Benjamin Hackl May 16, 2019 UNIVERSITÄT KLAGENFURT

Cutting Down and Growing Trees

Joint work with

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Der Wissenschaftsfonds.

Related Research

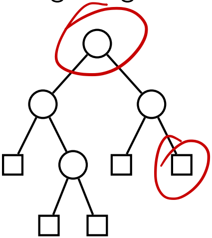
- Random edge removal (Meir–Moon, '70; Panholzer '06; . . .)
  - Choose and remove random edge
  - Keep component with root vertex
  - How long does tree survive?

- Tree percolation (Lyons, '90, . . .):
  - Infinite tree branching process
  - Remove edges (indep.) with probability p
  - How does "root component" look like?

Example: Trimming Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent



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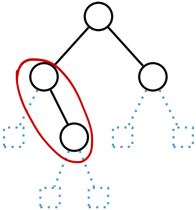

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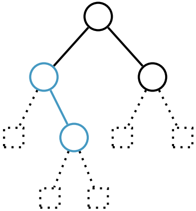

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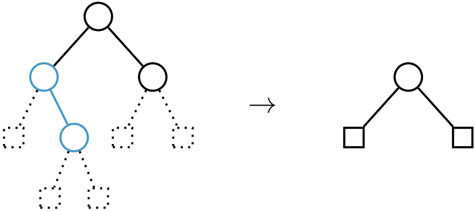

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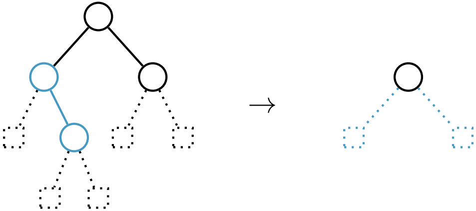

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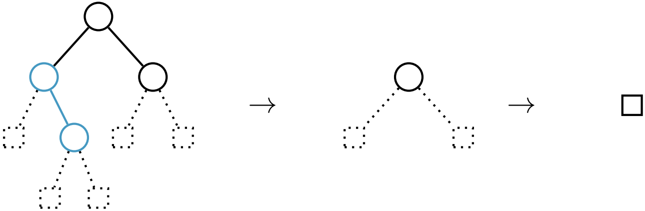
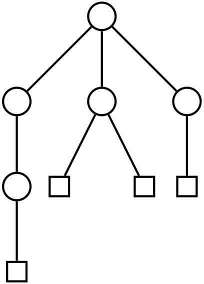


Example: Cutting Down Plane Trees

- Remove all leaves!



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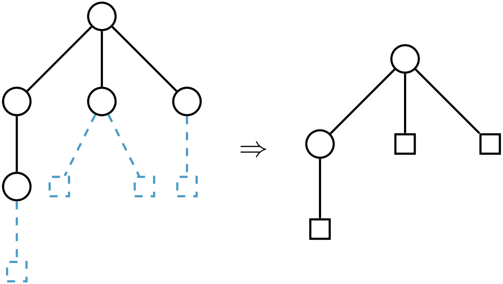

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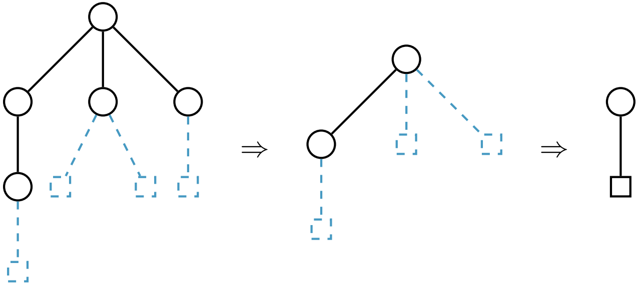

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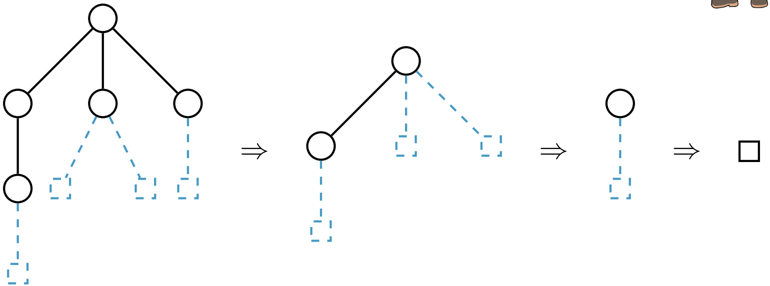

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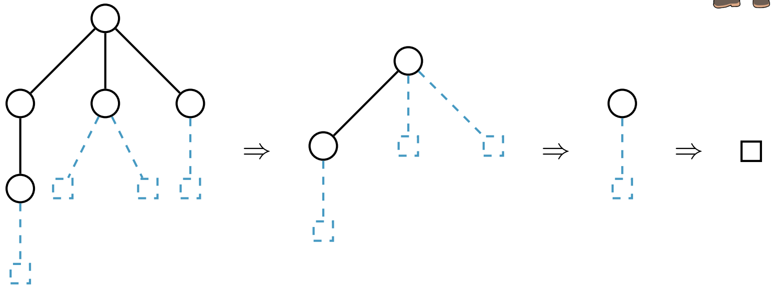

Parameters of Interest:

- Size of rth reduction
- Age: # of possible reductions

Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, **growing trees**

□

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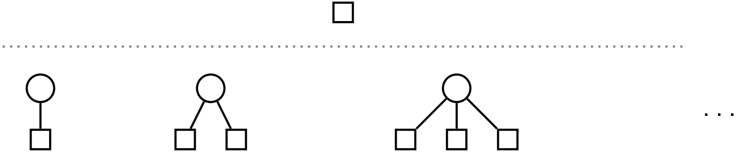

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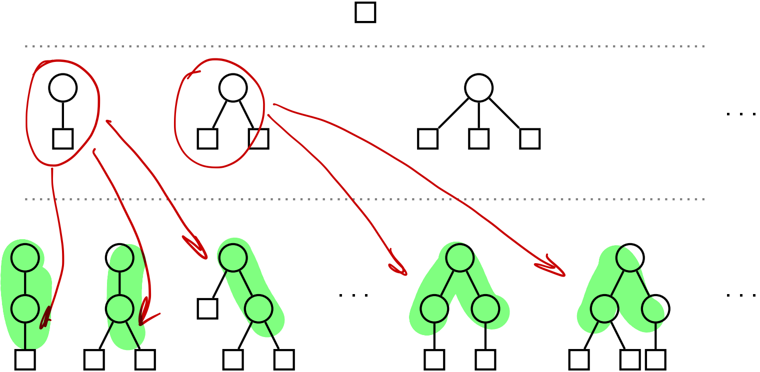

Expansion operators

- $F \ldots$ family of plane trees; bivariate generating function f
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

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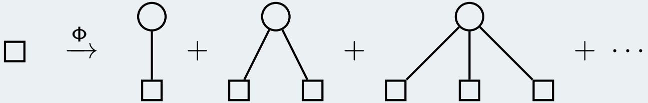
Leaf expansion Φ

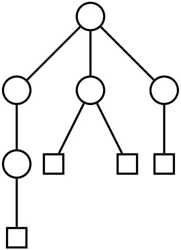
- inverse operation to leaf reduction
  - attach leaves to all current leaves (required)
  - attach leaves to inner nodes (optional)



$$\square \triangleq t,\ \bigcirc \triangleq z \quad \Rightarrow \quad \Phi(t) = zt + zt^2 + zt^3 + \cdots$$

Reductions on Plane Trees

Leaves



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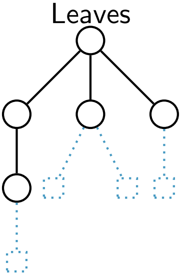

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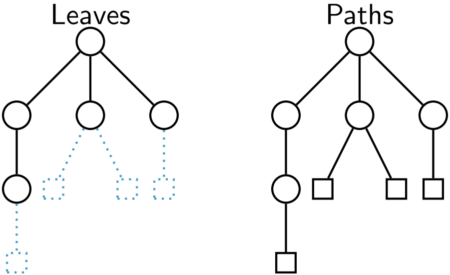

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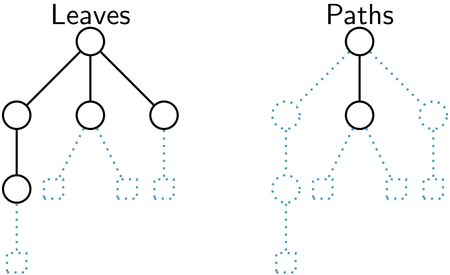

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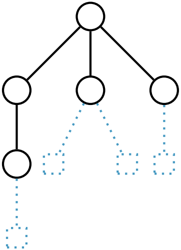
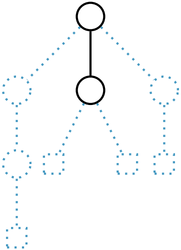
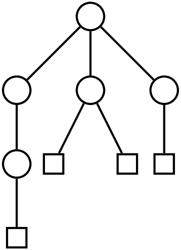


Paths



Old leaves



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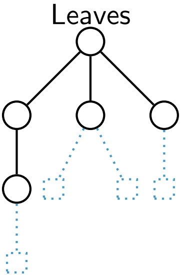

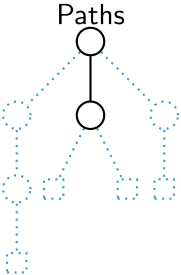

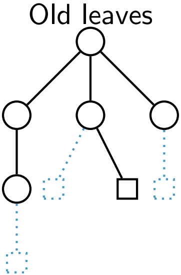

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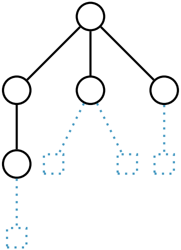

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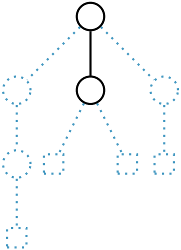

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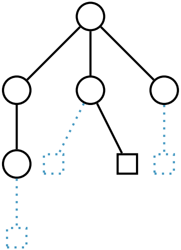

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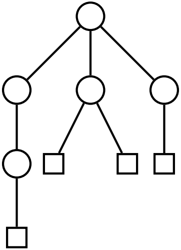

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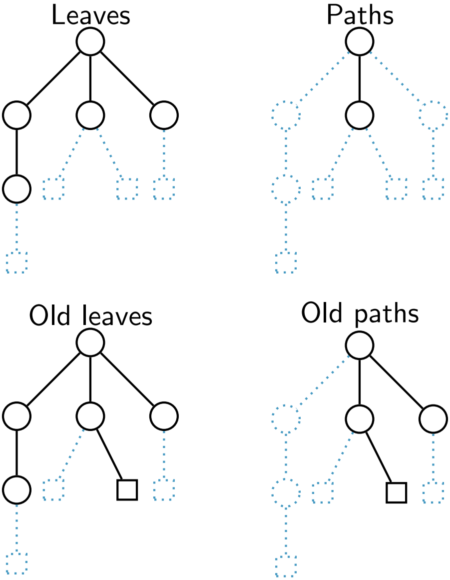

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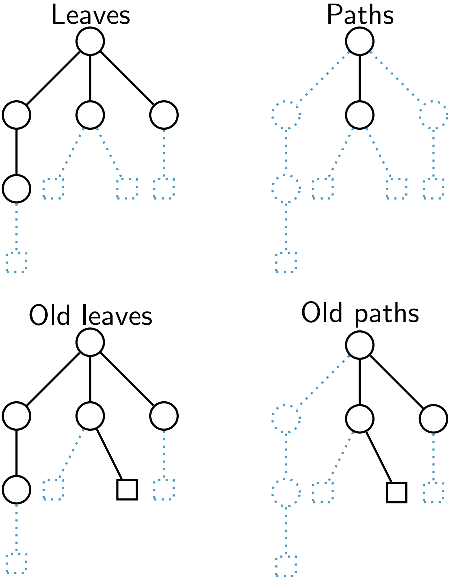

Parameters of Interest:

- tree size after r reductions
- cumulative reduction size

Bivariate Generating Function

Proposition

- $\mathcal{T}$ ... *rooted plane trees*

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- $\mathcal{T} \ldots$ *rooted plane trees*
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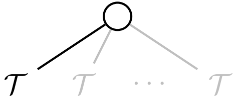
$$\Rightarrow T(z,t) = \frac{1-(z-t)-\sqrt{1-2(z+t)+(z-t)^2}}{2}$$

Proof. Symbolic equation

$$\mathcal{T} \quad = \quad \square \quad +$$



translates into

$$T(z,t) = t + z \cdot \frac{T(z,t)}{1-T(z,t)}$$

which can be solved explicitly.

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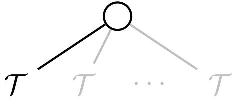

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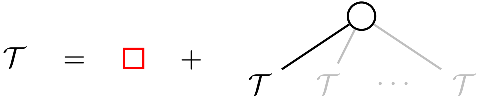

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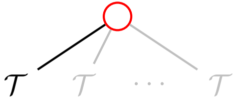

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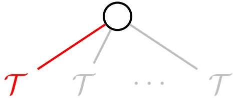

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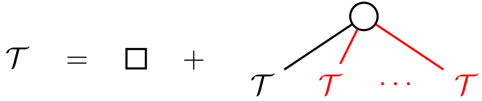

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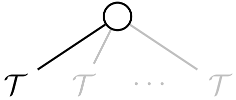

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SageMath Demo

Leaf expansion operator Φ

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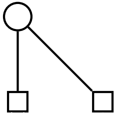

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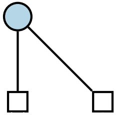

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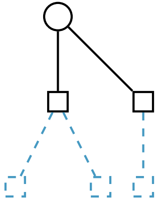

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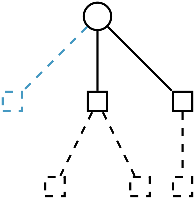

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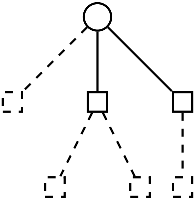

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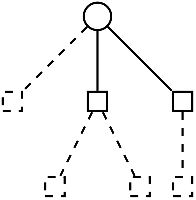

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- As Φ is linear, this proves the proposition.

Properties of Φ

- Functional equation: $T(z,t) = \Phi(T(z,t)) + t$

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- With $z = u/(1+u)^2$ and by some manipulations

$$\Phi^r(T(z,t))|_{t=z} = \frac{1-u^{r+2}}{(1-u^{r+1})(1+u)} T\Big(\frac{u(1-u^{r+1})^2}{(1-u^{r+2})^2}, \frac{u^{r+1}(1-u)^2}{(1-u^{r+2})^2}\Big)$$

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- BGF $G_r(z,v)$ for size comparison: z tracks original size, v size of r-fold reduced tree

Properties of Φ

- Functional equation: $T(z,t) = \Phi(T(z,t)) + t$
- With $z = u/(1+u)^2$ and by some manipulations

$$\Phi^r(T(z,t))|_{t=z} = \frac{1-u^{r+2}}{(1-u^{r+1})(1+u)} T\Big(\frac{u(1-u^{r+1})^2}{(1-u^{r+2})^2}, \frac{u^{r+1}(1-u)^2}{(1-u^{r+2})^2}\Big)$$

- BGF $G_r(z,v)$ for size comparison: z tracks original size, v size of r-fold reduced tree
- Intuition: v "remembers" size while tree family is expanded

$$G_r(z,v) = \frac{1-u^{r+2}}{(1-u^{r+1})(1+u)} T\Big(\frac{u(1-u^{r+1})^2}{(1-u^{r+2})^2} v, \frac{u^{r+1}(1-u)^2}{(1-u^{r+2})^2} v\Big)$$

Cutting leaves

Theorem (H.–Heuberger–Kropf–Prodinger)

After r reductions of a random tree of size n, the remaining size $X_{n,r}$ has **mean** *and* **variance**

$$\mathbb{E}X_{n,r} = \frac{n}{r+1} - \frac{r(r-1)}{6(r+1)} + O(n^{-1}),$$

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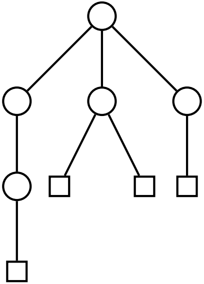
Proof insights:

- $\mathbb{E}X_{n,r}$ and $\mathbb{V}X_{n,r}$ follow via singularity analysis
- Asymptotic normality: $n - X_{n,r}$ is a tree parameter with small toll function, limit law by Wagner (2015)

Pruning

- Remove all paths that end in a leaf!



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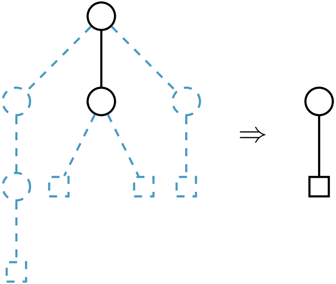
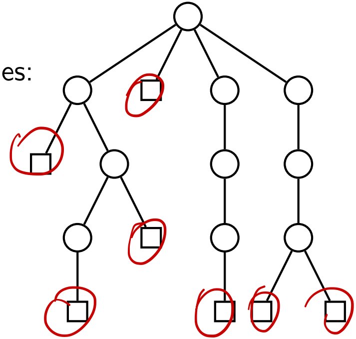


Branches in a Tree

- Trees can be partitioned into branches:



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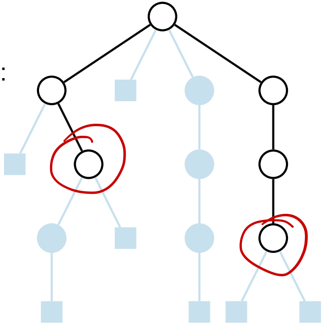

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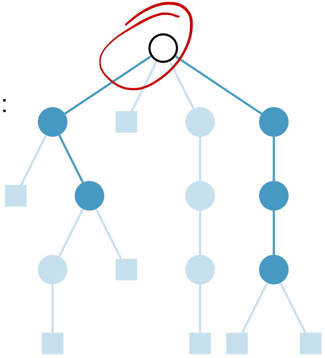

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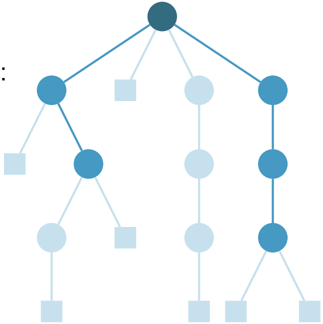

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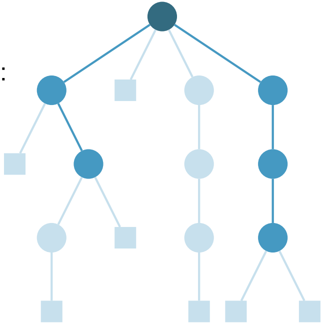

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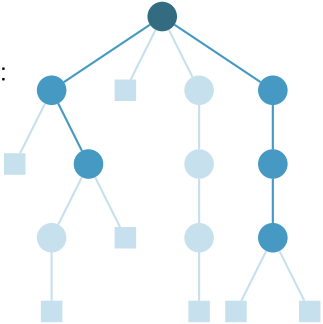

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Total # of branches $\triangleq$ # of leaves in all reduction stages

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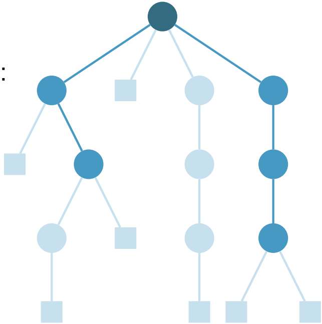

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Total # of branches $\triangleq$ # of leaves in all reduction stages

Proof: all branches end in exactly one leaf (at some point). □

Branches in a Tree – Result

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Theorem (H.–Heuberger–Kropf–Prodinger)

Average # of branches in a random plane tree of size n is

$$\alpha n + \frac{1}{6}\log_4 n + C + \delta(\log_4 n) + O(n^{-1/4}),$$

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- $\delta \ldots$ *periodic fluctuation:*

$$\delta(x) := \frac{1}{\log 2}\sum_{k\in\mathbb{Z}\setminus\{0\}} (-1+\chi_k)\Gamma(\chi_k/2)\zeta(-1+\chi_k)e^{2k\pi ix}, \quad \chi_k = \frac{2\pi ik}{\log 2}.$$

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Summary: Reductions on Plane Trees

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

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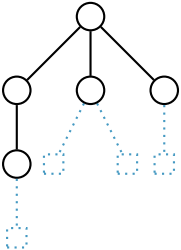
limit law: ✓



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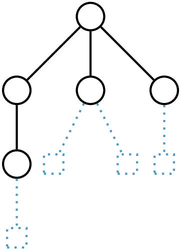

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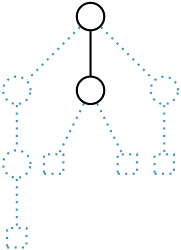

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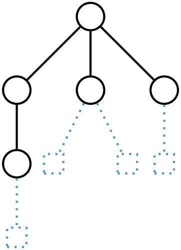

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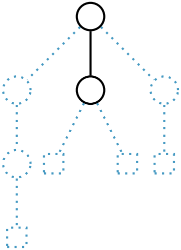

Old leaves

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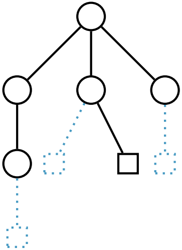

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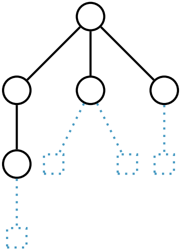

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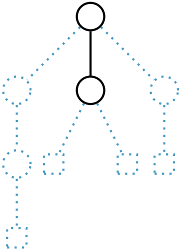

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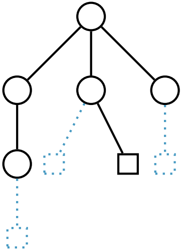

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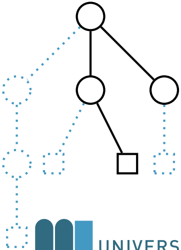

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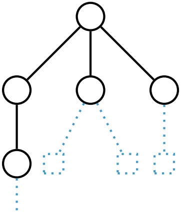

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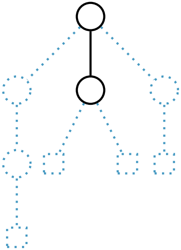

Disclaimer

Results are **not always** that nice!

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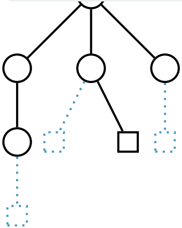

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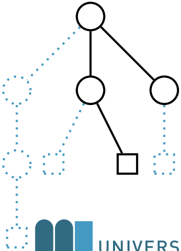

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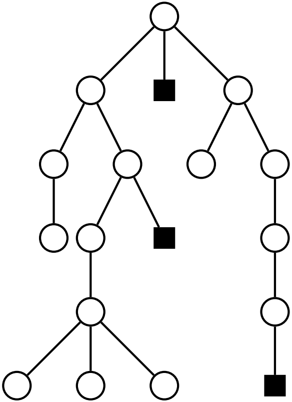
Counterexample: Catalan–Stanley trees

- Motivation: Stanley's Catalan interpretation #26
- Rightmost leaves in all branches of root have odd distance



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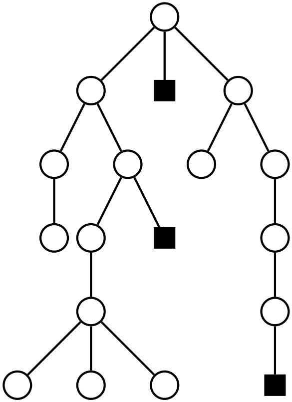

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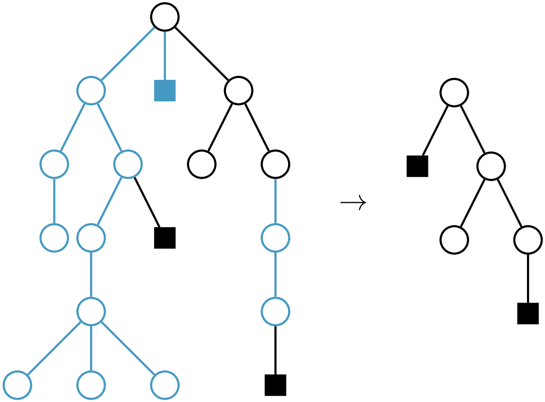

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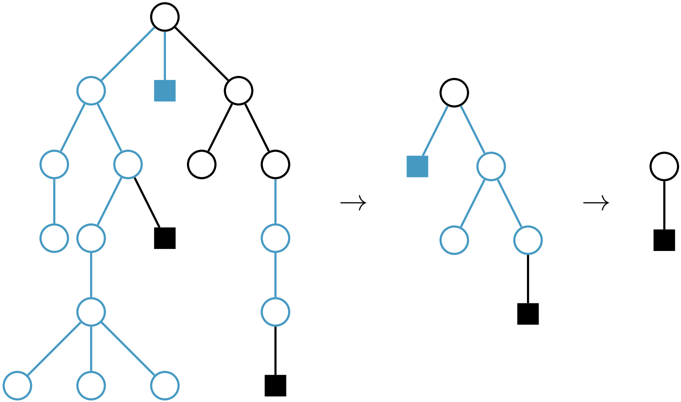

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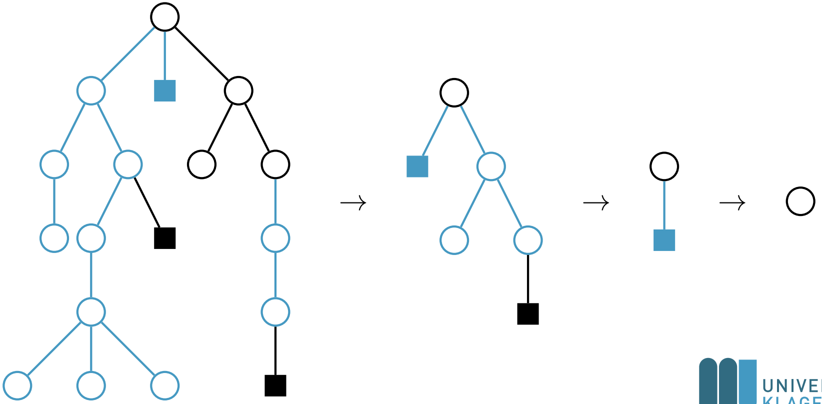

Counterexample: Results

- Reduction with different parameter behavior ✓

Age

Size of rth Reduction

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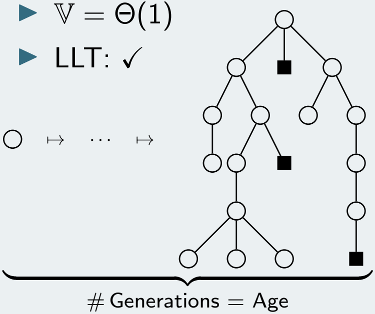

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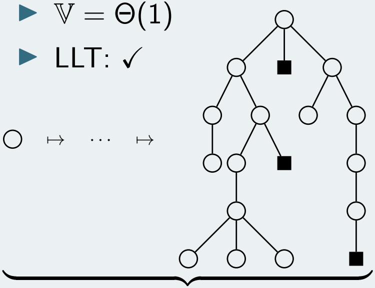
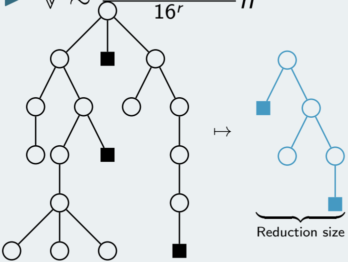


$\#$ Generations $=$ Age

Size of rth Reduction

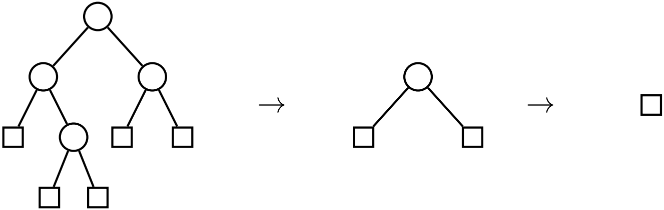
- $\mathbb{E} \sim \frac{1}{4^r} n$
- $\mathbb{V} \sim \frac{(2^r+1)(2^r-1)}{16^r} n^2$



Trimming Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent



Bonus: Touchard's Identity

Proposition

$$B(z) = 1 + \frac{z}{1-2z} B\Big(\frac{z^2}{(1-2z)^2}\Big)$$

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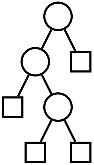

- Binary trees consist of...
  - ...just a leaf,
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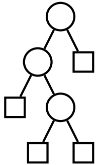

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Corollary (Touchard's Identity)

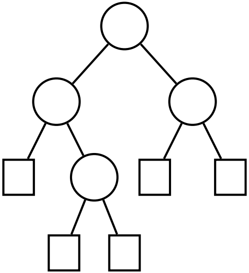
The Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ satisfy

$$C_{n+1} = \sum_{0 \le k \le n/2} C_k 2^{n-2k} \binom{n}{2k}.$$

How “old” do the nodes get?

We label the nodes according to the following rules:

- Leaves $\rightarrow$ 0
- age(left child) $=$ age(right child) $\rightarrow$ increase by 1
- Otherwise: maximum of children



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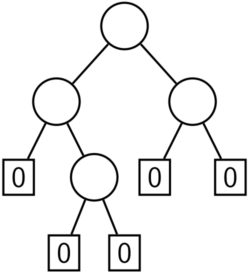

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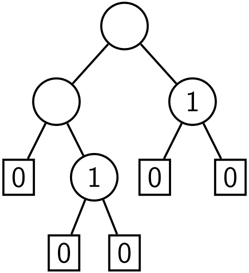

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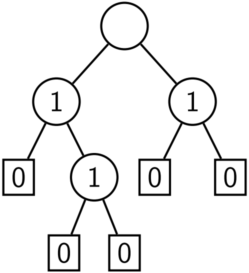

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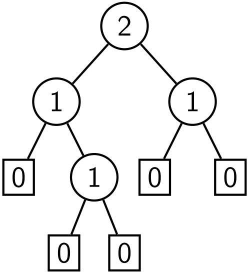

The Register Function

Age $\rightsquigarrow$ *Register function* (*Horton-Strahler*-Index)

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- Applications:

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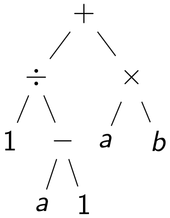

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- Applications:
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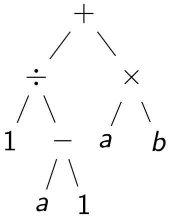
    - Branching complexity of river networks (e.g. Danube: 9)



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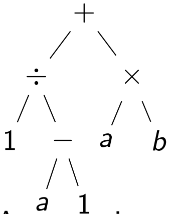
  - Required stack size for evaluating arithmetic expressions
  - Branching complexity of river networks (e.g. Danube: 9)



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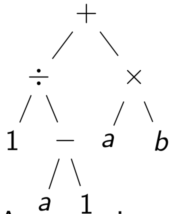

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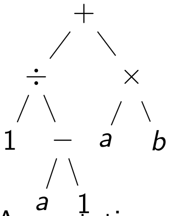

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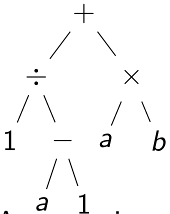

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  - r-branches, Numerics: Yamamoto, Yamazaki (2009)

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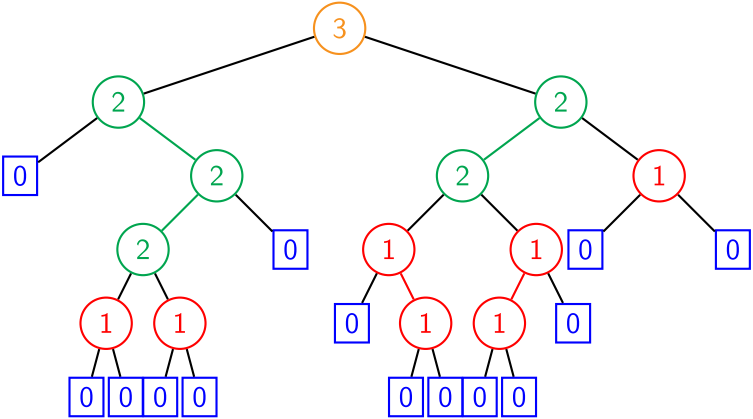
Local Structures – "*r*-branches"



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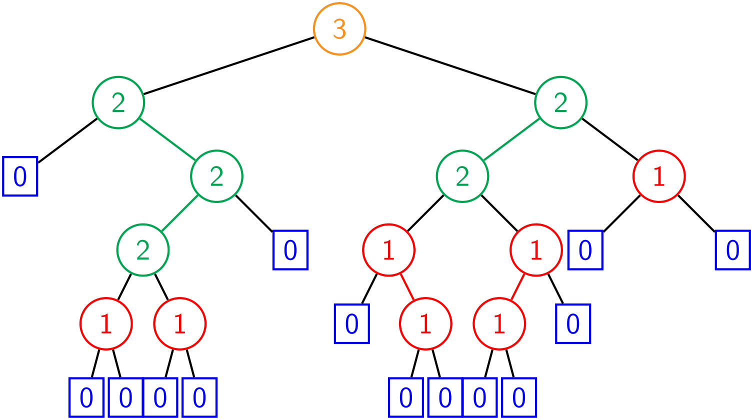

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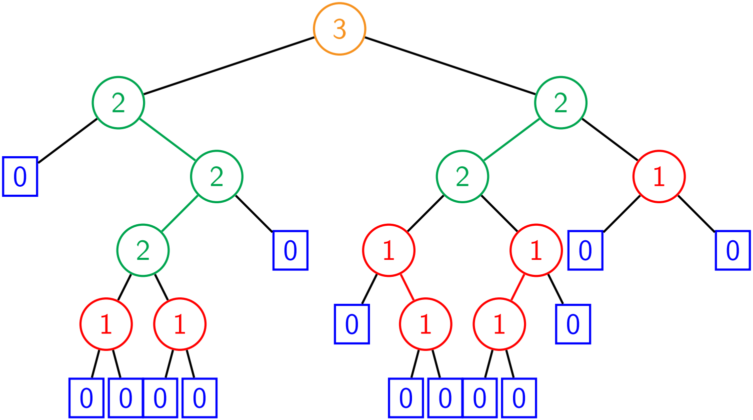

- Number / Distribution of (r-)branches?
- Example:

r	0	1	2	3
# r-branches	14	5	2	1

“r-branches” – Results

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