Benjamin Hackl

February 27, 2019

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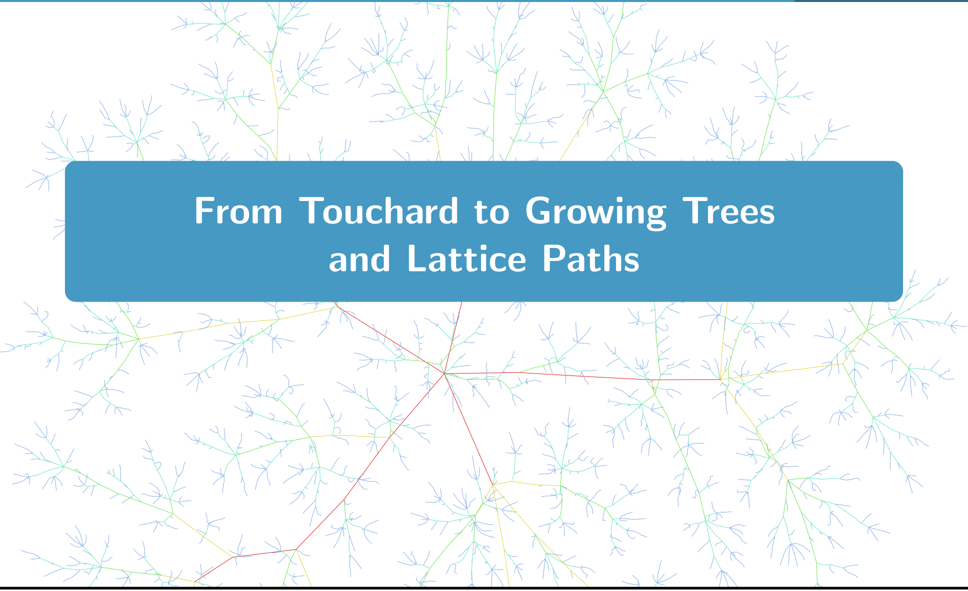


# From Touchard to Growing Trees and Lattice Paths

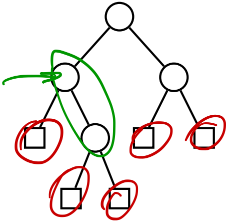


FWF
Der Wissenschaftsfonds.

# Example: Trimming Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent



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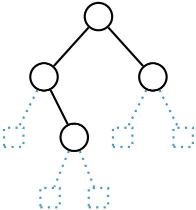

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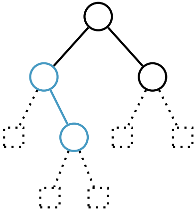

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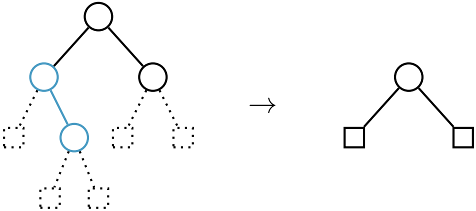

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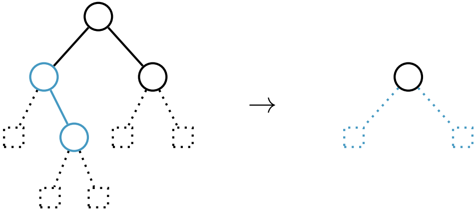

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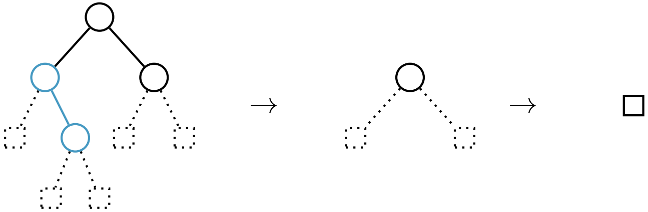

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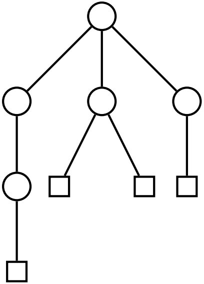
- Remove all leaves!



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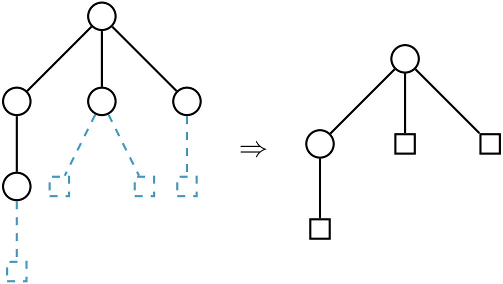

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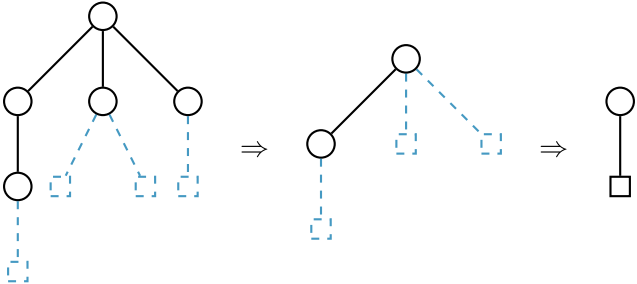

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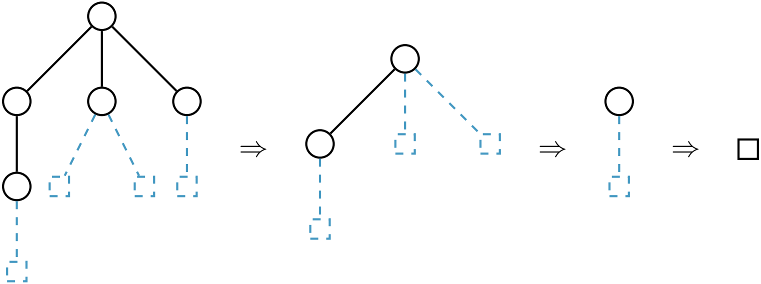

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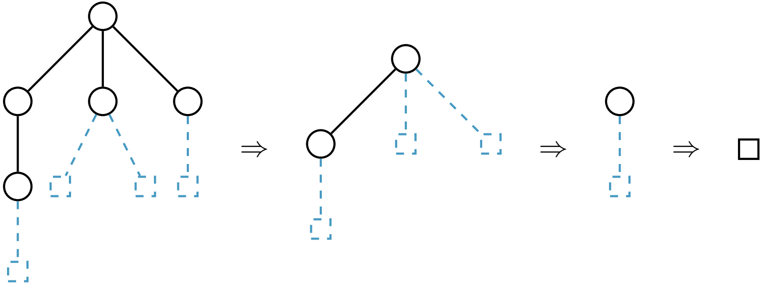

**Parameters of Interest:**

- Size of $r$th reduction
- Age: # of possible reductions

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# Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, **growing trees**

$\square$

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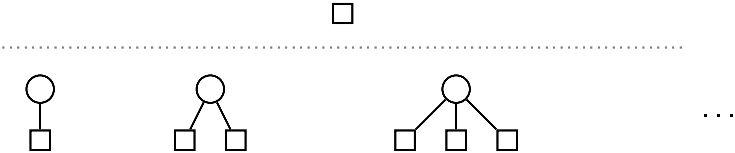

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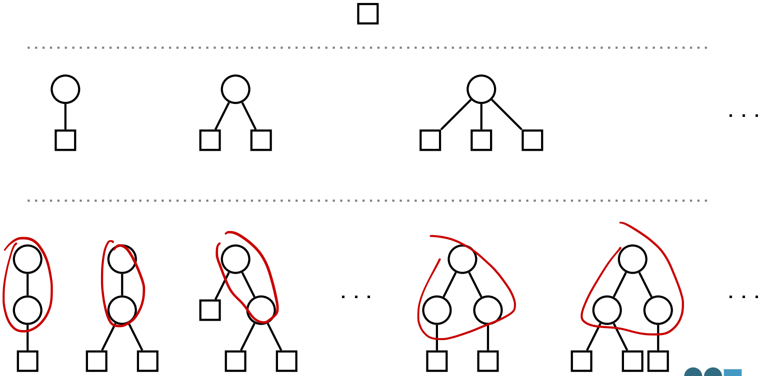

# Expansion operators

- $F$... family of plane trees; bivariate generating function $f$
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

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- $F$ . . . family of plane trees; bivariate generating function $f$
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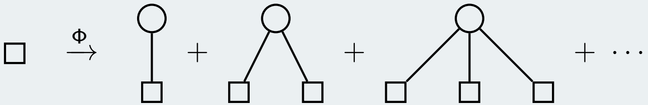
## Leaf expansion $\Phi$

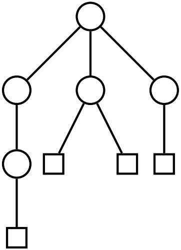
- inverse operation to leaf reduction
  - attach leaves to all current leaves (required)
  - attach leaves to inner nodes (optional)



$$\square \triangleq t, \ \bigcirc \triangleq z \quad \Rightarrow \quad \Phi(t) = zt + zt^2 + zt^3 + \cdots$$

# Reductions on Plane Trees

Leaves



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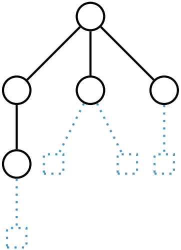

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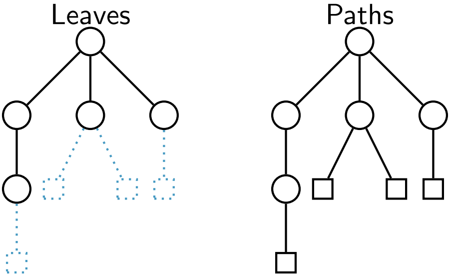

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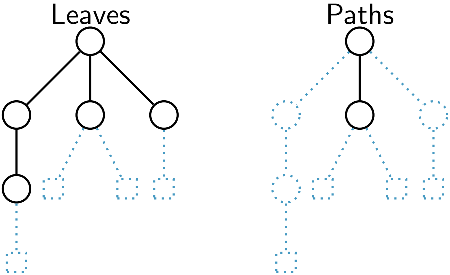

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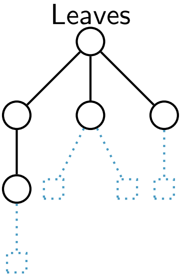


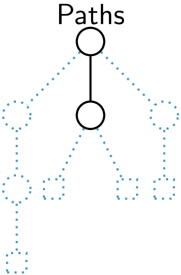


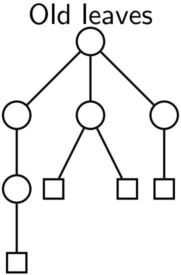

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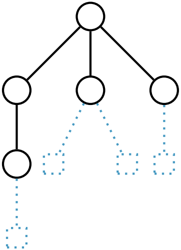
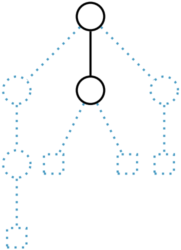
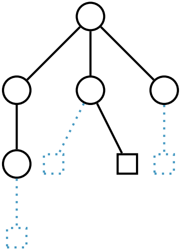


Paths



Old leaves



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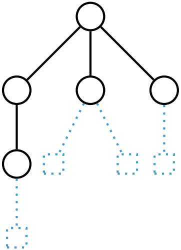

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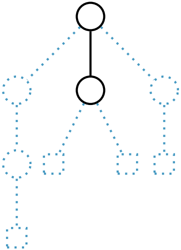

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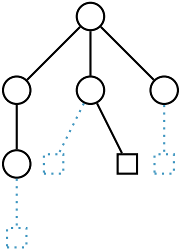

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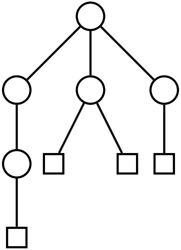

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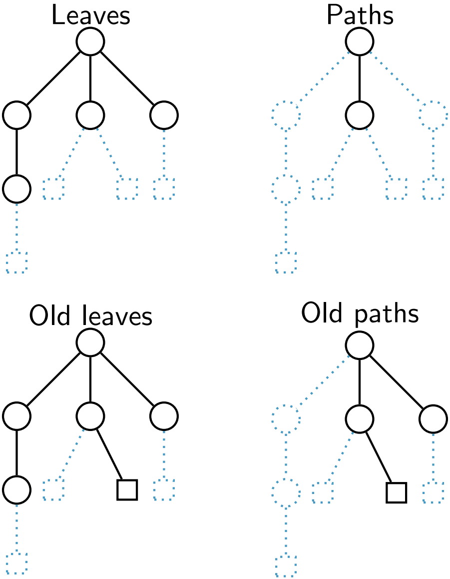

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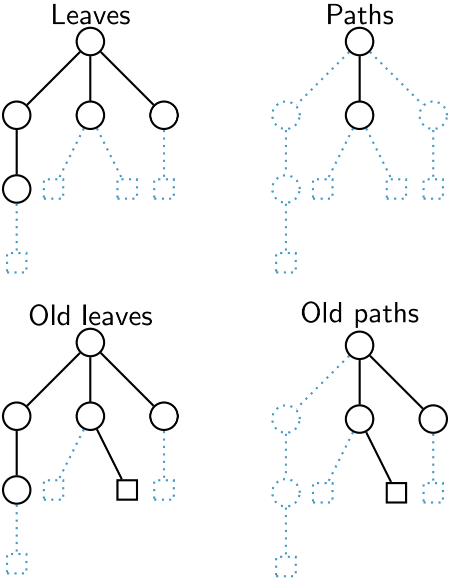


**Parameters of Interest:**

- tree size after $r$ reductions
- cumulative reduction size

# Bivariate Generating Function

## Proposition

- *$\mathcal{T}$ . . . rooted plane trees*

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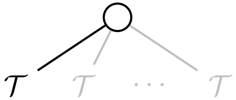
$$\Rightarrow T(z,t) = \frac{1-(z-t)-\sqrt{1-2(z+t)+(z-t)^2}}{2}$$

**Proof.** Symbolic equation

$$\mathcal{T} \quad = \quad \square \quad +$$



translates into

$$T(z,t) = t + z \cdot \frac{T(z,t)}{1-T(z,t)}$$

which can be solved explicitly.

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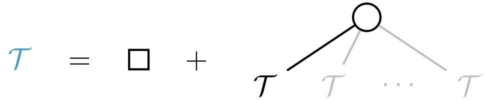

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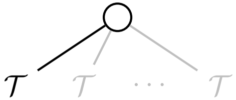


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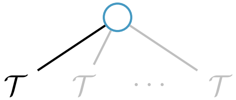


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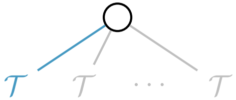


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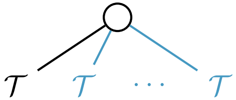


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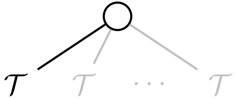


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# Leaf expansion operator $\Phi$

**Proposition**

$$\Phi(f(z,t)) = (1-t)f\left(\frac{z}{(1-t)^2}, \frac{zt}{(1-t)^2}\right)$$

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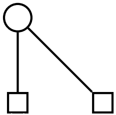

- In total:

$\Phi(z^n t^k) =$

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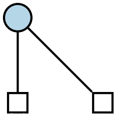

- In total:

$$\Phi(z^n t^k) = z^n \cdot$$

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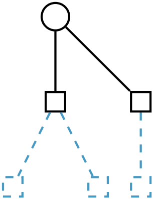

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$$\Phi(z^n t^k) = z^n \cdot \left(\frac{zt}{1-t}\right)^k.$$

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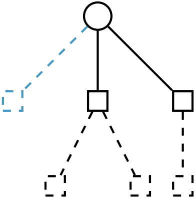

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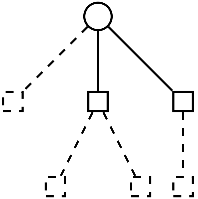

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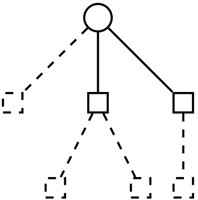

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- As $\Phi$ is linear, this proves the proposition.

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- Functional equation: $T(z,t) = \Phi(T(z,t)) + t$

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$$\Phi^r(T(z,t))|_{t=z} = \frac{1-u^{r+2}}{(1-u^{r+1})(1+u)} T\Big(\frac{u(1-u^{r+1})^2}{(1-u^{r+2})^2}, \frac{u^{r+1}(1-u)^2}{(1-u^{r+2})^2}\Big)$$

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- Intuition: $v$ “remembers” size while tree family is expanded

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# Cutting leaves

## Theorem (H.–Heuberger–Kropf–Prodinger)

*After $r$ reductions of a random tree of size $n$, the remaining size $X_{n,r}$ has* **mean** *and* **variance**

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**Proof insights:**

- $\mathbb{E}X_{n,r}$ and $\mathbb{V}X_{n,r}$ follow via singularity analysis
- Asymptotic normality: $n - X_{n,r}$ is a tree parameter with small toll function, limit law by Wagner (2015)

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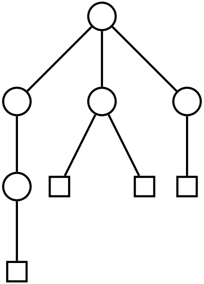
# Pruning

- Remove all paths that end in a leaf!



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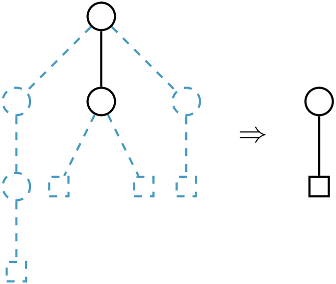

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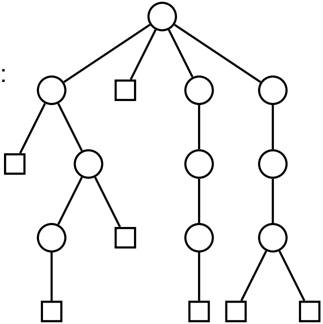

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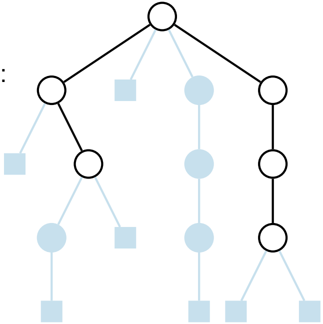

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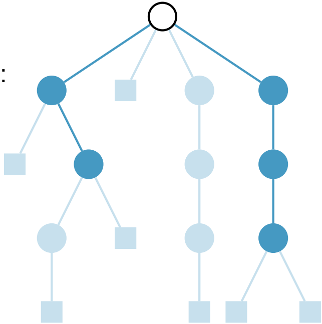

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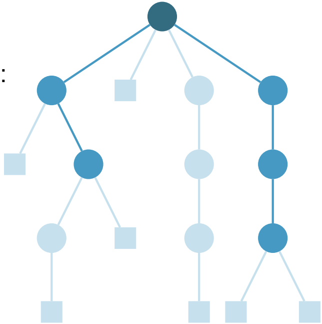

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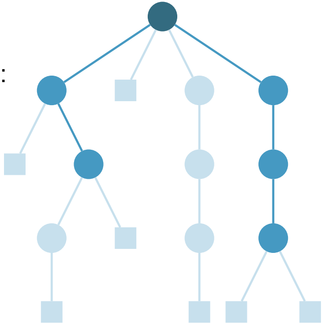

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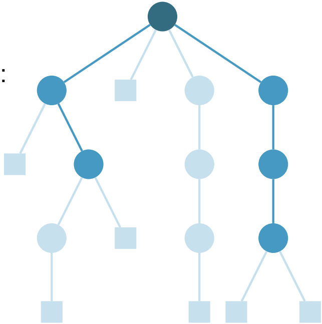

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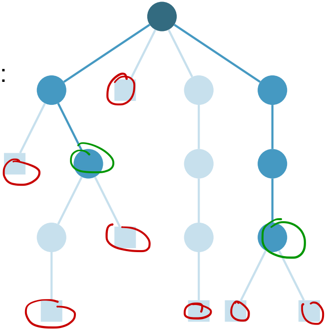

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Total # of branches $\triangleq$ # of leaves in all reduction stages

**Proof:** all branches end in exactly one leaf (at some point). $\square$

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**Theorem (H.–Heuberger–Kropf–Prodinger)**

*Average # of branches in a random plane tree of size n is*

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- $\delta$... *periodic fluctuation:*

$$\delta(x) := \frac{1}{\log 2}\sum_{k\in\mathbb{Z}\setminus\{0\}} (-1+\chi_k)\Gamma(\chi_k/2)\zeta(-1+\chi_k)e^{2k\pi i x}, \quad \chi_k = \frac{2\pi i k}{\log 2}.$$

# Summary: Reductions on Plane Trees

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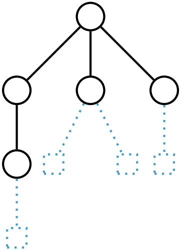
limit law: ✓



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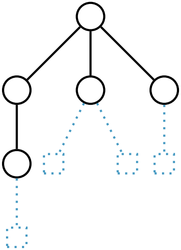

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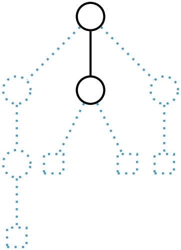

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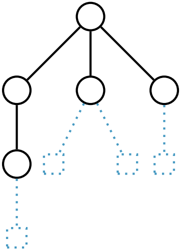

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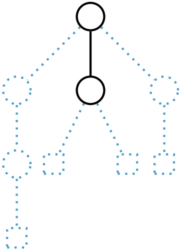

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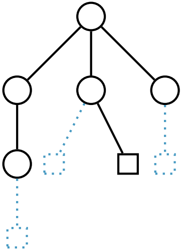

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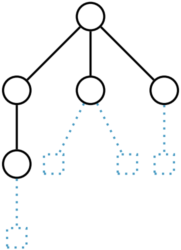

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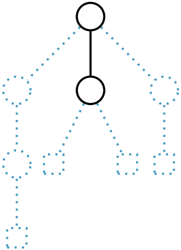

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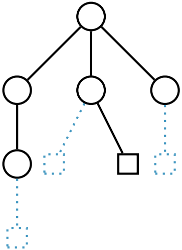

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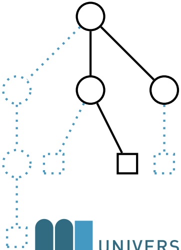

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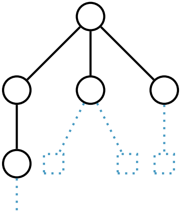

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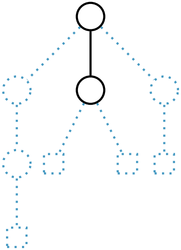

**Disclaimer**

Results are **not always** that nice!

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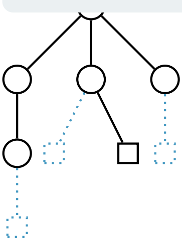

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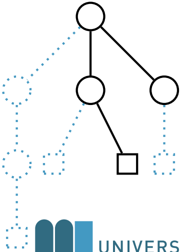

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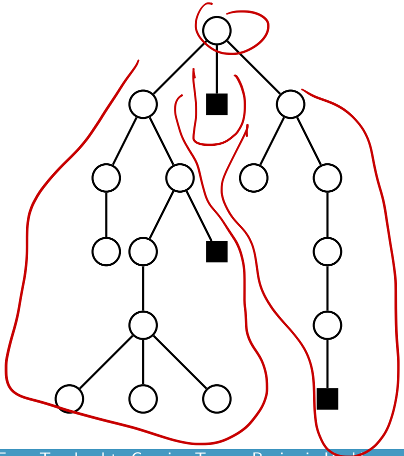
# Counterexample: Catalan–Stanley trees

- Motivation: Stanley's Catalan interpretation #26
- Rightmost leaves in all branches of root have odd distance



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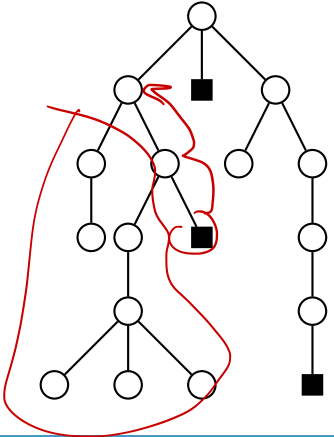

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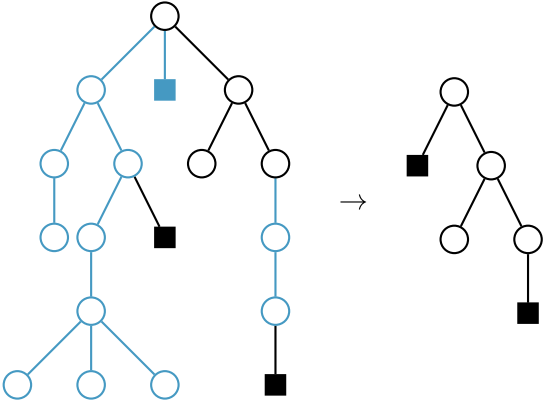

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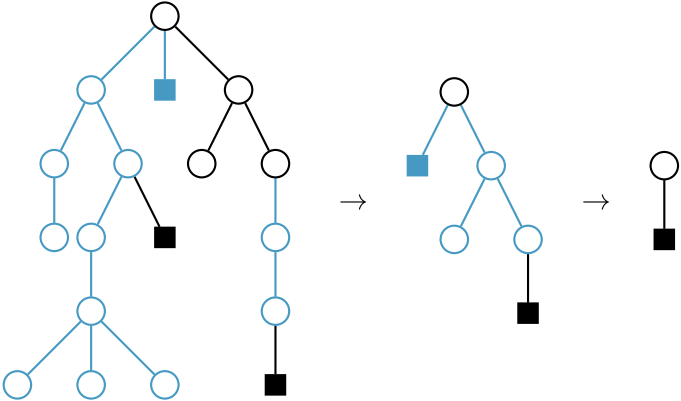

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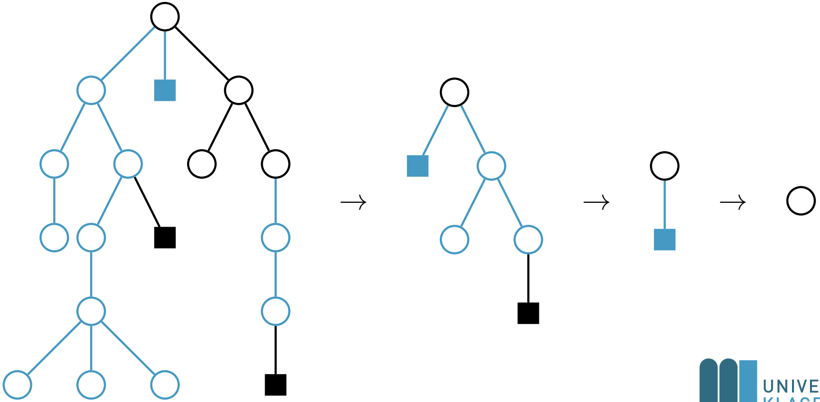

# Counterexample: Results

- Reduction with different parameter behavior ✓

## Age

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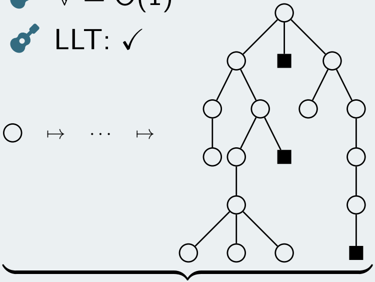

# Generations = Age

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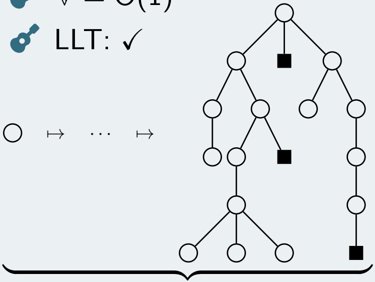

# Generations = Age

## Size of $r$th Reduction

- $\mathbb{E} \sim \frac{1}{4^r} n$
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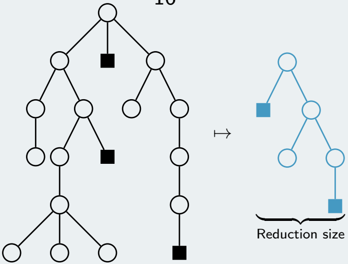
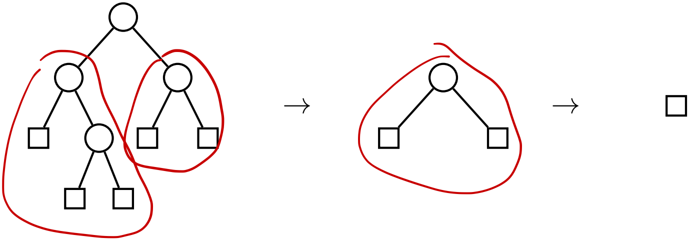


# Trimming Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent



# Bonus: Touchard's Identity

## Proposition

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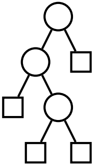

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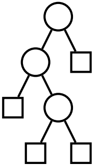
$\frac{z}{1-2z} \rightsquigarrow$ 

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## Corollary (Touchard's Identity)

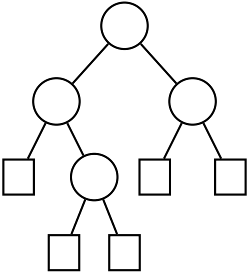
The Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ satisfy

$$C_{n+1} = \sum_{0\le k\le n/2} C_k 2^{n-2k}\binom{n}{2k}.$$

# How "old" do the nodes get?

We label the nodes according to the following rules:

- Leaves $\to 0$
- $\text{age(left child)} = \text{age(right child)} \to \text{increase by } 1$
- Otherwise: maximum of children



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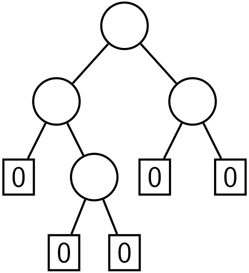

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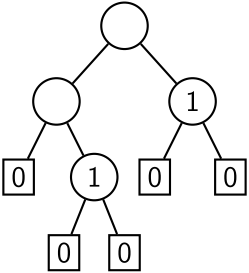

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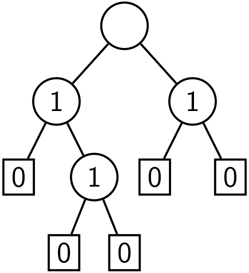

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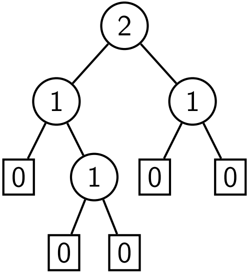

# The Register Function

Age $\rightsquigarrow$ *Register function* (*Horton-Strahler*-Index)

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- Applications:

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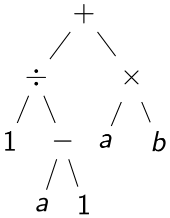
- Applications:
  - Required stack size for evaluating arithmetic expressions



# The Register Function

Age $\rightsquigarrow$ *Register function* (*Horton-Strahler*-Index)

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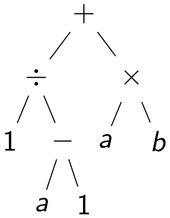
  - Branching complexity of river networks (e.g. Danube: 9)

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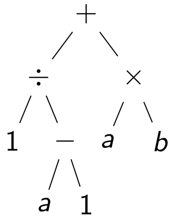

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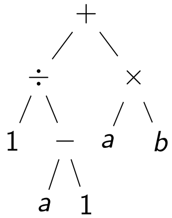


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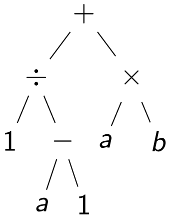


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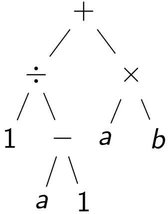


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  - $r$-branches, Numerics: Yamamoto, Yamazaki (2009)

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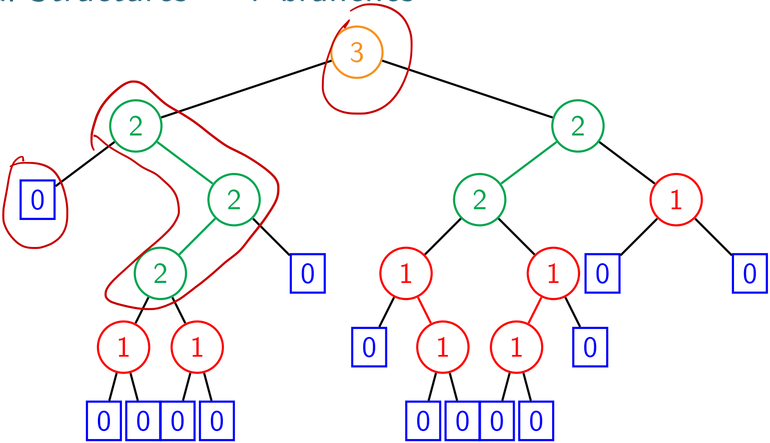
# Local Structures – “$r$-branches”



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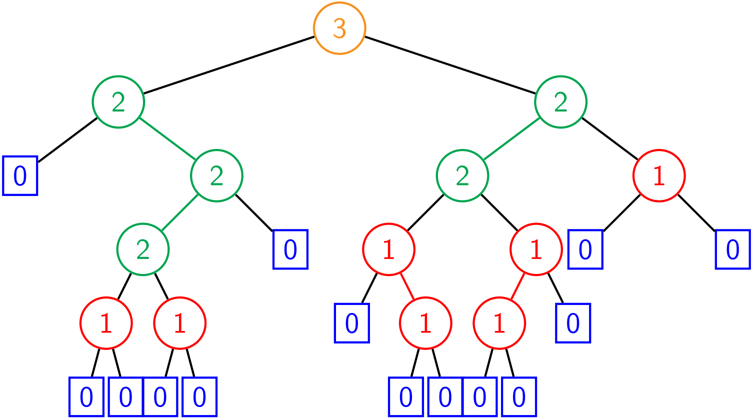


Number / Distribution of ($r$-)branches?

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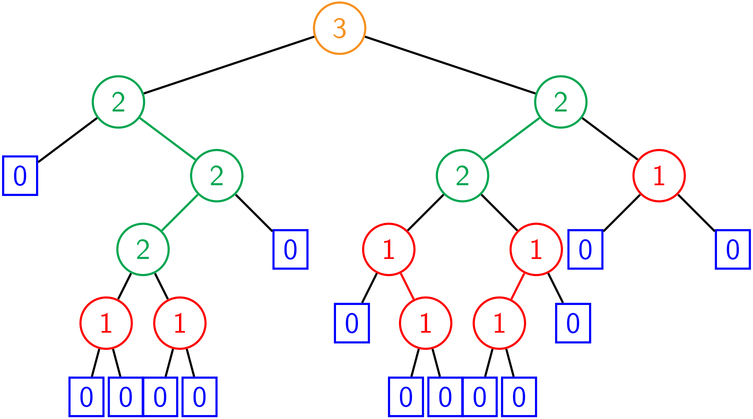


- Number / Distribution of ($r$-)branches?
- Example:

| $r$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| # $r$-branches | 14 | 5 | 2 | 1 |

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# "$r$-branches" – Results

**Theorem (H.–Heuberger–Prodinger)**

*In a random binary tree of size $n$...*

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- *with* **mean** *and* **variance**

$$\mathbb{E} = \frac{n}{4^r} + \frac{1}{6}\Big(1 + \frac{5}{4^r}\Big) + O(n^{-1}), \qquad \mathbb{V} = \frac{4^r - 1}{3 \cdot 16^r} n + O(1)$$

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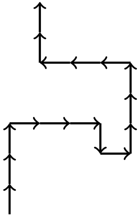
- **expected total #** *of branches is*

$$\frac{4}{3} n + \frac{1}{6} \log_4 n + C + \delta(\log_4 n) + O(n^{-1} \log n),$$

  - $C \approx 1.36190$, $\delta \ldots$ *periodic fluctuation*

# Reduction of lattice paths

Reduction of a simple, two-dimensional lattice path (i.e. a sequence of $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$):



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- Consider the pairs of horizontal-vertical segments:
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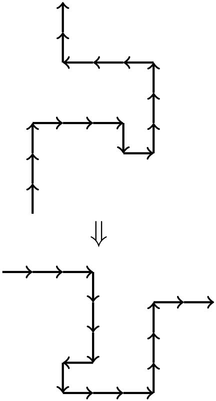

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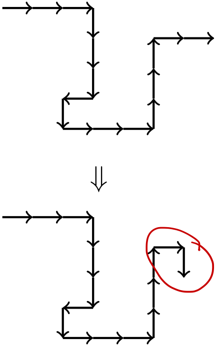

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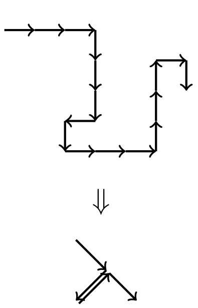

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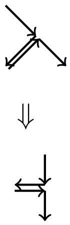

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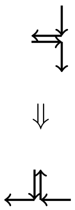

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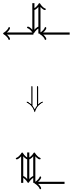

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$$\nwarrow$$

$$\Downarrow$$

$$\uparrow$$

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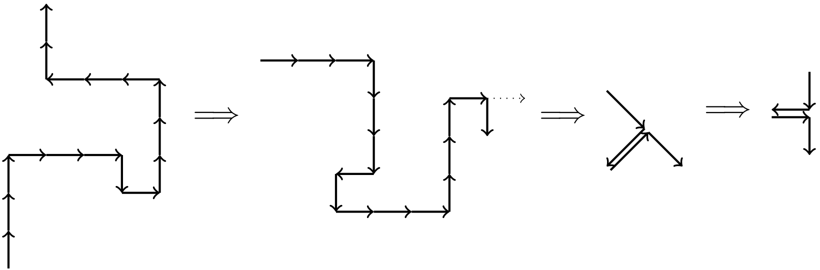
# Reduction – Example



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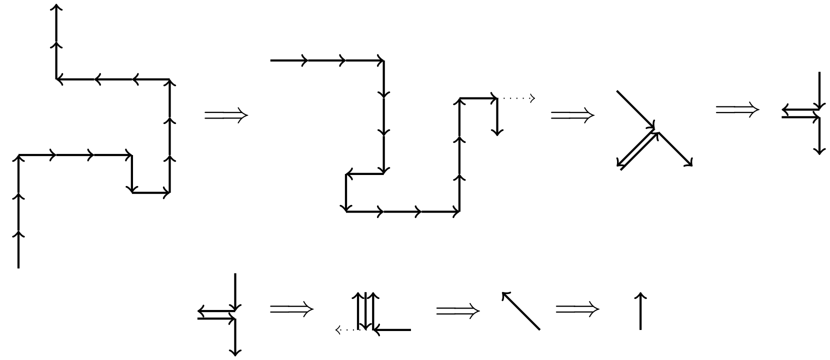

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*The generating function of simple two-dimensional lattice paths of length $\geq 1$, $L(z) = \frac{4z}{1-4z}$, satisfies the functional equation*

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Can be checked directly—or proven combinatorially!

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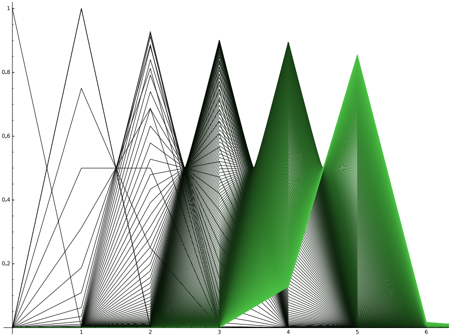
- Probability densities of $X_1$ up to $X_{512}$:



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# Reduction Degree – Expectation

**Theorem (H.–Heuberger–Prodinger)**

*The expected compactification degree among all simple 2D lattice paths of length n admits the asymptotic expansion*

$$\mathbb{E}X_n = \log_4 n + \frac{\gamma + 2 - 3\log 2}{2\log 2} + \delta_1(\log_4 n) + O(n^{-1})$$

*where*

$$\delta_1(x) = \frac{1}{\log 2} \sum_{k\neq 0} \frac{\Gamma(2+\chi_k)\zeta(1+\chi_k)}{\Gamma(1+\chi_k/2)} e^{2k\pi i x}$$

*is a small 1-periodic fluctuation.*

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# Reduction Degree – Variance

**Theorem (H.–Heuberger–Prodinger)**

*The corresponding variance is given by*

$$\mathbb{V}X_n = \frac{\pi^2 - 24\log^2\pi - 48\zeta''(0) - 24}{24\log^2 2} - \frac{2\log\pi}{\log 2} - \frac{11}{12} + \delta_2(\log_4 n) + \frac{\gamma + 2 - 3\log 2}{\log 2}\delta_1(\log_4 n) + \delta_1^2(\log_4 n) + O\Big(\frac{1}{\log n}\Big),$$

*where $\delta_1(x)$ is defined as above and $\delta_2(x)$ is a small 1-periodic fluctuation as well.*

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