Cutting Down Plane Trees

- Remove all leaves!
Cutting Down Plane Trees

- Remove all leaves!

![Diagram showing the process of removing leaves from a tree structure.](image-url)
Cutting Down Plane Trees

- Remove all leaves!
Cutting Down Plane Trees

- Remove all leaves!
Cutting Down Plane Trees

- Remove all leaves!

Parameters of Interest:
- Size of $r$th reduction
- Age: $\#$ of possible reductions
Reduction $\rightarrow$ Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, growing trees
Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, **growing trees**

```
  □
```

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![Tree diagrams](image)
Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, growing trees

\[ \text{\ldots} \]

\[ \text{\ldots} \]
Expansion operators

- $F \ldots$ family of plane trees; bivariate generating function $f$
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees
Expansion operators

- $F$... family of plane trees; bivariate generating function $f$
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

**Leaf expansion $\Phi$**

- inverse operation to leaf reduction
  - attach leaves to all current leaves (required)
  - attach leaves to inner nodes (optional)

\[ \Box \xrightarrow{\Phi} \bigcirc + \bigcirc + \bigcirc + \cdots \]
\[ \Box \triangleq t, \bigcirc \triangleq z \Rightarrow \Phi(t) = zt + zt^2 + zt^3 + \cdots \]
Reductions on Plane Trees

Leaves

Parameters of Interest:
▶ tree size after $r$ reductions
▶ cumulative reduction size
Reductions on Plane Trees

Leaves

Parameters of Interest:
- Tree size after $r$ reductions
- Cumulative reduction size
Reductions on Plane Trees

Leaves

Paths

Parameters of Interest:
- Tree size after $r$ reductions
- Cumulative reduction size
Reductions on Plane Trees

Leaves

Paths
Reductions on Plane Trees

Leaves

 Paths

Old leaves
Reductions on Plane Trees

- Leaves
- Paths
- Old leaves

Parameters of Interest:
- Tree size after \( r \) reductions
- Cumulative reduction size
Reductions on Plane Trees

Leaves

Old leaves

Paths

Old paths

Parameters of Interest:
- tree size after $r$ reductions
- cumulative reduction size
Reductions on Plane Trees

Leaves

Old leaves

Paths

Old paths
Reductions on Plane Trees

Parameters of Interest:
- tree size after $r$ reductions
- cumulative reduction size
Bivariate Generating Function

Proposition

\( T \ldots \text{rooted plane trees} \)
Bivariate Generating Function

Proposition

- $\mathcal{T}$ ... rooted plane trees
- $T(z, t)$ ... BGF for $\mathcal{T}$ ($z \sim$ inner nodes, $t \sim$ leaves)
Bivariate Generating Function

**Proposition**

- $\mathcal{T}$... *rooted plane trees*
- $T(z, t)$... *BGF for $\mathcal{T}$ (z $\leadsto$ inner nodes, t $\leadsto$ leaves)*

\[ T(z, t) = \frac{1 - (z - t) - \sqrt{1 - 2(z + t) + (z - t)^2}}{2} \]

**Proof.** Symbolic equation

\[ \mathcal{T} = \square + \mathcal{T} \]

translates into

\[ T(z, t) = t + z \cdot \frac{T(z, t)}{1 - T(z, t)} \]

which can be solved explicitly.
Bivariate Generating Function

**Proposition**

- $\mathcal{T}$… rooted plane trees
- $T(z, t)$… BGF for $\mathcal{T}$ ($z \mapsto$ inner nodes, $t \mapsto$ leaves)

\[
\Rightarrow T(z, t) = \frac{1 - (z - t) - \sqrt{1 - 2(z + t) + (z - t)^2}}{2}
\]

**Proof.** Symbolic equation

\[
\mathcal{T} = \quad +
\]

translates into

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T(z, t) = t + z \cdot \frac{T(z, t)}{1 - T(z, t)}
\]

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Bivariate Generating Function

Proposition

- $\mathcal{T} \ldots$ rooted plane trees
- $T(z, t) \ldots$ BGF for $\mathcal{T}$ ($z \sim$ inner nodes, $t \sim$ leaves)

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\[ \mathcal{T} = \text{square} + \mathcal{T} \]

translates into

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Bivariate Generating Function

Proposition

- $\mathcal{T}$ ... rooted plane trees
- $T(z, t)$ ... BGF for $\mathcal{T}$ ($z \sim$ inner nodes, $t \sim$ leaves)

\[ T(z, t) = \frac{1 - (z - t) - \sqrt{1 - 2(z + t) + (z - t)^2}}{2} \]

Proof. Symbolic equation

\[ \mathcal{T} = \square + \mathcal{T} + \mathcal{T} \]

translates into

\[ T(z, t) = t + z \cdot \frac{T(z, t)}{1 - T(z, t)} \]

which can be solved explicitly.
Bivariate Generating Function

Proposition

- $\mathcal{T}$... rooted plane trees
- $T(z, t)$... BGF for $\mathcal{T}$ ($z \rightsquigarrow$ inner nodes, $t \rightsquigarrow$ leaves)

\[
T(z, t) = \frac{1 - (z - t) - \sqrt{1 - 2(z + t) + (z - t)^2}}{2}
\]

Proof. Symbolic equation

\[
\mathcal{T} = \square + \mathcal{T}
\]

translates into

\[
T(z, t) = t + z \cdot \frac{T(z, t)}{1 - T(z, t)}
\]

which can be solved explicitly.
Bivariate Generating Function

**Proposition**

- $\mathcal{T} \ldots$ rooted plane trees
- $T(z, t) \ldots$ BGF for $\mathcal{T}$ ($z \leadsto$ inner nodes, $t \leadsto$ leaves)

\[ T(z, t) = \frac{1 - (z - t) - \sqrt{1 - 2(z + t) + (z - t)^2}}{2} \]

**Proof.** Symbolic equation

\[ \mathcal{T} = \square + \mathcal{N} \stackrel{\mathcal{T}}{\longrightarrow} \ldots \mathcal{T} \]

translates into

\[ T(z, t) = t + z \cdot \frac{T(z, t)}{1 - T(z, t)} \]

which can be solved explicitly.
Bivariate Generating Function

**Proposition**

- $\mathcal{T} \ldots$ rooted plane trees
- $T(z, t) \ldots$ BGF for $\mathcal{T}$ ($z \rightsquigarrow$ inner nodes, $t \rightsquigarrow$ leaves)

$$\Rightarrow T(z, t) = \frac{1 - (z - t) - \sqrt{1 - 2(z + t) + (z - t)^2}}{2}$$

**Proof.** Symbolic equation

$$\mathcal{T} = \square + \sum_{\mathcal{T}}$$

translates into

$$T(z, t) = t + z \cdot \frac{T(z, t)}{1 - T(z, t)}$$

which can be solved explicitly.
Leaf expansion operator $\Phi$

**Proposition**

$$\Phi(f(z, t)) = (1 - t)f\left(\frac{z}{(1 - t)^2}, \frac{zt}{(1 - t)^2}\right)$$
Leaf expansion operator $\Phi$

**Proposition**

$$\Phi(f(z, t)) = (1 - t)f\left(\frac{z}{(1 - t)^2}, \frac{zt}{(1 - t)^2}\right)$$

- Tree with $n$ inner nodes and $k$ leaves $\leadsto z^n t^k$
- **Expansion:**
  - In total:
  $$\Phi(z^n t^k) =$$
Leaf expansion operator $\Phi$

**Proposition**

\[
\Phi(f(z, t)) = (1 - t)f\left(\frac{z}{(1 - t)^2}, \frac{zt}{(1 - t)^2}\right)
\]

- Tree with $n$ inner nodes and $k$ leaves $\sim z^n t^k$
- **Expansion:**
  - inner nodes stay inner nodes

- In total:

\[
\Phi(z^n t^k) = z^n.
\]
Leaf expansion operator $\Phi$

**Proposition**

$$\Phi(f(z, t)) = (1 - t)f\left(\frac{z}{(1 - t)^2}, \frac{zt}{(1 - t)^2}\right)$$

- Tree with $n$ inner nodes and $k$ leaves $\leadsto z^n t^k$
- **Expansion:**
  - inner nodes stay inner nodes
  - attach a non-empty sequence of leaves to all current leaves

- In total:
  $$\Phi(z^n t^k) = z^n \cdot \left(\frac{zt}{1 - t}\right)^k.$$
Leaf expansion operator $\Phi$

**Proposition**

$$\Phi(f(z, t)) = (1 - t)f\left(\frac{z}{(1 - t)^2}, \frac{zt}{(1 - t)^2}\right)$$

- Tree with $n$ inner nodes and $k$ leaves $\sim z^nt^k$
- **Expansion:**
  - inner nodes stay inner nodes
  - attach a non-empty sequence of leaves to all current leaves
  - there are $2n + k - 1$ positions where sequences of leaves can be inserted
- In total:
  $$\Phi(z^nt^k) = z^n \cdot \left(\frac{zt}{1 - t}\right)^k \cdot \frac{1}{(1 - t)^{2n+k-1}}$$
Leaf expansion operator \( \Phi \)

**Proposition**

\[
\Phi(f(z, t)) = (1 - t)f\left(\frac{z}{(1-t)^2}, \frac{zt}{(1-t)^2}\right)
\]

- Tree with \( n \) inner nodes and \( k \) leaves \( \rightsquigarrow z^n t^k \)
- **Expansion:**
  - inner nodes stay inner nodes
  - attach a non-empty sequence of leaves to all current leaves
  - there are \( 2n + k - 1 \) positions where sequences of leaves can be inserted
- In total:
  \[
  \Phi(z^n t^k) = z^n \cdot \left(\frac{zt}{1-t}\right)^k \cdot \frac{1}{(1-t)^{2n+k-1}} = (1-t)\left(\frac{z}{(1-t)^2}\right)^n \left(\frac{zt}{(1-t)^2}\right)^k
  \]

As \( \Phi \) is linear, this proves the proposition.
Leaf expansion operator $\Phi$

**Proposition**

$$\Phi(f(z, t)) = (1 - t)f\left(\frac{z}{(1 - t)^2}, \frac{zt}{(1 - t)^2}\right)$$

- Tree with $n$ inner nodes and $k$ leaves $\leadsto z^n t^k$

- **Expansion:**
  - Inner nodes stay inner nodes
  - Attach a non-empty sequence of leaves to all current leaves
  - There are $2n + k - 1$ positions where sequences of leaves can be inserted

- In total:

$$\Phi(z^n t^k) = z^n \cdot \left(\frac{zt}{1 - t}\right)^k \cdot \frac{1}{(1 - t)^{2n+k-1}} = (1-t)\left(\frac{z}{(1 - t)^2}\right)^n \left(\frac{zt}{(1 - t)^2}\right)^k$$

- As $\Phi$ is linear, this proves the proposition.
Properties of $\Phi$

- Functional equation: $T(z, t) = \Phi(T(z, t)) + t$
Properties of $\Phi$

- Functional equation: $T(z, t) = \Phi(T(z, t)) + t$
- With $z = u/(1 + u)^2$ and by some manipulations

\[
\Phi^r(T(z, t))|_{t=z} = \frac{1 - u^{r+2}}{(1 - u^{r+1})(1 + u)} T\left(\frac{u(1 - u^{r+1})^2}{(1 - u^{r+2})^2}, \frac{u^{r+1}(1 - u)^2}{(1 - u^{r+2})^2}\right)
\]
Properties of $\Phi$

- Functional equation: $T(z, t) = \Phi(T(z, t)) + t$
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- BGF $G_r(z, v)$ for size comparison: $z$ tracks original size, $v$ size of $r$-fold reduced tree
Properties of $\Phi$

- **Functional equation:** $T(z, t) = \Phi(T(z, t)) + t$
- **With** $z = u/(1 + u)^2$ and by some manipulations

$$\Phi_r(T(z, t))|_{t=z} = \frac{1 - u^{r+2}}{(1 - u^{r+1})(1 + u)} T\left(\frac{u(1 - u^{r+1})^2}{(1 - u^{r+2})^2}, \frac{u^{r+1}(1 - u)^2}{(1 - u^{r+2})^2}\right)$$

- **BGF** $G_r(z, \nu)$ for size comparison: $z$ tracks original size, $\nu$ size of $r$-fold reduced tree
- **Intuition:** $\nu$ “remembers” size while tree family is expanded

$$G_r(z, \nu) = \frac{1 - u^{r+2}}{(1 - u^{r+1})(1 + u)} T\left(\frac{u(1 - u^{r+1})^2}{(1 - u^{r+2})^2} \nu, \frac{u^{r+1}(1 - u)^2}{(1 - u^{r+2})^2} \nu\right)$$
<table>
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<th>Cutting Leaves</th>
<th>Cutting Paths</th>
<th>Results</th>
<th>Outlook</th>
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**SageMath Demo**
Theorem (H.–Heuberger–Kropf–Prodinger)

After \( r \) reductions of a random tree of size \( n \), the remaining size \( X_{n,r} \) has mean and variance

\[
\mathbb{E}X_{n,r} \approx \frac{n}{r+1} - \frac{r(r-1)}{6(r+1)} + O(n^{-1}),
\]

\[
\text{Var}X_{n,r} \approx \frac{r(r+2)}{6(r+1)^2}n + O(1),
\]

and \( X_{n,r} \) is asymptotically normally distributed.
Cutting leaves

Theorem (H. – Heuberger – Kropf – Prodinger)

After $r$ reductions of a random tree of size $n$, the remaining size $X_{n,r}$ has mean and variance

$$\mathbb{E} X_{n,r} = \frac{n}{r+1} - \frac{r(r-1)}{6(r+1)} + O(n^{-1}),$$

$$\mathbb{V} X_{n,r} = \frac{r(r+2)}{6(r+1)^2} n + O(1),$$

and $X_{n,r}$ is asymptotically normally distributed.

Proof insights:

- $\mathbb{E} X_{n,r}$ and $\mathbb{V} X_{n,r}$ follow via singularity analysis
- Asymptotic normality: $n - X_{n,r}$ is a tree parameter with small toll function, limit law by Wagner (2015)
Pruning

- Remove all paths that end in a leaf!
Pruning

- Remove all paths that end in a leaf!
Branches in a Tree

Trees can be partitioned into branches:
Branches in a Tree

> Trees can be partitioned into branches:

Trees can be partitioned into branches:
Branches in a Tree

- Trees can be partitioned into branches:

▶ Trees can be partitioned into branches:
Branches in a Tree

- Trees can be partitioned into branches:
Branches in a Tree

▶ Trees can be partitioned into branches:
▶ \textbf{Q:} How many branches are there?
Branches in a Tree

- Trees can be partitioned into branches:
- Q: How many branches are there?

Observation

Total # of branches $\triangleq$ # of leaves in all reduction stages
Branches in a Tree

- Trees can be partitioned into branches:
- **Q:** How many branches are there?

Observation

Total # of branches $\geq$ # of leaves in all reduction stages

**Proof:** all branches end in exactly one leaf (at some point).
Branches in a Tree – Result

Theorem (H.–Heuberger–Kropf–Prodinger)
Average # of branches in a random plane tree of size $n$ is

$$\alpha n + \frac{1}{6} \log_4 n + C + \delta(\log_4 n) + O\left(n^{-\frac{1}{4}}\right),$$

where

$$\alpha = \sum_{k \geq 2} \frac{1}{2^{k-1}} \approx 0.60669,$$

$$C = -\gamma + 4 \alpha \log 2 + \log 2 + 24 \zeta'(-1) + \frac{1}{12} \log 2 \approx -0.11811,$$

and

$$\delta$$ is periodic fluctuation:

$$\delta(x) := \frac{1}{\log 2} \sum_{k \in \mathbb{Z}\setminus\{0\}} (-1 + \chi_k) \Gamma(\chi_k/2) \zeta(-1 + \chi_k) e^{2k\pi i x},$$

with

$$\chi_k = 2\pi i k \log 2.$$
Branches in a Tree – Result

Theorem (H.–Heuberger–Kropf–Prodinger)

Average # of branches in a random plane tree of size $n$ is

$$\alpha n + \frac{1}{6} \log_4 n + C + \delta(\log_4 n) + O(n^{-1/4}),$$

where

- $\alpha = \sum_{k \geq 2} \frac{1}{2^k - 1} \approx 0.60669$
- $C = -\gamma + 4\alpha \log 2 + \log 2 + 24\zeta'(1) + \frac{1}{12} \log 2 \approx -0.11811$
- $\delta$ is a periodic fluctuation:

$$\delta(x) := \frac{1}{\log 2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \left(1 + \chi_k\right)\Gamma\left(\chi_k/2\right)\zeta\left(1 + \chi_k\right)e^{2k\pi ix},$$

with $\chi_k = \frac{2\pi i k}{\log 2}$.
Branches in a Tree – Result

**Theorem (H.–Heuberger–Kropf–Prodinger)**

Average # of branches in a random plane tree of size $n$ is

$$\alpha n + \frac{1}{6} \log_4 n + C + \delta(\log_4 n) + O(n^{-1/4}),$$

where

$$\alpha = \sum_{k \geq 2} \frac{1}{2^k - 1} \approx 0.60669,$$

and $C$, $\delta$, and $O$ are constants.
Branches in a Tree – Result

Theorem (H.-Heuberger–Kropf–Prodinger)

Average # of branches in a random plane tree of size \(n\) is

\[
\alpha n + \frac{1}{6} \log_4 n + C + \delta(\log_4 n) + O(n^{-1/4}),
\]

- \(\alpha = \sum_{k \geq 2} \frac{1}{2^k - 1} \approx 0.60669,\)
- \(C = -\frac{\gamma + 4\alpha \log 2 + \log 2 + 24\zeta'(-1) + 2}{12 \log 2} \approx -0.11811,\)
Branches in a Tree – Result

Theorem (H.–Heuberger–Kropf–Prodinger)

Average # of branches in a random plane tree of size $n$ is

$$\alpha n + \frac{1}{6} \log_4 n + C + \delta(\log_4 n) + O(n^{-1/4}),$$

- $\alpha = \sum_{k \geq 2} \frac{1}{2^k - 1} \approx 0.60669,$
- $C = -\frac{\gamma + 4\alpha \log 2 + \log 2 + 24\zeta'(-1) + 2}{12 \log 2} \approx -0.11811,$
- $\delta \ldots periodic fluctuation:
  $$\delta(x) := \frac{1}{\log 2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( -1 + \chi_k \right) \Gamma(\chi_k/2) \zeta(-1 + \chi_k) e^{2k\pi ix}, \quad \chi_k = \frac{2\pi ik}{\log 2}. $$
Summary: Reductions on Plane Trees

Leaves

\[ E \sim \frac{n}{r+1} \]
\[ V \sim \frac{r(r+2)}{6(r+1)^2} n \]

limit law: ✓
Summary: Reductions on Plane Trees

**Leaves**

\[ E \sim \frac{n}{r+1} \]

\[ V \sim \frac{r(r+2)}{6(r+1)^2} n \]

Limit law: ✓

**Paths**

\[ E \sim \frac{n}{2^{r+1} - 1} \]

\[ V \sim \frac{2^{r+1}(2^r - 1)}{3(2^{r+1} - 1)^2} n \]

Limit law: ✓
## Summary: Reductions on Plane Trees

### Leaves

- Expected value of leaves: 
  \[ E \sim \frac{n}{r+1} \]
- Variance of leaves: 
  \[ \mathbb{V} \sim \frac{r(r+2)}{6(r+1)^2} n \]
- Limit law: ✓

### Paths

- Expected value of paths: 
  \[ E \sim \frac{n}{2^{r+1} - 1} \]
- Variance of paths: 
  \[ \mathbb{V} \sim \frac{2^{r+1}(2^r - 1)}{3(2^{r+1} - 1)^2} n \]
- Limit law: ✓

### Old leaves

- Expected value of old leaves: 
  \[ E \sim (2 - B_{r-1}(1/4))n \]
- Variance of old leaves: 
  \[ \mathbb{V} = \Theta(n) \]
- Limit law: ✓
Summary: Reductions on Plane Trees

**Leaves**

\[ E \sim \frac{n}{r+1} \]
\[ V \sim \frac{r(r+2)}{6(r+1)^2} n \]

limit law: ✓

**Paths**

\[ E \sim \frac{n}{2^{r+1}-1} \]
\[ V \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n \]

limit law: ✓

**Old leaves**

\[ E \sim (2 - B_{r-1}(1/4)) n \]
\[ V = \Theta(n) \]

limit law: ✓

**Old paths**

\[ E \sim \frac{2n}{r+2} \]
\[ V \sim \frac{2r(r+1)}{3(r+2)^2} n \]

limit law: ✓
Summary: Reductions on Plane Trees

Leaves
\[ E \sim \frac{n}{r+1} \]
\[ V \sim \frac{r(r+2)}{6(r+1)^2} n \]
limit law: \( \checkmark \)

Paths
\[ E \sim \frac{n}{2r+1-1} \]
\[ V \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n \]
limit law: \( \checkmark \)

Old leaves
\[ E \sim (2 - B_{r-1}(1/4))n \]
\[ V = \Theta(n) \]
limit law: \( \checkmark \)

Old paths
\[ E \sim \frac{2n}{r+2} \]
\[ V \sim \frac{2r(r+1)}{3(r+2)^2} n \]
limit law: \( \checkmark \)

Disclaimer
Results are not always that nice!
Catalan–Stanley trees

- Motivation: Stanley’s Catalan interpretation #26
- Rightmost leaves in all branches of root have odd distance
Catalan–Stanley trees

► Motivation: Stanley’s Catalan interpretation #26
► Rightmost leaves in all branches of root have odd distance
► Reduction: remove parent & grandparent (except root) of ■
Catalan–Stanley trees

- **Motivation:** Stanley’s Catalan interpretation #26
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Catalan–Stanley trees

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Catalan–Stanley trees

- Motivation: Stanley’s Catalan interpretation #26
- Rightmost leaves in all branches of root have odd distance
- **Reduction:** remove parent & grandparent (except root) of □
Counterexample: Results

- Reduction with different parameter behavior ✓

<table>
<thead>
<tr>
<th>Age</th>
<th>Size of $r$th Reduction</th>
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</tbody>
</table>

- $E = \Theta(1)$
- $V = \Theta(1)$
- $LLT: \implies \rightarrow \cdots \rightarrow$

$\# \text{ Generations} = \text{Age}$

- $E \sim \frac{1}{4^r}$
- $V \sim \frac{(2^r + 1)(2^r - 1)}{16^r}$

$\rightarrow \text{Reduction size}$
Counterexample: Results

- Reduction with different parameter behavior ✓

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$E = \Theta(1)$</td>
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</tr>
<tr>
<td>$V = \Theta(1)$</td>
<td>$V \sim \frac{(2^r + 1)(2^r - 1)}{16^r n^2}$</td>
</tr>
<tr>
<td>LLT: ✓</td>
<td></td>
</tr>
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</table>

$\# \text{ Generations} = \text{Age}$
Counterexample: Results

- Reduction with different parameter behavior ✓

**Age**

- $E = \Theta(1)$
- $V = \Theta(1)$
- LLT: ✓

**Size of $r$th Reduction**

- $E \sim \frac{1}{4^r} n$
- $V \sim \frac{(2^r + 1)(2^r - 1)}{16^r} n^2$
A Reduction on Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent
A Reduction on Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent
A Reduction on Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent
A Reduction on Binary Trees

Cutting strategy:
- Remove Leaves
- Merge single children with their corresponding parent

![Diagram of a binary tree reduction](image-url)
A Reduction on Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent
A Reduction on Binary Trees

Cutting strategy:
- Remove Leaves
- Merge single children with their corresponding parent
How “old” do the nodes get?

We label the nodes according to the following rules:

- Leaves $\rightarrow$ 0
- $\text{age(left child)} = \text{age(right child)}$ $\rightarrow$ increase by 1
- Otherwise: maximum of children
How “old” do the nodes get?

We label the nodes according to the following rules:

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- \( \text{age(left child)} = \text{age(right child)} \) → increase by 1
- Otherwise: maximum of children
How “old” do the nodes get?

We label the nodes according to the following rules:

- Leaves $\rightarrow 0$
- $\text{age(left child)} = \text{age(right child)} \rightarrow$ increase by 1
- Otherwise: maximum of children

![Diagram of a tree with node ages labeled according to the rules.]

Cutting Down and Growing Trees – Benjamin Hackl
The Register Function

\[ \text{Age} \sim \text{Register function (Horton-Strahler-Index)} \]
The Register Function

Age \sim Register function (Horton-Strahler-Index)

- Applications:
The Register Function

Age $\leadsto \textit{Register function (Horton-Strahler-Index)}$

- Applications:
  - Required stack size for evaluating arithmetic expressions

```
+   
|   |   |
|   |   |
| ÷   X |
| /   /   |
| /   /   |
| /   /   |
| 1   a   b |
| /   /   |
| /   /   |
| a   1 |
```
The Register Function

Age $\sim$ Register function (Horton-Strahler-Index)

Applications:
- Required stack size for evaluating arithmetic expressions
- Branching complexity of river networks (e.g. Danube: 9)
The Register Function

Age $\sim$ Register function (Horton-Strahler-Index)

- Applications:
  - Required stack size for evaluating arithmetic expressions
  - Branching complexity of river networks (e.g. Danube: 9)

Asymptotic analysis:
The Register Function

Age $\sim$ *Register function* (*Horton-Strahler*-Index)

- **Applications:**
  - Required stack size for evaluating arithmetic expressions
  - Branching complexity of river networks (e.g. Danube: 9)

\[
\begin{array}{c}
+ \\
\frac{\div}{\times} \\
\frac{1}{\frac{a}{b}} \\
\frac{a}{1}
\end{array}
\]

- **Asymptotic analysis:**
  - Flajolet, Raoult, Vuillemin (1979)
The Register Function

Age $\sim$ Register function (Horton-Strahler-Index)

- **Applications:**
  - Required stack size for evaluating arithmetic expressions
  - Branching complexity of river networks (e.g. Danube: 9)

- **Asymptotic analysis:**
  - Flajolet, Raoult, Vuillemin (1979)
  - Flajolet, Prodinger (1986)
The Register Function

\[ \text{Age} \sim \text{Register function (Horton-Strahler-Index)} \]

- **Applications:**
  - Required stack size for evaluating arithmetic expressions
  - Branching complexity of river networks (e.g. Danube: 9)

\[
\begin{align*}
+ & \quad \div \quad \times \\
\quad \downarrow & \quad \downarrow & \quad \downarrow \\
\quad \quad \quad & \quad a & \quad b \\
\frac{1}{a} & \quad \frac{1}{b} \\
& \quad a & \quad 1
\end{align*}
\]

- **Asymptotic analysis:**
  - Flajolet, Raoult, Vuillemin (1979)
  - Flajolet, Prodinger (1986)
  - \( r \)-branches, Numerics: Yamamoto, Yamazaki (2009)
Leaf Reduction: More Tree Families

**Core Idea:** \# removed vertices \ldots additive tree parameter

- \( \tau \in \mathcal{T} \) tree; \( \tau_1, \tau_2, \ldots, \tau_k \) branches of \( \tau \)
Leaf Reduction: More Tree Families

Core Idea: \# removed vertices . . . additive tree parameter

- \( \tau \in \mathcal{T} \) tree; \( \tau_1, \tau_2, \ldots, \tau_k \) branches of \( \tau \)
- \( F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau) \),
  - with toll function \( f : \mathcal{T} \to \mathbb{R} \)
Leaf Reduction: More Tree Families

Core Idea: \# removed vertices \ldots additive tree parameter

\[ F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau), \]

\[ with \ toll \ function \ f: \mathcal{T} \rightarrow \mathbb{R} \]
Leaf Reduction: More Tree Families

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Leaf Reduction: More Tree Families

**Core Idea:** \# removed vertices ... *additive tree parameter*

- \( \tau \in \mathcal{T} \) tree; \( \tau_1, \tau_2, \ldots, \tau_k \) branches of \( \tau \)
- \( F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau) \)
  - with *toll function* \( f: \mathcal{T} \to \mathbb{R} \)
Leaf Reduction: More Tree Families

Core Idea: \# removed vertices \ldots additive tree parameter

- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of $\tau$
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
  - with toll function $f: \mathcal{T} \to \mathbb{R}$
Leaf Reduction: More Tree Families

Core Idea: \# removed vertices \ldots additive tree parameter

- \( \tau \in T \) tree; \( \tau_1, \tau_2, \ldots, \tau_k \) branches of \( \tau \)
- \( F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau) \),
  - with toll function \( f: T \to \mathbb{R} \)

- Wagner (2015), Janson (2016), Wagner et al. (2018) \ldots:
  - \( \tau_n \) random tree, size \( n \); \( f \) suitable
    \( \leadsto F(\tau_n) \) asymptotically Gaussian
Leaf Reduction: More Tree Families

Core Idea: \# removed vertices \ldots additive tree parameter

- \( \tau \in \mathcal{T} \) tree; \( \tau_1, \tau_2, \ldots, \tau_k \) branches of \( \tau \)
- \( F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau) \),
  - with toll function \( f : \mathcal{T} \to \mathbb{R} \)

Families suitable for this approach:
- simply generated
- Pólya
- non-crossing
- \ldots (?)

- Wagner (2015), Janson (2016), Wagner et al. (2018)\ldots:
  - \( \tau_n \) random tree, size \( n \); \( f \) suitable
  \( \rightsquigarrow F(\tau_n) \) asymptotically Gaussian