Benjamin Hackl

June 5, 2019

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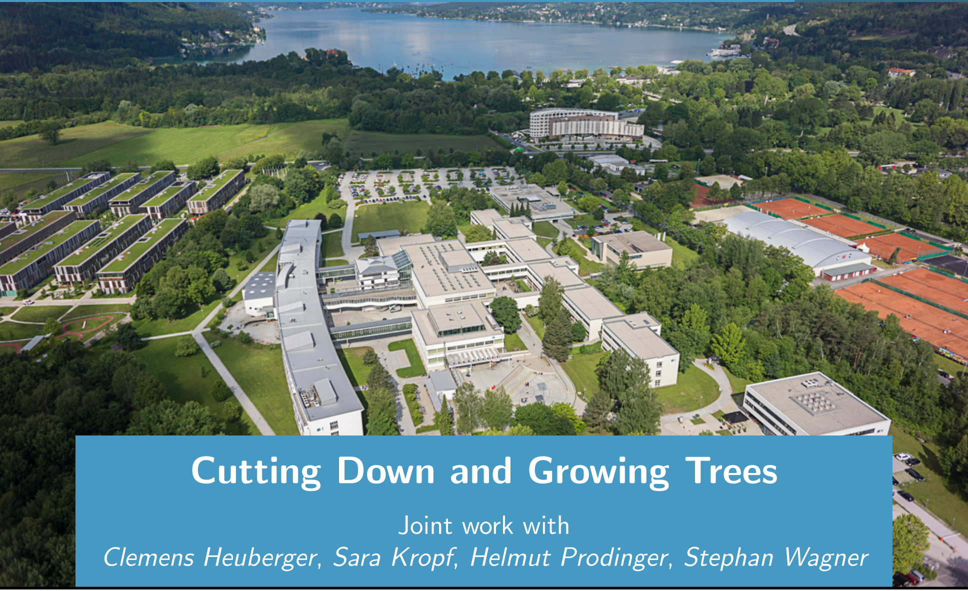


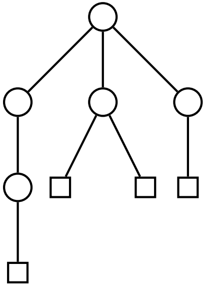
Cutting Down and Growing Trees

Joint work with
Clemens Heuberger, Sara Kropf, Helmut Prodinger, Stephan Wagner

FWF Der Wissenschaftsfonds.

Cutting Down Plane Trees

- Remove all leaves!



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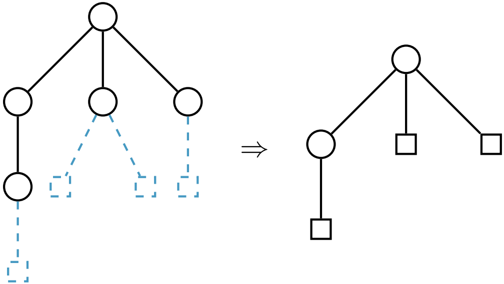

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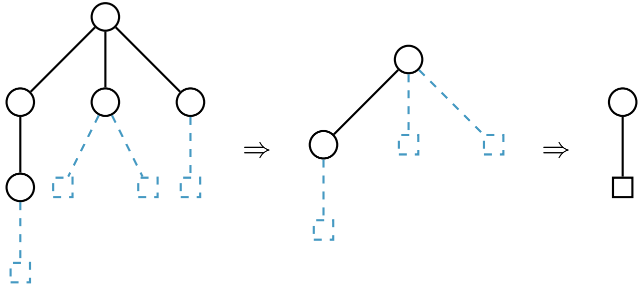

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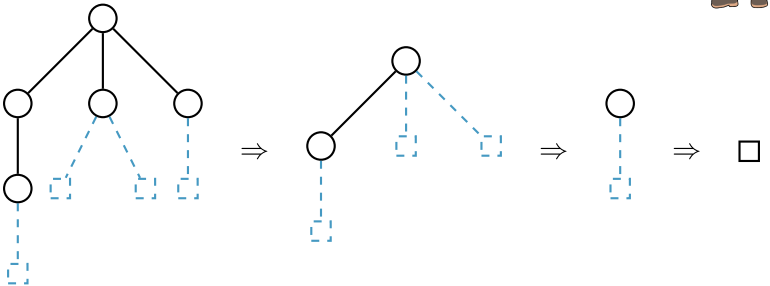

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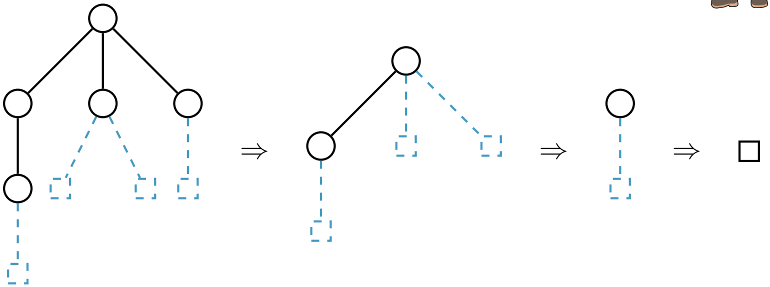

Parameters of Interest:

- Size of rth reduction
- Age: # of possible reductions

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Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, **growing trees**

$\square$

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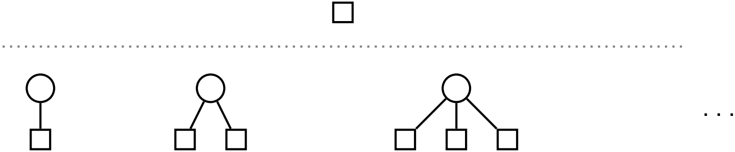

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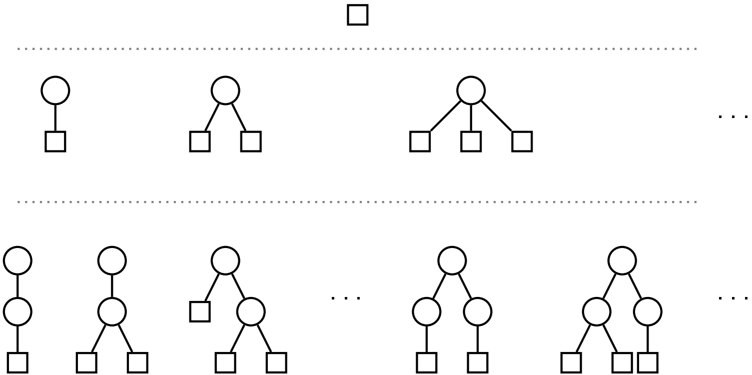

Expansion operators

- $F\ldots$ family of plane trees; bivariate generating function f
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

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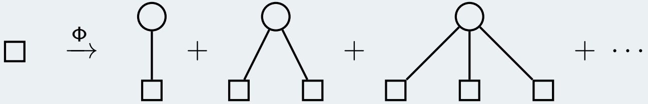
Leaf expansion Φ

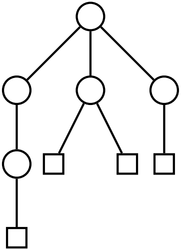
- inverse operation to leaf reduction
  - attach leaves to all current leaves (required)
  - attach leaves to inner nodes (optional)



$$\square \triangleq t,\ \bigcirc \triangleq z \quad \Rightarrow \quad \Phi(t) = zt + zt^2 + zt^3 + \cdots$$

Reductions on Plane Trees

Leaves



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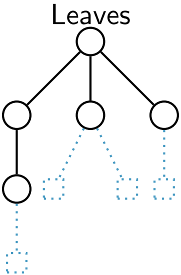

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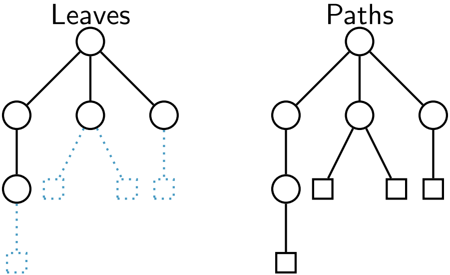

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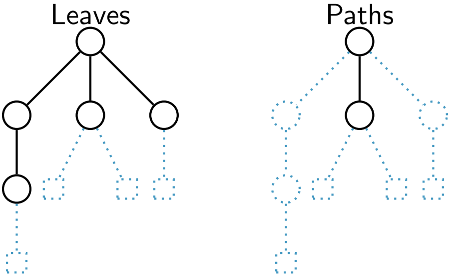

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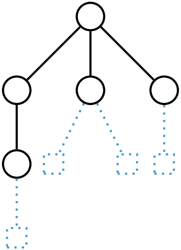
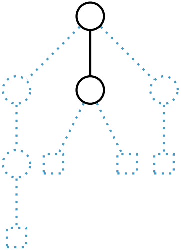
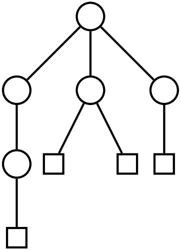


Paths



Old leaves



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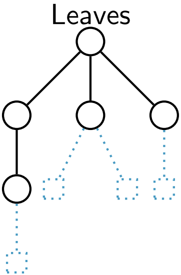

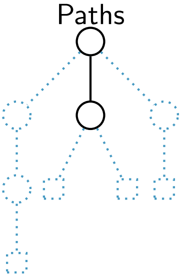

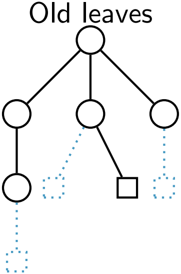

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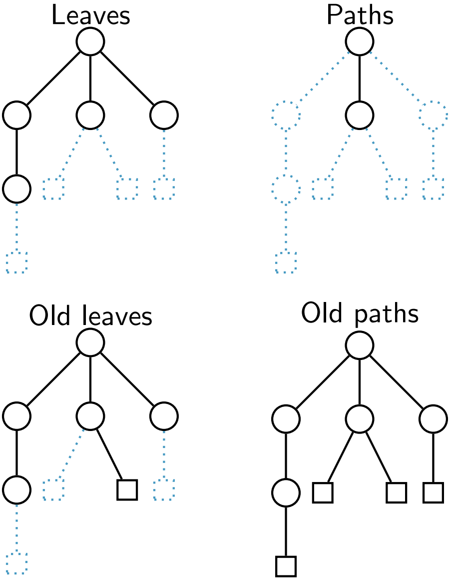

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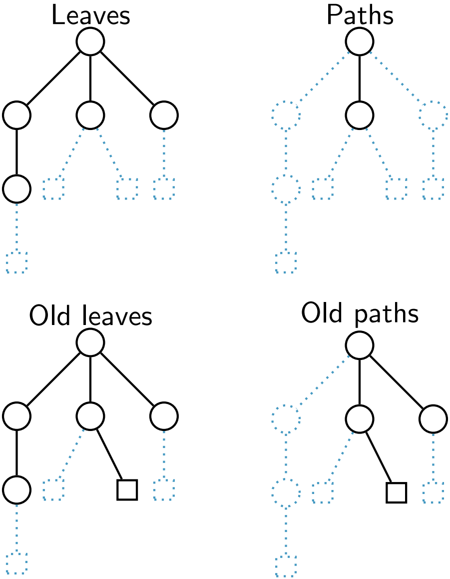

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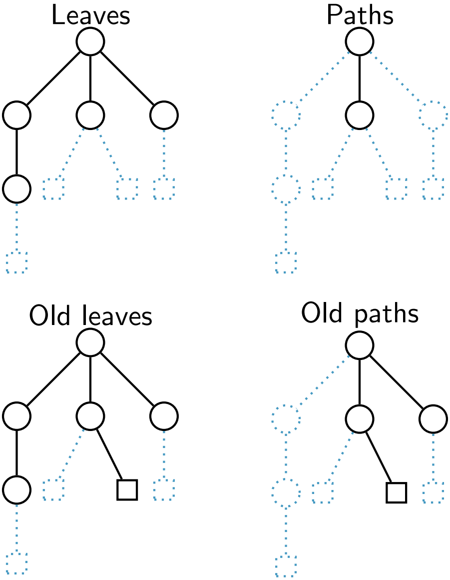

Parameters of Interest:

- tree size after r reductions
- cumulative reduction size

Bivariate Generating Function

Proposition

- $\mathcal{T}\ldots$ *rooted plane trees*

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- $\mathcal{T}$ ... *rooted plane trees*
- $T(z,t)$... *BGF for* $\mathcal{T}$ *(*$z \rightsquigarrow$ *inner nodes,* $t \rightsquigarrow$ *leaves)*

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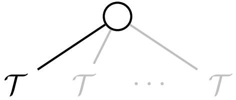
$$\Rightarrow T(z,t) = \frac{1-(z-t)-\sqrt{1-2(z+t)+(z-t)^2}}{2}$$

Proof. Symbolic equation

$\mathcal{T} \quad = \quad \square \quad +$ 

translates into

$$T(z,t) = t + z \cdot \frac{T(z,t)}{1-T(z,t)}$$

which can be solved explicitly.

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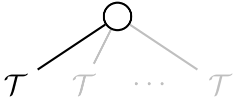

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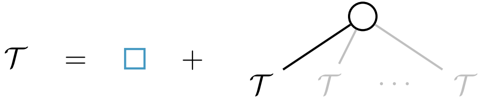

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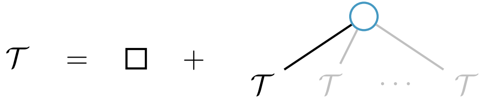

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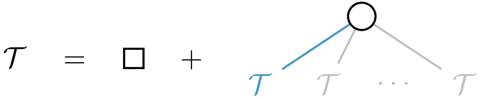

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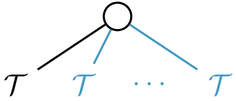
$\mathcal{T} \quad = \quad \square \quad +$ 

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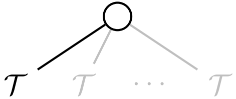

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Leaf expansion operator Φ

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$$\Phi(f(z,t)) = (1-t)f\left(\frac{z}{(1-t)^2}, \frac{zt}{(1-t)^2}\right)$$

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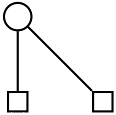
- Tree with n inner nodes and k leaves $\rightsquigarrow z^n t^k$
- **Expansion:**



- In total:

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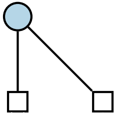

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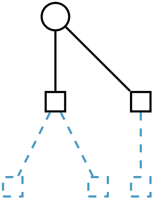

- In total:

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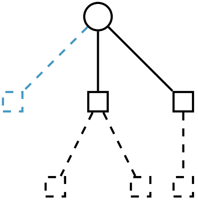
  - there are $2n + k - 1$ positions where sequences of leaves can be inserted



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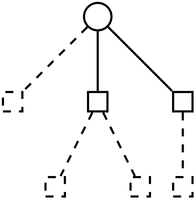

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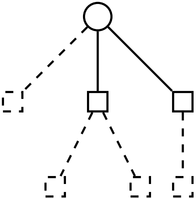

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- As Φ is linear, this proves the proposition.

Properties of Φ

- Functional equation: $T(z,t) = \Phi(T(z,t)) + t$

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- With $z = u/(1+u)^2$ and by some manipulations

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- BGF $G_r(z,v)$ for size comparison: z tracks original size, v size of r-fold reduced tree
- Intuition: v "remembers" size while tree family is expanded

$$G_r(z,v) = \frac{1-u^{r+2}}{(1-u^{r+1})(1+u)} T\Big(\frac{u(1-u^{r+1})^2}{(1-u^{r+2})^2} v, \frac{u^{r+1}(1-u)^2}{(1-u^{r+2})^2} v\Big)$$

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SageMath Demo

Cutting leaves

Theorem (H.–Heuberger–Kropf–Prodinger)

After r reductions of a random tree of size n, the remaining size $X_{n,r}$ has **mean** *and* **variance**

$$\mathbb{E}X_{n,r} = \frac{n}{r+1} - \frac{r(r-1)}{6(r+1)} + O(n^{-1}),$$

$$\mathbb{V}X_{n,r} = \frac{r(r+2)}{6(r+1)^2} n + O(1),$$

and $X_{n,r}$ is **asymptotically normally distributed**.

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Proof insights:

- $\mathbb{E}X_{n,r}$ and $\mathbb{V}X_{n,r}$ follow via singularity analysis
- Asymptotic normality: $n - X_{n,r}$ is a tree parameter with small toll function, limit law by Wagner (2015)

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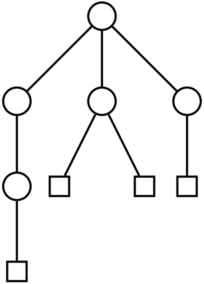
Pruning

- Remove all paths that end in a leaf!



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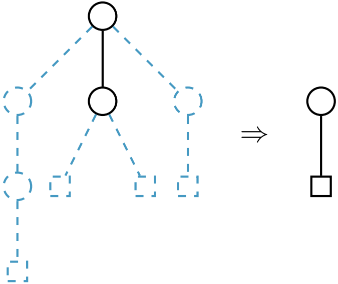
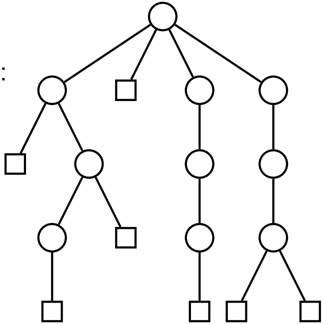


Branches in a Tree

- Trees can be partitioned into branches:



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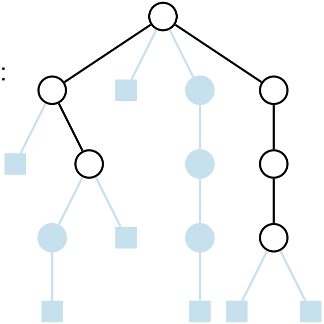

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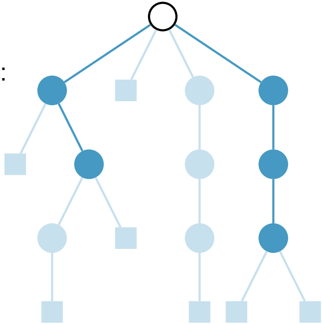

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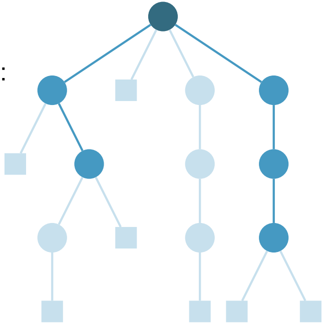

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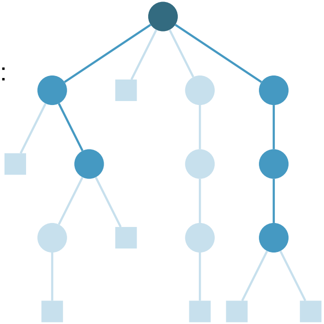

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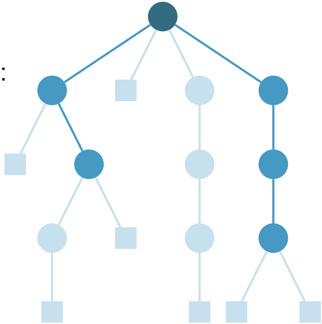

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Total # of branches $\triangleq$ # of leaves in all reduction stages

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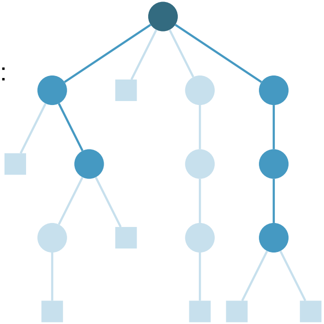

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Proof: all branches end in exactly one leaf (at some point). $\square$

Branches in a Tree – Result

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Theorem (H.–Heuberger–Kropf–Prodinger)

Average # of branches in a random plane tree of size n is

$$\alpha n + \frac{1}{6}\log_4 n + C + \delta(\log_4 n) + O(n^{-1/4}),$$

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- $\delta\ldots$ *periodic fluctuation:*

$$\delta(x) := \frac{1}{\log 2}\sum_{k\in\mathbb{Z}\setminus\{0\}} (-1+\chi_k)\Gamma(\chi_k/2)\zeta(-1+\chi_k)e^{2k\pi ix}, \quad \chi_k = \frac{2\pi ik}{\log 2}.$$

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Summary: Reductions on Plane Trees

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

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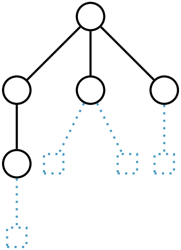
limit law: ✓



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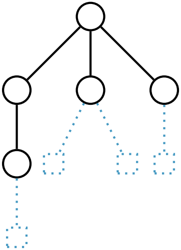

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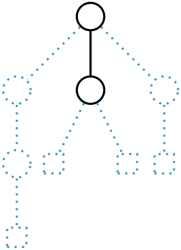

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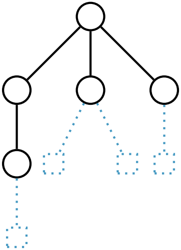

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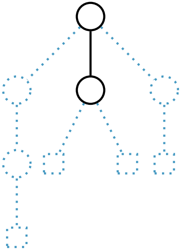

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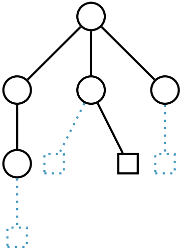

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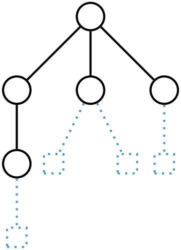

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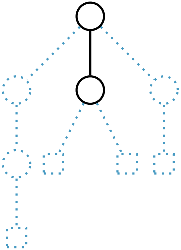

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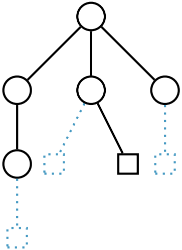

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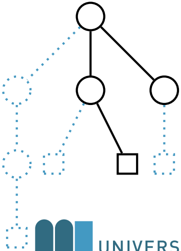

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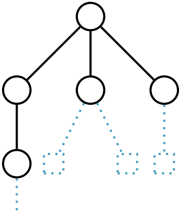

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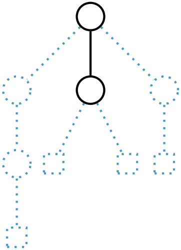

Disclaimer

Results are **not always** that nice!

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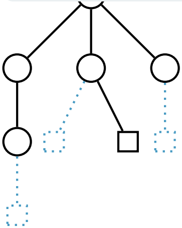

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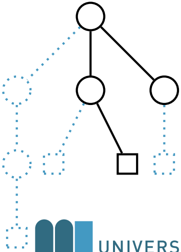

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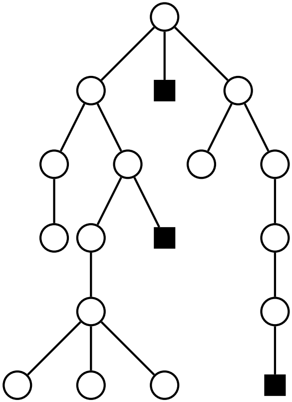
Catalan–Stanley trees

- Motivation: Stanley's Catalan interpretation #26
- Rightmost leaves in all branches of root have odd distance



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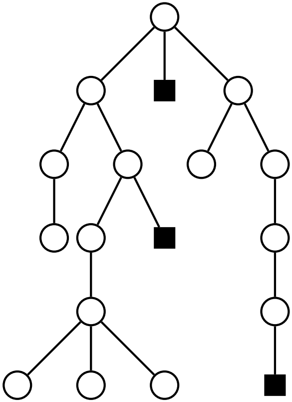

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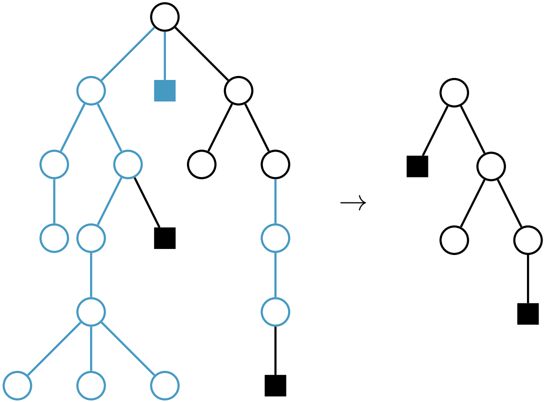

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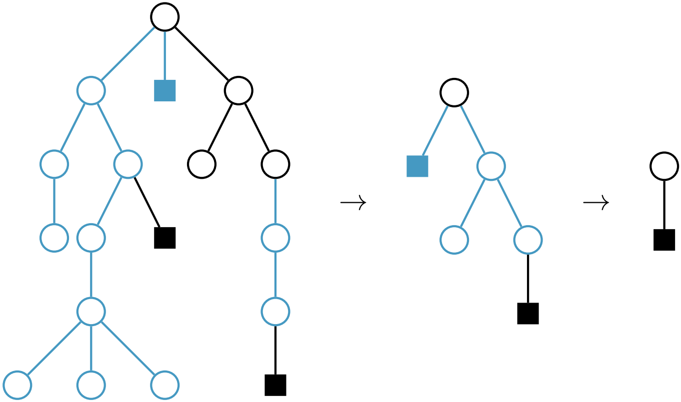

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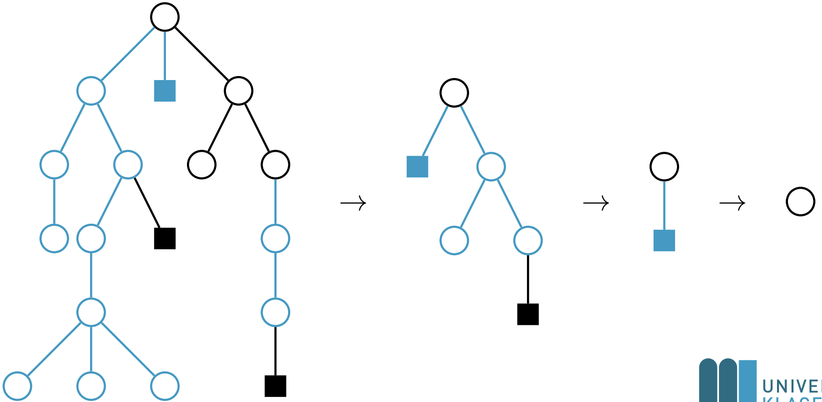

Counterexample: Results

- Reduction with different parameter behavior ✓

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Size of rth Reduction

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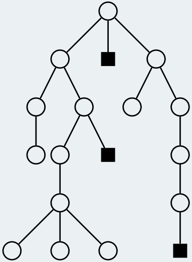
$\bigcirc \mapsto \cdots \mapsto$



$\#$ Generations $=$ Age

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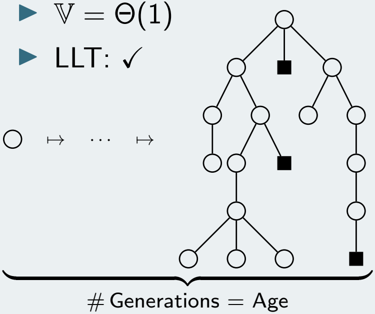

Size of rth Reduction

- $\mathbb{E} \sim \frac{1}{4^r} n$
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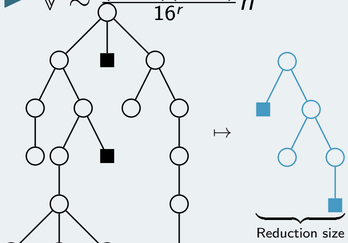

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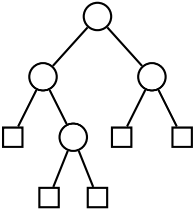
A Reduction on Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent



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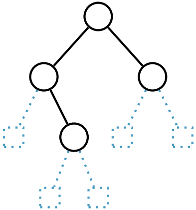

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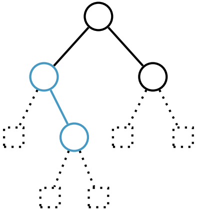

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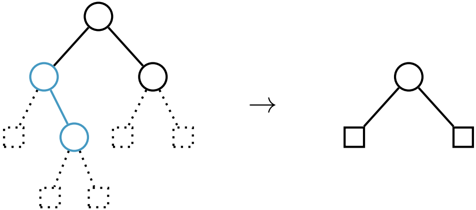

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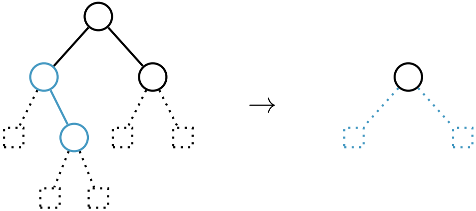

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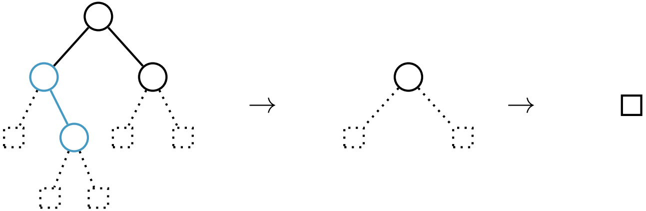

How “old” do the nodes get?

We label the nodes according to the following rules:

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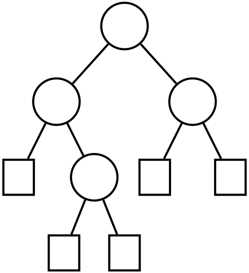

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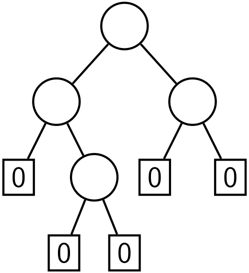

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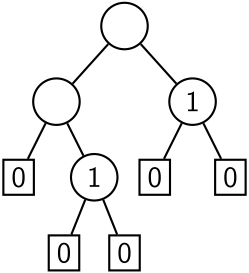

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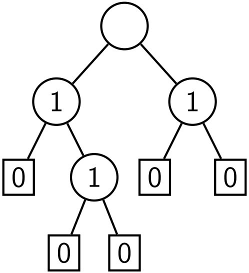

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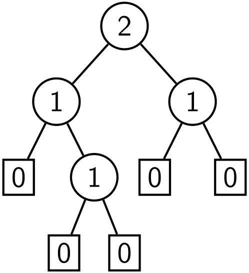

The Register Function

Age $\rightsquigarrow$ *Register function* (*Horton-Strahler*-Index)

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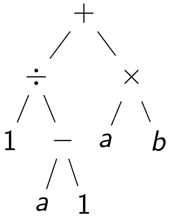
  - Required stack size for evaluating arithmetic expressions



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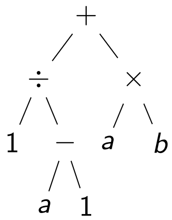

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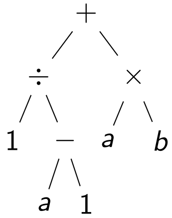

- Asymptotic analysis:

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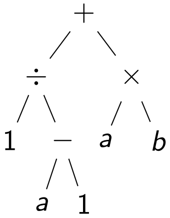

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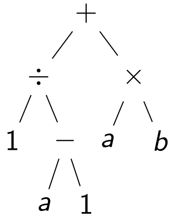

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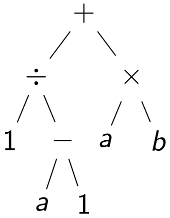

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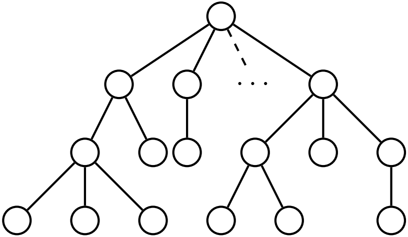
Leaf Reduction: More Tree Families

Core Idea: # removed vertices . . . *additive tree parameter*

- $\tau \in \mathcal{T}$ tree; τ_1, τ_2, . . . , τ_k branches of τ



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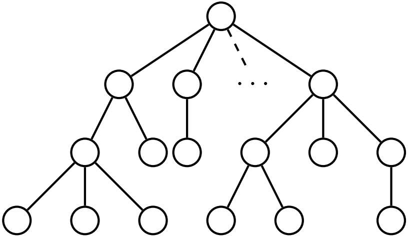
  - with *toll function* $f : \mathcal{T} \to \mathbb{R}$



UNIVERSITÄT KLAGENFURT

Leaf Reduction: More Tree Families

Core Idea: # removed vertices . . . *additive tree parameter*

- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of τ
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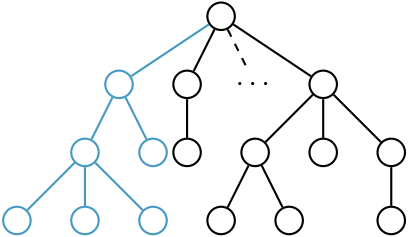

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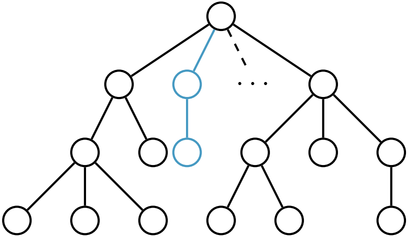

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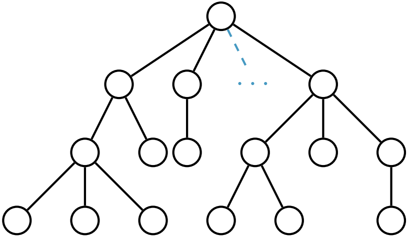

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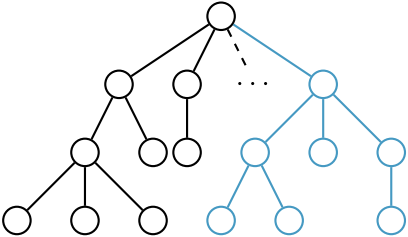

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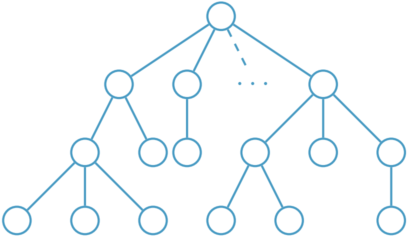

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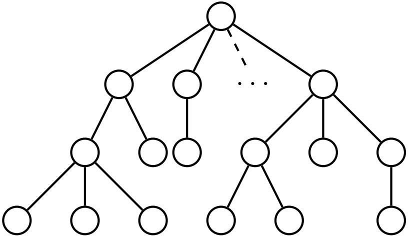

- Wagner (2015), Janson (2016), Wagner et al. (2018). . . :
  - τ_n random tree, size n; f *suitable*
    $\rightsquigarrow F(\tau_n)$ asymptotically Gaussian

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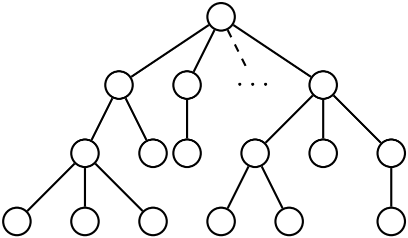

Families suitable for this approach:

- simply generated
- Pólya
- non-crossing
- . . . (?)

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