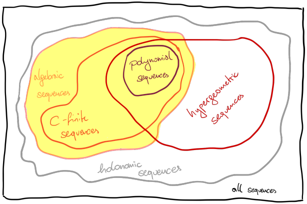
Specialization: $\emptyset \mathrel{\hat{=}} z^0,\ \square \mathrel{\hat{=}} z,\ \boxminus \mathrel{\hat{=}} z^2$

$$T(z) = \frac{z^0}{z^0 - (z + z^2)} = \frac{1}{1 - (z + z^2)}. \longrightarrow [z^n]\,T(z) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

(Binet's formula.)

Algebraic Sequences

- $(a_n)_{n\geq 0}$ is **algebraic** $:\Longleftrightarrow$
$$p_0(z) + p_1(z)A(z) + \cdots + p_d(z)A(z)^d = 0$$
for GF $A(z) = \sum_{n\geq 0} a_n z^n$ and polynomials $p_0, \ldots, p_d$



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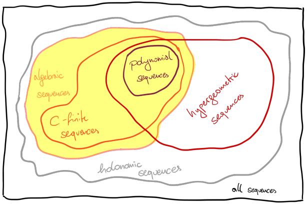

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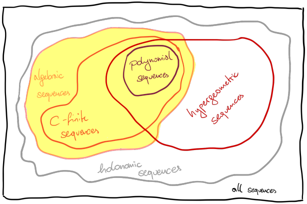
Algebraic Sequences



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Counting Sequence

$(a_n)_{n\geq 0}$ satisfies linear recurrence with polynomial coefficients

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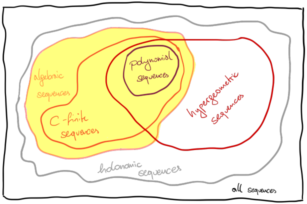

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Analytic structure

- Puiseux series!

$$A(z) \overset{z\to\zeta}{=} \sum_{k\geq k_0} \tilde{a}_k (1 - z/\zeta)^{k/r}$$

- *"Singularity Analysis"*

Counting Sequence

$(a_n)_{n\geq 0}$ satisfies linear recurrence with polynomial coefficients

NB: Rational Exponent!

$$[z^n]\left(1-z/\zeta\right)^{k/r} = \frac{n^{-1-k/r}}{\Gamma(-k/r)}\zeta^{-n}\left(1+O\left(\tfrac{1}{n}\right)\right)$$

→ typical Asymptotics!

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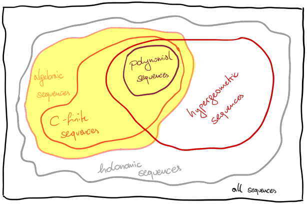
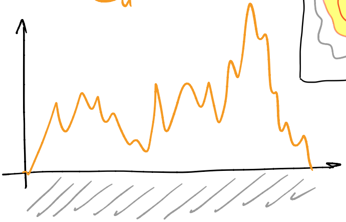
Algebraic Example: Catalan

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$



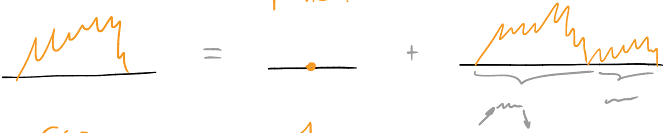
Count Dyck-Paths:



Sequence of steps ↗, ↘.

Avoiding area under axis.

Generating func.: via Decomposition!



$C(z) = 1 + z \cdot C(z) \cdot C(z)$

$$C(z) = 1 + zC(z)^2$$

$\Downarrow$

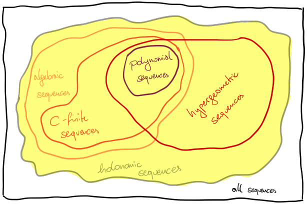
$\vdots$

$\Downarrow$

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

Singularity Analysis: … $\to$ $\displaystyle \frac{-2\,n^{-3/2}}{\Gamma(-3/2)} (1/4)^{-n} \left(1 + O(1/n)\right) = \frac{4^n n^{-3/2}}{\sqrt{\pi}}.$

Holonomic Sequences



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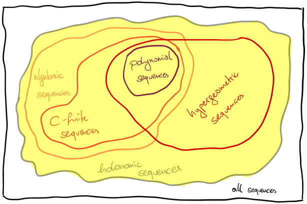

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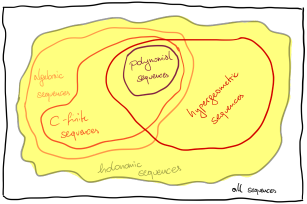

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Generating Functions:

- $A(z) = \sum_{n\geq 0} a_n z^n \rightsquigarrow$ linear differential equation with polynomial coefficients
- Solutions: "generalized series", shape

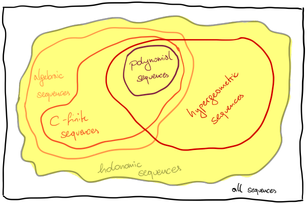
$$z^\alpha \exp(q(z^{-1/s}))(a_0(z) + \log(z)a_1(z) + \cdots + \log(z)^m a_m(z))$$

Strategy: ① Identify singularity closest to 0.

② Use ODE, extract gen. series

③ Singularity Analysis!

Holonomic Example



- **Harmonic Numbers:** $H_n = \sum_{1\le k\le n} \frac{1}{k}$

$$H_{n+1} = H_n + \frac{1}{n+1} \qquad \Big|\cdot(n+1)$$

$$(n+1)H_{n+1} = (n+1)H_n + 1 \qquad \Big|\cdot(-1)$$

$$(n+2)H_{n+2} = (n+2)H_{n+1} + 1 \qquad +$$

$$(n+2)H_{n+2} - (2n+3)H_{n+1} + (n+1)H_n = 0$$

Holonomic equation!

$$\log(1+z) = \sum_{n\ge1} \frac{(-1)^{n+1}}{n} z^n \implies \log(1-z) = -\sum_{n\ge1} \frac{1}{n} z^n$$

$$\implies \log\left(\frac{1}{1-z}\right) = -\log(1-z) = \sum_{n\ge1} \frac{z^n}{n}. \qquad \Big|\cdot \frac{1}{1-z}$$

partial sums!

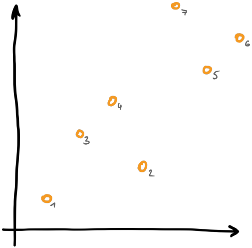
$$H = \frac{1}{1-z}\log\left(\frac{1}{1-z}\right) = \sum_{n\ge1} H_n z^n \longrightarrow H' = \frac{1}{(1-z)^2}\log\left(\frac{1}{1-z}\right) + \frac{1}{1-z}\cdot(1-z)\cdot\frac{1}{(1-z)^2}$$

$$= \frac{1}{1-z} H + \frac{1}{(1-z)^2} \qquad \Big|\cdot(1-z)^2$$

$$\implies (1-z)^2 H' = (1-z)H + 1.$$

A Bit About Permutations

- **Permutations:** bijective maps $\sigma\colon [n] \to [n]$
- **Run / Fall:** consecutive elements getting larger / smaller



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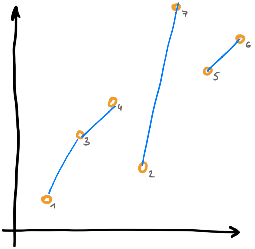
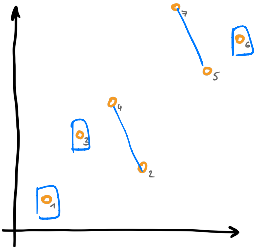


134|27|56 (runs)

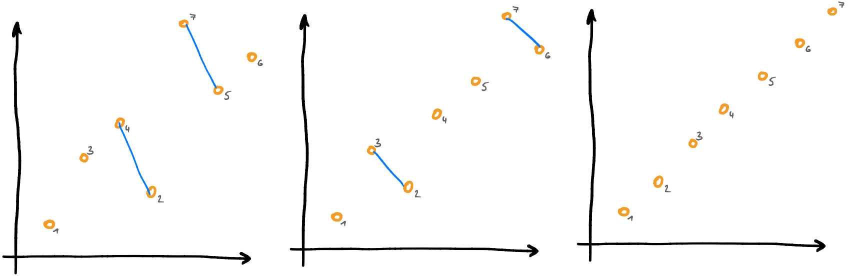


1|3|42|75|6 (falls)

Flip-Sorting

Repeat until sorted:

- Partition σ into falls
- Reverse ("flip") all falls

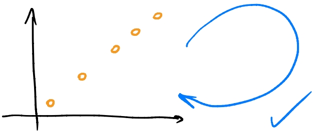


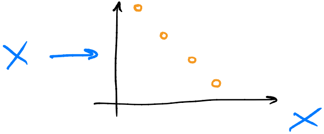
Theorem (Ungar, 1982)

Any permutation of $[n]$ is sorted after at most $n-1$ flip-rounds.

Pop-Stacked Permutations joint work with Andrei Asinowski, Cyril Banderier, Sara Billey, Svante Linusson

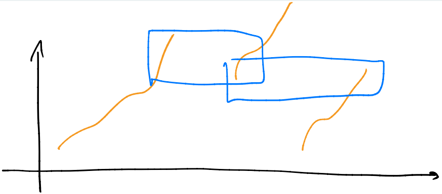
- Permutations are **pop-stacked** if they result from a flip-round





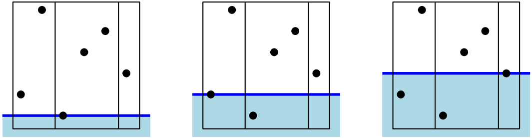
Theorem (Asinowski–Banderier–Billey–H.–Linusson, 2019+)

1. π is pop-stacked $\iff$ consecutive runs overlap
2. number of pop-stacked permutations with k runs $\to$ **C-finite**



Pop-Stacked with k Runs – Proof

Bijection: permutations with k runs $\longleftrightarrow \mathcal{L}_k \subset$ words over $[k]$



$$261453 \mapsto 213221$$

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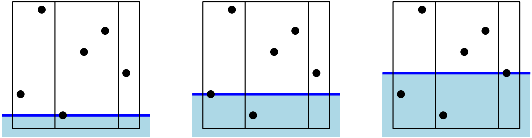

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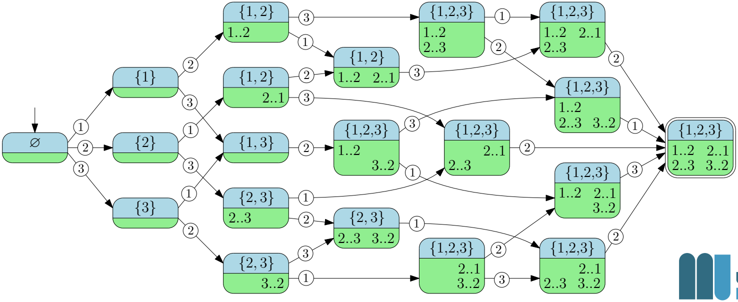
Strategy: construct DFA recognizing $\mathcal{L}_k \Rightarrow$ sequence C-finite



Pop-Stacked: Experimental Observations

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$$P_1(z) = \frac{z}{1-z}$$

$$P_2(z) = \frac{2z^3}{(1-z)^2(1-2z)}$$

$$P_3(z) = \frac{2z^4(1+3z-6z^2)}{(1-z)^3(1-2z)^2(1-3z)},$$

$$P_4(z) = \frac{2z^6(21-74z+5z^2+180z^3-144z^4)}{(1-z)^4(1-2z)^3(1-3z)^2(1-4z)},$$

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$$
\begin{aligned}
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\end{aligned}
$$

Conjecture:

$$P_k(z) = \frac{q_k(z)}{\prod_{1\leq j\leq k}(1-jz)^{k-j+1}}$$

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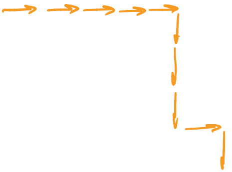
A Bit About (2D) Lattice Walks

- **Step set:** $\mathcal{S} \subseteq \mathbb{Z}^2$
- **Lattice walk:** sequence over $\mathcal{S}$

$S = \{\uparrow, \rightarrow, \downarrow, \leftarrow\}$



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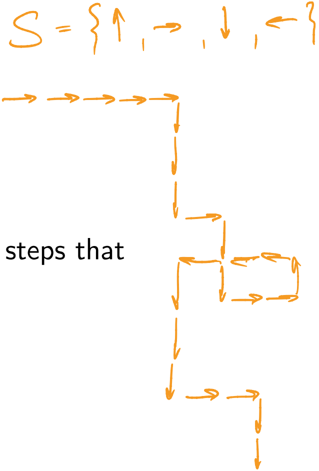

Typical Questions: find # of walks with n steps that

- end in given point
- avoid given region

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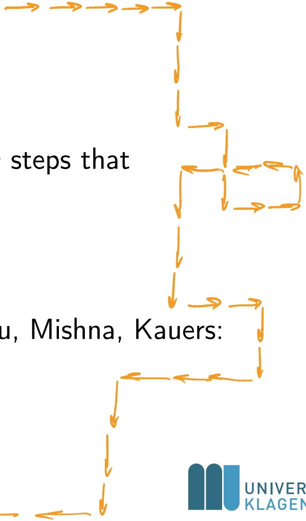

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Relevant Work by Bostan, Bousquet-Mélou, Mishna, Kauers: classification of lattice walks

- with "small steps"
- remaining in the quarter plane

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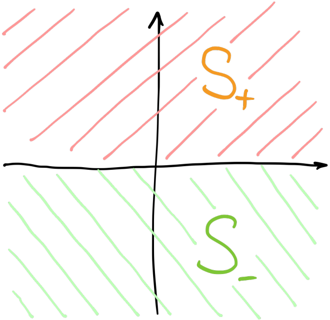
Step-Changing Lattice Walks joint work with Manuel Kauers

- **Idea:** step set depends on current position



- Unrestricted walks: functional equation $\rightsquigarrow$ **algebraic**
- **Goal:** systematic analysis of walks *avoiding left half plane*
  - $2\times$ small steps $\to (2^8)^2 = 65536$ different families!

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Experimental Results

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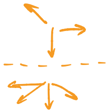
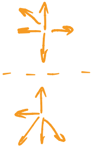
Endpoint at (0, 0)	(x, 0)	(0, y)	*	% of walks
✓	✓	✓	✓	5.174
✓	✓	✓		0.264
✓	✓		✓	0.031
✓	✓			5.087
✓		✓	✓	0.329
✓		✓		2.687
✓			✓	0.028
✓				2.413
	✓	✓	✓	0.000
	✓	✓		0.000
	✓		✓	0.000
	✓			0.008
		✓	✓	2.605
		✓		0.000
			✓	0.023
				81.352

e.g.



e.g.



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