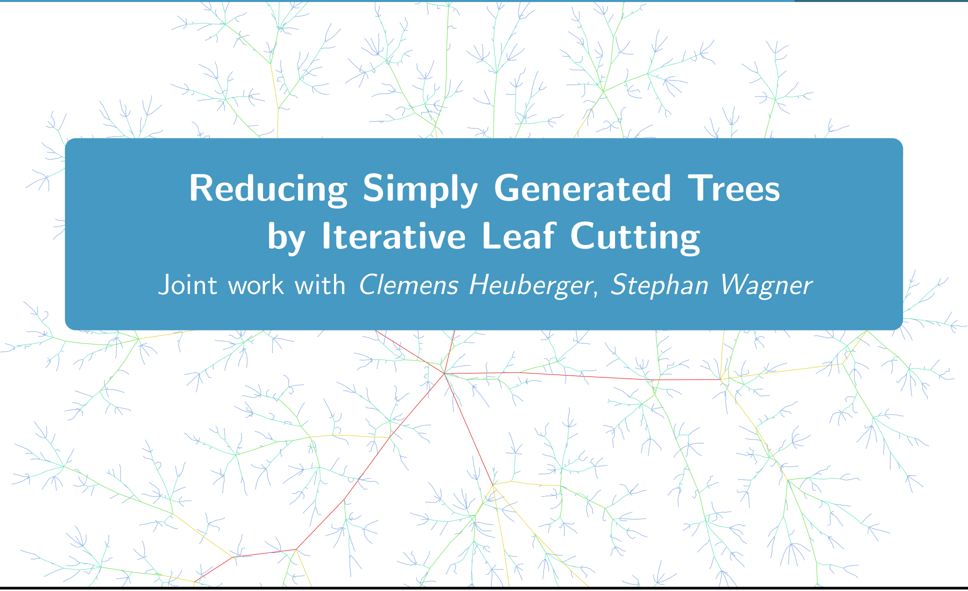


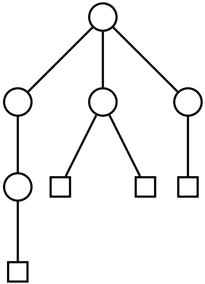
Reducing Simply Generated Trees by Iterative Leaf Cutting

Joint work with *Clemens Heuberger*, *Stephan Wagner*

Der Wissenschaftsfonds.

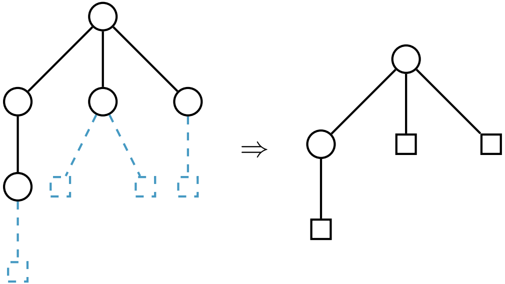
Example: Deterministic Plane Tree Reduction

- Remove all leaves!



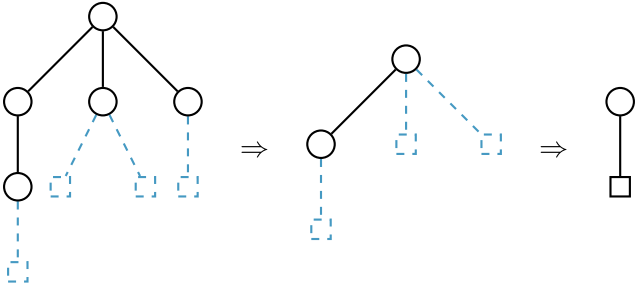
Example: Deterministic Plane Tree Reduction

- Remove all leaves!



Example: Deterministic Plane Tree Reduction

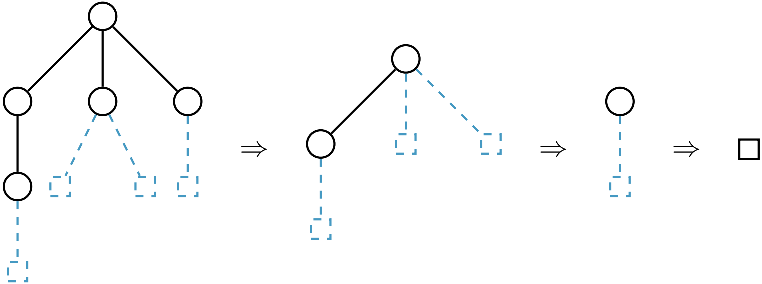
- Remove all leaves!



UNIVERSITÄT KLAGENFURT

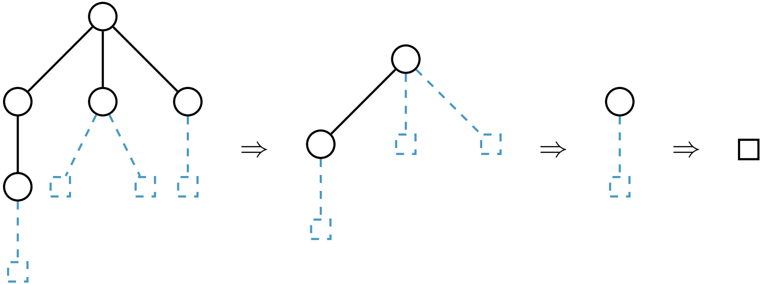
Example: Deterministic Plane Tree Reduction

- Remove all leaves!



Example: Deterministic Plane Tree Reduction

- Remove all leaves!



Parameter of Interest:

- Size of rth reduction $\longleftrightarrow$ # of removed nodes

UNIVERSITÄT KLAGENFURT

Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, **growing trees**

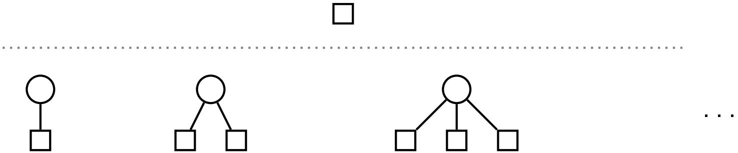
Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, **growing trees**

□

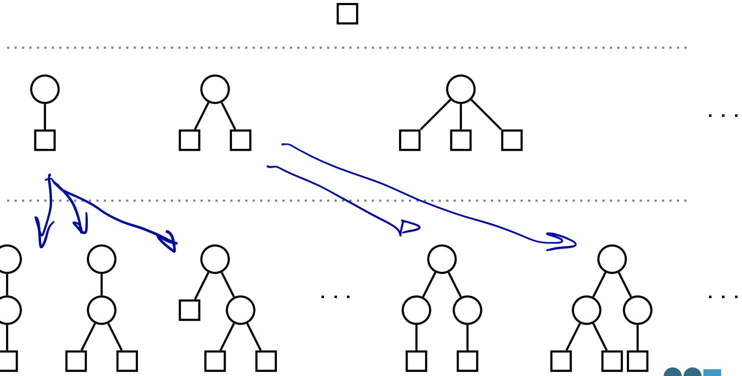
Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, **growing trees**



Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, **growing trees**



Expansion operators

- $F \ldots$ family of plane trees; bivariate generating function f
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

Expansion operators

- $F \dots$ family of plane trees; bivariate generating function f
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

Leaf expansion Φ

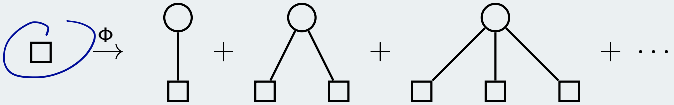
- inverse operation to leaf reduction

Expansion operators

- F . . . family of plane trees; bivariate generating function f
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

Leaf expansion Φ

- inverse operation to leaf reduction
  - attach leaves to all current leaves (required)
  - attach leaves to inner nodes (optional)

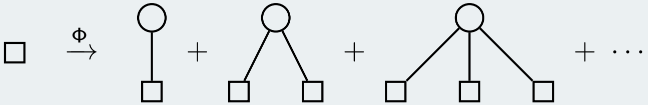


Expansion operators

- F . . . family of plane trees; bivariate generating function f
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

Leaf expansion Φ

- inverse operation to leaf reduction
  - attach leaves to all current leaves (required)
  - attach leaves to inner nodes (optional)



$$\square \triangleq t,\ \bigcirc \triangleq z \quad \Rightarrow \quad \Phi(t) = zt + zt^2 + zt^3 + \cdots$$

UNIVERSITÄT KLAGENFURT

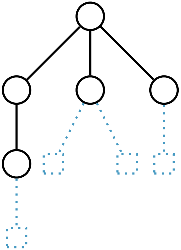
Reductions on Plane Trees

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

$\mathbb{V} \sim \frac{r(r+2)}{6(r+1)^2} n$

limit law: ✓



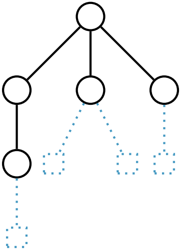
Reductions on Plane Trees

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

$\mathbb{V} \sim \frac{r(r+2)}{6(r+1)^2} n$

limit law: ✓

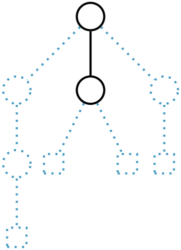


Paths

$\mathbb{E} \sim \frac{n}{2^{r+1}-1}$

$\mathbb{V} \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n$

limit law: ✓



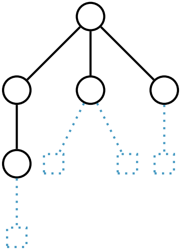
Reductions on Plane Trees

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

$\mathbb{V} \sim \frac{r(r+2)}{6(r+1)^2} n$

limit law: ✓

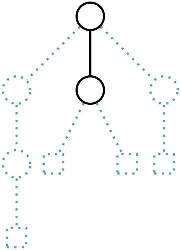


Paths

$\mathbb{E} \sim \frac{n}{2^{r+1}-1}$

$\mathbb{V} \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n$

limit law: ✓

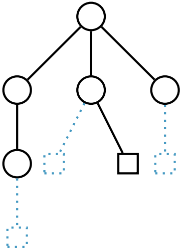


Old leaves

$\mathbb{E} \sim (2 - B_{r-1}(1/4))n$

$\mathbb{V} = \Theta(n)$

limit law: ✓



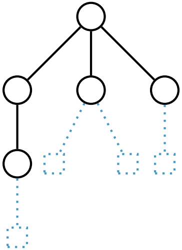
Reductions on Plane Trees

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

$\mathbb{V} \sim \frac{r(r+2)}{6(r+1)^2} n$

limit law: ✓

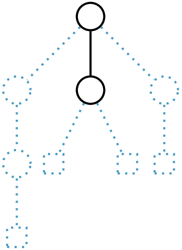


Paths

$\mathbb{E} \sim \frac{n}{2^{r+1}-1}$

$\mathbb{V} \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n$

limit law: ✓

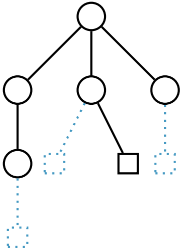


Old leaves

$\mathbb{E} \sim (2 - B_{r-1}(1/4))n$

$\mathbb{V} = \Theta(n)$

limit law: ✓

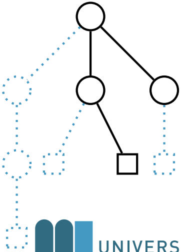


Old paths

$\mathbb{E} \sim \frac{2n}{r+2}$

$\mathbb{V} \sim \frac{2r(r+1)}{3(r+2)^2} n$

limit law: ✓



UNIVERSITÄT KLAGENFURT

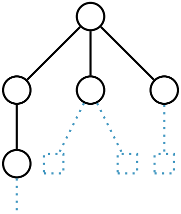
Reductions on Plane Trees

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

$\mathbb{V} \sim \frac{r(r+2)}{6(r+1)^2} n$

limit law: ✓

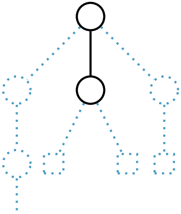


Paths

$\mathbb{E} \sim \frac{n}{2^{r+1}-1}$

$\mathbb{V} \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n$

limit law: ✓



Question

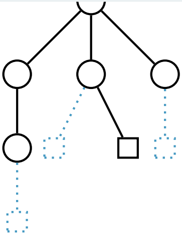
What can be done for other tree classes?

Old leaves

$\mathbb{E} \sim (2 - B_{r-1}(1/4))n$

$\mathbb{V} = \Theta(n)$

limit law: ✓

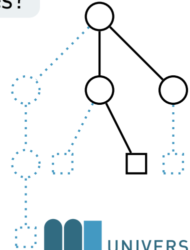


Old paths

$\mathbb{E} \sim \frac{2n}{r+2}$

$\mathbb{V} \sim \frac{2r(r+1)}{3(r+2)^2} n$

limit law: ✓

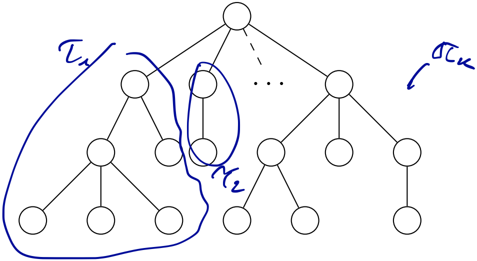


UNIVERSITÄT KLAGENFURT

Additive Tree Parameters

$\mathcal{T}$ ... rooted trees. $F\colon \mathcal{T} \to \mathbb{R}$ ... *additive tree parameter*, if

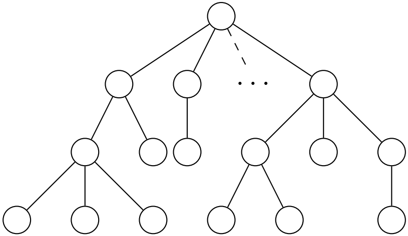
- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of τ



Additive Tree Parameters

$\mathcal{T}\dots$ rooted trees. $F\colon \mathcal{T} \to \mathbb{R}$ … *additive tree parameter*, if

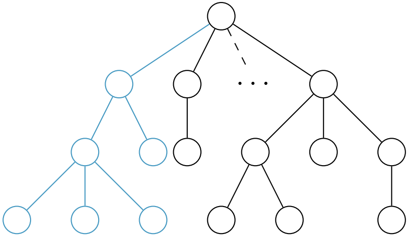
- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \dots, \tau_k$ branches of τ
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
  - with *toll function* $f\colon \mathcal{T} \to \mathbb{R}$



Additive Tree Parameters

$\mathcal{T}$ ... rooted trees. $F\colon \mathcal{T} \to \mathbb{R}$ ... *additive tree parameter*, if

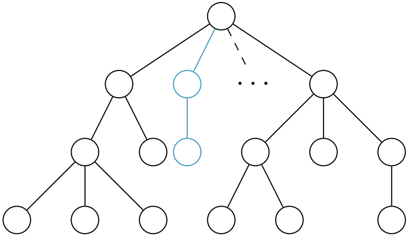
- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of τ
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
  - with *toll function* $f\colon \mathcal{T} \to \mathbb{R}$



Additive Tree Parameters

$\mathcal{T}$ ... rooted trees. $F\colon \mathcal{T} \to \mathbb{R}$ ... *additive tree parameter*, if

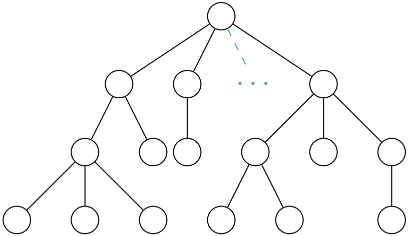
- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of τ
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
  - with *toll function* $f\colon \mathcal{T} \to \mathbb{R}$



Additive Tree Parameters

$\mathcal{T}$ ... rooted trees. $F\colon \mathcal{T} \to \mathbb{R}$ ... *additive tree parameter*, if

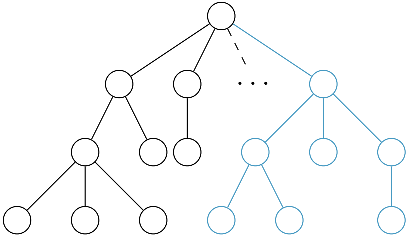
- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of τ
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
  - with *toll function* $f\colon \mathcal{T} \to \mathbb{R}$



Additive Tree Parameters

$\mathcal{T}$ ... rooted trees. $F\colon \mathcal{T} \to \mathbb{R}$ ... *additive tree parameter*, if

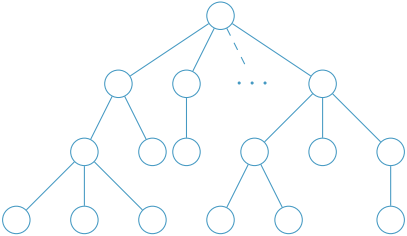
- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of τ
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
  - with *toll function* $f\colon \mathcal{T} \to \mathbb{R}$



Additive Tree Parameters

$\mathcal{T}$ ... rooted trees. $F\colon \mathcal{T} \to \mathbb{R}$ ... *additive tree parameter*, if

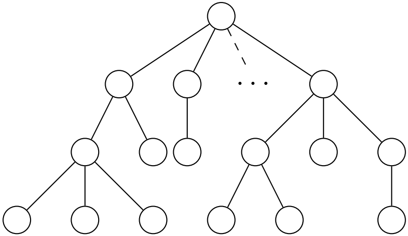
- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of τ
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
  - with *toll function* $f\colon \mathcal{T} \to \mathbb{R}$



Additive Tree Parameters

$\mathcal{T}\ldots$ rooted trees. $F\colon \mathcal{T} \to \mathbb{R}$ ... *additive tree parameter*, if

- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of τ
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
    - with *toll function* $f\colon \mathcal{T} \to \mathbb{R}$



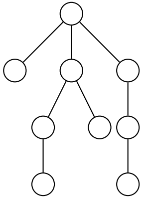
- Wagner (2015), Janson (2016), Wagner et al. (2018)...:
    - τ_n random tree, size n; f *suitable*
      $\rightsquigarrow F(\tau_n)$ asymptotically Gaussian

Example: Removed Leaves

- $a_r(\tau) \rightsquigarrow \#$ of removed nodes when cutting leaves r times

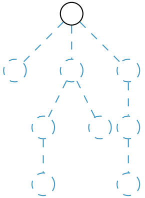
Example: Removed Leaves

- $a_r(\tau) \rightsquigarrow \#$ of removed nodes when cutting leaves r times
- **Example:** $r = 3$



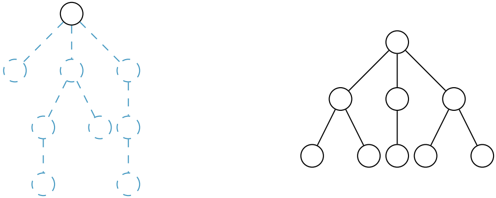
Example: Removed Leaves

- $a_r(\tau) \rightsquigarrow \#$ of removed nodes when cutting leaves r times
- **Example:** $r = 3$



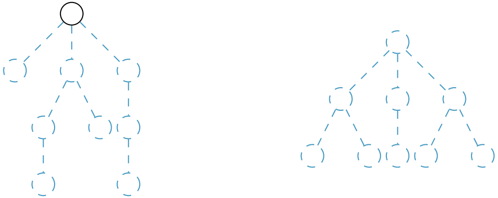
Example: Removed Leaves

- $a_r(\tau) \rightsquigarrow \#$ of removed nodes when cutting leaves r times
- **Example:** $r = 3$



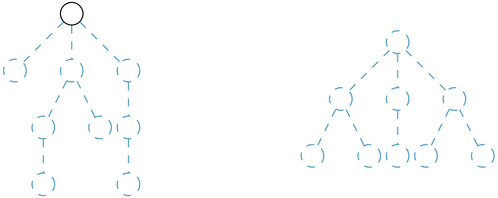
Example: Removed Leaves

- $a_r(\tau) \rightsquigarrow \#$ of removed nodes when cutting leaves r times
- **Example:** $r = 3$



Example: Removed Leaves

- $a_r(\tau) \rightsquigarrow$ # of removed nodes when cutting leaves r times
- **Example:** $r = 3$



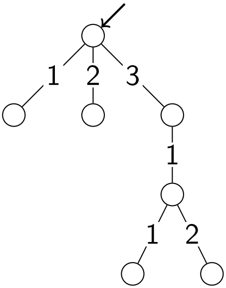
"Cutting Leaves" – Toll Function

$$f(\tau) = \begin{cases} 0 & \text{if } \tau \text{ has height } \geq r, \\ 1 & \text{if } \tau \text{ has height } < r. \end{cases}$$

Simply Generated Trees

Plane Trees:

- *rooted*: designated root node
- *ordered*: order of children matters



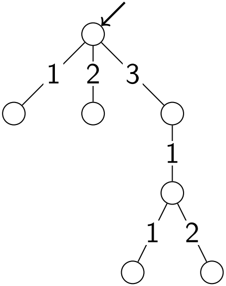
Simply Generated Trees

Plane Trees:

- *rooted*: designated root node
- *ordered*: order of children matters

Generalization:

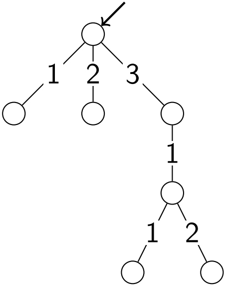
- *weight sequence*: $\mathbf{w} = (w_k)_{k\geq 0} \subseteq \mathbb{R}_{\geq 0}$



Simply Generated Trees

Plane Trees:

- *rooted*: designated root node
- *ordered*: order of children matters



Generalization:

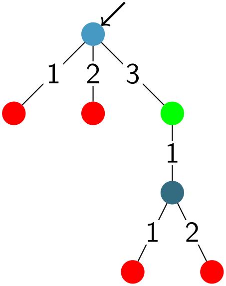
- *weight sequence*: $\mathbf{w} = (w_k)_{k\geq 0} \subseteq \mathbb{R}_{\geq 0}$
- *weight of tree* τ:

$$w(\tau) := \prod_{v \text{ node in } \tau} w_{(\#\text{ children of } v)}$$

Simply Generated Trees

Plane Trees:

- *rooted*: designated root node
- *ordered*: order of children matters



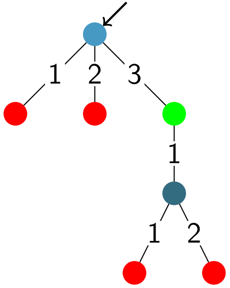
Generalization:

- *weight sequence*: $\mathbf{w} = (w_k)_{k\geq 0} \subseteq \mathbb{R}_{\geq 0}$
- *weight of tree* τ:

$$w(\tau) := \prod_{v \text{ node in } \tau} w_{(\#\text{ children of } v)}$$

$$w(\tau) = w_0^4 w_1 w_2 w_3$$

Simply Generated Trees



Plane Trees:

- *rooted*: designated root node
- *ordered*: order of children matters

Generalization:

- *weight sequence*: $\mathbf{w} = (w_k)_{k\geq 0} \subseteq \mathbb{R}_{\geq 0}$
- *weight of tree* τ:

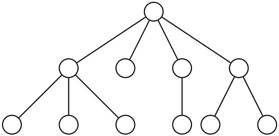
$$w(\tau) := \prod_{v \text{ node in } \tau} w_{(\#\text{ children of } v)} \qquad w(\tau) = w_0^4 w_1 w_2 w_3$$

- *partition function* and *probability distribution*:

$$Z_n = Z_n(\mathbf{w}) := \sum_{\tau \in \mathcal{T}_n} w(\tau) \quad \Rightarrow \quad \mathbb{P}(\tau_n = \tau) = \frac{w(\tau)}{Z_n}$$

Simply Generated Trees: Examples

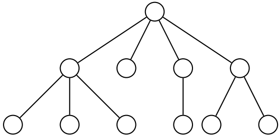
Plane Trees



$$w_0 = w_1 = \cdots = 1$$

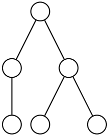
Simply Generated Trees: Examples

Plane Trees



$w_0 = w_1 = \cdots = 1$

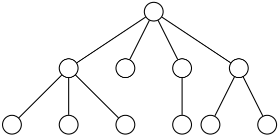
Unary-Binary Trees



$w_0 = w_1 = w_2 = 1$

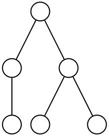
Simply Generated Trees: Examples

Plane Trees



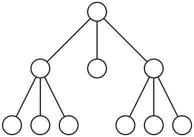
$$w_0 = w_1 = \cdots = 1$$

Unary-Binary Trees



$$w_0 = w_1 = w_2 = 1$$

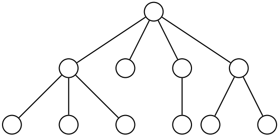
d-ary Trees



$$w_0 = w_d = 1$$

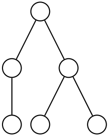
Simply Generated Trees: Examples

Plane Trees



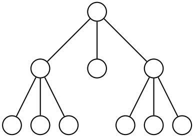
$$w_0 = w_1 = \cdots = 1$$

Unary-Binary Trees



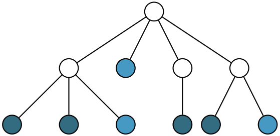
$$w_0 = w_1 = w_2 = 1$$

d-ary Trees



$$w_0 = w_d = 1$$

k-colored leaves



$$w_0 = k,\ w_1 = \cdots = 1$$

UNIVERSITÄT KLAGENFURT

Recursive Characterization

Proposition

- $T(z, u) \ldots$ *BWGF (z... tree size, u... removed nodes)*

Recursive Characterization

Proposition

- *$T(z,u)\ldots$ BWGF ($z\ldots$ tree size, $u\ldots$ removed nodes)*
- $\Phi(t)\ldots$ *GF of* $\mathbf{w}$, $\Phi(t) = \sum_{k\geq 0} w_k t^k$

Recursive Characterization

Proposition

- $T(z, u) \ldots$ *BWGF (z… tree size, u… removed nodes)*
- $\Phi(t) \ldots$ *GF of* $\mathbf{w}$, $\Phi(t) = \sum_{k \geq 0} w_k t^k$
- $T_r(z) \ldots$ *WGF of trees of height* $< r$

Recursive Characterization

Proposition

- *$T(z, u)\ldots$ BWGF ($z\ldots$ tree size, $u\ldots$ removed nodes)*
- *$\Phi(t)\ldots$ GF of $\mathbf{w}$, $\Phi(t) = \sum_{k\geq 0} w_k t^k$*
- *$T_r(z)\ldots$ WGF of trees of height $< r$*

Then:

$$T(z, u) = z\Phi(T(z, u)) + \left(1 - \frac{1}{u}\right) T_r(zu)$$

Recursive Characterization

Proposition

- $T(z,u)\ldots$ *BWGF* ($z\ldots$ *tree size*, $u\ldots$ *removed nodes*)
- $\Phi(t)\ldots$ *GF of* $\mathbf{w}$, $\Phi(t) = \sum_{k\geq 0} w_k t^k$
- $T_r(z)\ldots$ *WGF of trees of height* $< r$

Then:

$$T(z,u) = z\Phi(T(z,u)) + \left(1 - \frac{1}{u}\right) T_r(zu)$$

Observation. $\mathcal{T}\ldots\mathbf{w}$-simply generated trees, GF $T(z) = T(z,1)$

Recursive Characterization

Proposition

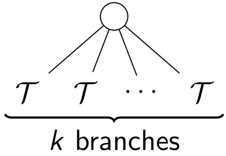
- $T(z,u)\ldots$ *BWGF ($z\ldots$ tree size, $u\ldots$ removed nodes)*
- $\Phi(t)\ldots$ *GF of* $\mathbf{w}$, $\Phi(t) = \sum_{k\geq 0} w_k t^k$
- $T_r(z)\ldots$ *WGF of trees of height* $< r$

Then:

$$T(z,u) = z\Phi(T(z,u)) + \left(1 - \frac{1}{u}\right) T_r(zu)$$

Observation. $\mathcal{T}\ldots\mathbf{w}$-simply generated trees, GF $T(z) = T(z,1)$

$$\mathcal{T} = \sum_{\substack{k\geq 0\\ w_k\neq 0}}$$



$T(z) = z \cdot \sum_{k\geq 0} w_k \cdot T(z)^k$

Recursive Characterization

Proposition

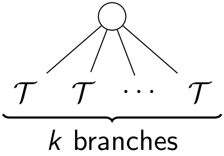
- $T(z,u)\ldots$ *BWGF ($z\ldots$ tree size, $u\ldots$ removed nodes)*
- $\Phi(t)\ldots$ *GF of* $\mathbf{w}$, $\Phi(t) = \sum_{k\geq 0} w_k t^k$
- $T_r(z)\ldots$ *WGF of trees of height* $< r$

Then:

$$T(z,u) = z\Phi(T(z,u)) + \left(1 - \frac{1}{u}\right) T_r(zu)$$

Observation. $\mathcal{T}\ldots$ **w**-simply generated trees, GF $T(z) = T(z,1)$

$$\mathcal{T} = \sum_{\substack{k\geq 0\\ w_k\neq 0}}$$



$$\Rightarrow \qquad T(z) = z\Phi(T(z))$$

Recursive Characterization – Proof

Functional Equation

$$T(z,u) = z\Phi(T(z,u)) + \left(1 - \frac{1}{u}\right) T_r(zu)$$

Proof. $\mathcal{T}_r \ldots$ trees with height less than r

Recursive Characterization – Proof

Functional Equation

$$T(z,u) = z\Phi(T(z,u)) + \left(1 - \frac{1}{u}\right) T_r(zu)$$

Proof. $\mathcal{T}_r \ldots$ trees with height less than r

$$T(z,u) = \sum_{\tau\in\mathcal{T}} w(\tau) z^{|\tau|} u^{a_r(\tau)} =$$

$$= \sum_{k\geq 0} z \cdot w_k \sum_{\tau_1} \sum_{\tau_2} \cdots \sum_{\tau_k} \left(\prod w(\tau_j)\right) z^{\sum|\tau_j|} u^{a_r(\tau)}$$

Recursive Characterization – Proof

Functional Equation

$$T(z,u) = z\Phi(T(z,u)) + \left(1 - \frac{1}{u}\right) T_r(zu)$$

Proof. $\mathcal{T}_r \ldots$ trees with height less than r

$$\begin{aligned} T(z,u) &= \sum_{\tau\in\mathcal{T}} w(\tau) z^{|\tau|} u^{a_r(\tau)} \\ &= \sum_{k\geq 0} w_k \sum_{\tau_1} \cdots \sum_{\tau_k} \left(\prod_{j=1}^{k} w(\tau_j)\right) z^{1+\sum|\tau_j|} u^{\sum a_r(\tau_j)} \\ &\quad + \sum_{\tau\in\mathcal{T}_r} w(\tau) z^{|\tau|} (u^{|\tau|} - u^{|\tau|-1}) \end{aligned}$$

Recursive Characterization – Proof

Functional Equation

$$T(z,u) = z\Phi(T(z,u)) + \left(1 - \frac{1}{u}\right) T_r(zu)$$

Proof. $\mathcal{T}_r \ldots$ trees with height less than r

$$\begin{aligned}
T(z,u) &= \sum_{\tau \in \mathcal{T}} w(\tau) z^{|\tau|} u^{a_r(\tau)} \\
&= \sum_{k \geq 0} w_k \sum_{\tau_1} \cdots \sum_{\tau_k} \left(\prod_{j=1}^{k} w(\tau_j)\right) z^{1+\sum|\tau_j|} u^{\sum a_r(\tau_j)} \\
&\quad + \sum_{\tau \in \mathcal{T}_r} w(\tau) z^{|\tau|} (u^{|\tau|} - u^{|\tau|-1}) \\
&= \cdots = z\Phi(T(z,u)) + \left(1 - \frac{1}{u}\right) T_r(zu).
\end{aligned}$$

UNIVERSITÄT KLAGENFURT

Singular Expansion

- **Goal:** extract information from functional equation

Singular Expansion

- **Goal:** extract information from functional equation
- In particular: partial derivatives $\partial_u T(z,1)$, $\partial_{uu} T(z,1)$

Singular Expansion

- **Goal:** extract information from functional equation
- In particular: partial derivatives $\partial_u T(z, 1)$, $\partial_{uu} T(z, 1)$
- Via implicit differentiation:

$$\partial_u T(z, 1) = \frac{zT'(z) T_r(z)}{T(z)}$$

UNIVERSITÄT KLAGENFURT

Singular Expansion

- **Goal:** extract information from functional equation
- In particular: partial derivatives $\partial_u T(z,1)$, $\partial_{uu} T(z,1)$
- Via implicit differentiation:

$$\partial_u T(z,1) = \frac{zT'(z)T_r(z)}{T(z)}$$

- $T(z)$ satisfies $T(z) = z\Phi(T(z)) \rightsquigarrow$ singular inversion!

UNIVERSITÄT KLAGENFURT

Singular Expansion

- **Goal:** extract information from functional equation
- In particular: partial derivatives $\partial_u T(z,1)$, $\partial_{uu} T(z,1)$
- Via implicit differentiation:

$$\partial_u T(z,1) = \frac{zT'(z)T_r(z)}{T(z)}$$

- $T(z)$ satisfies $T(z) = z\Phi(T(z))$ $\rightsquigarrow$ singular inversion!
  - *fundamental constant* $\tau > 0$: solution of $\tau\Phi'(\tau) - \Phi(\tau) = 0$

UNIVERSITÄT KLAGENFURT

Singular Expansion

- **Goal:** extract information from functional equation
- In particular: partial derivatives $\partial_u T(z,1)$, $\partial_{uu} T(z,1)$
- Via implicit differentiation:

$$\partial_u T(z,1) = \frac{zT'(z)T_r(z)}{T(z)}$$

- $T(z)$ satisfies $T(z) = z\Phi(T(z)) \rightsquigarrow$ singular inversion!
  - *fundamental constant* $\tau > 0$: solution of $\tau\Phi'(\tau) - \Phi(\tau) = 0$
  - T has square root singularity at $\rho = \tau/\Phi(\tau)$

UNIVERSITÄT KLAGENFURT

Singular Expansion

- **Goal:** extract information from functional equation
- In particular: partial derivatives $\partial_u T(z,1)$, $\partial_{uu} T(z,1)$
- Via implicit differentiation:

$$\partial_u T(z,1) = \frac{zT'(z)T_r(z)}{T(z)}$$

- $T(z)$ satisfies $T(z) = z\Phi(T(z)) \rightsquigarrow$ singular inversion!
  - *fundamental constant* $\tau > 0$: solution of $\tau\Phi'(\tau) - \Phi(\tau) = 0$
  - T has square root singularity at $\rho = \tau/\Phi(\tau)$

$$T(z) \overset{z\to\rho}{=} \tau - \sqrt{\frac{2\tau}{\rho\Phi''(\tau)}}\sqrt{1 - z/\rho} + O(1 - z/\rho)$$

Result – Simply Generated Trees

Theorem (H.–Heuberger–Wagner, '18+)

- *weight sequence* **w**, *fundamental constant* τ, $\rho = \tau/\Phi(\tau)$
- $X_{n,r} \dots \#$ *of removed nodes after cutting leaves* r *times*

Then:

Result – Simply Generated Trees

Theorem (H.–Heuberger–Wagner, '18+)

- *weight sequence* $\mathbf{w}$*, fundamental constant* τ*,* $\rho = \tau/\Phi(\tau)$
- $X_{n,r} \ldots \#$ *of removed nodes after cutting leaves r times*

Then:

- $X_{n,r}$ *is asymptotically normally distributed for* $n \to \infty$

UNIVERSITÄT KLAGENFURT

Result – Simply Generated Trees

Theorem (H.–Heuberger–Wagner, '18+)

- *weight sequence* $\mathbf{w}$, *fundamental constant* τ, $\rho = \tau/\Phi(\tau)$
- $X_{n,r} \ldots \#$ *of removed nodes after cutting leaves* r *times*

Then:

- $X_{n,r}$ *is asymptotically normally distributed for* $n \to \infty$ *with* **mean** *and* **variance**

$$\mathbb{E}X_{n,r} = \frac{T_r(\rho)}{\tau} \cdot n + O(1), \quad \mathbb{V}X_{n,r} = \sigma_r^2 \cdot n + O(1)$$

Result – Simply Generated Trees

Theorem (H.–Heuberger–Wagner, '18+)

- *weight sequence* $\mathbf{w}$, *fundamental constant* τ, $\rho = \tau/\Phi(\tau)$
- $X_{n,r} \ldots$ *# of removed nodes after cutting leaves r times*

Then:

- $X_{n,r}$ *is asymptotically normally distributed for* $n \to \infty$ *with* **mean** *and* **variance**

$$\mathbb{E}X_{n,r} = \frac{T_r(\rho)}{\tau} \cdot n + O(1), \quad \mathbb{V}X_{n,r} = \sigma_r^2 \cdot n + O(1)$$

- *and for* $r \to \infty$ *we have*

$$\frac{F_r(\rho)}{\tau} = 1 - \frac{2}{\rho\tau\Phi''(\tau)} r^{-1} + o(r^{-1}), \quad \sigma_r^2 = \frac{1}{3\rho\tau\Phi''(\tau)} + o(1).$$

UNIVERSITÄT KLAGENFURT

Result – Simply Generated Trees

Theorem (H.–Heuberger–Wagner, '18+)

- *weight sequence* $\mathbf{w}$, *fundamental constant* τ, $\rho = \tau/\Phi(\tau)$
- $X_{n,r} \ldots$ *# of removed nodes after cutting leaves r times*

Then:

- $X_{n,r}$ *is asymptotically normally distributed for* $n \to \infty$ *with* **mean** *and* **variance**

$$\mathbb{E}X_{n,r} = \frac{T_r(\rho)}{\tau} \cdot n + O(1), \quad \mathbb{V}X_{n,r} = \sigma_r^2 \cdot n + O(1)$$

- *and for* $r \to \infty$ *we have*

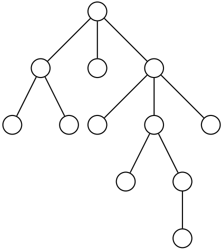
$$\frac{F_r(\rho)}{\tau} = 1 - \frac{2}{\rho\tau\Phi''(\tau)} r^{-1} + o(r^{-1}), \quad \sigma_r^2 = \frac{1}{3\rho\tau\Phi''(\tau)} + o(1).$$

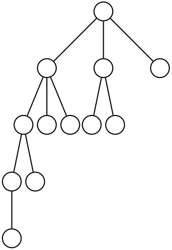
Technical Detail. $\mathbf{w}$ periodic $\rightsquigarrow$ parity!

UNIVERSITÄT KLAGENFURT

Pólya Trees

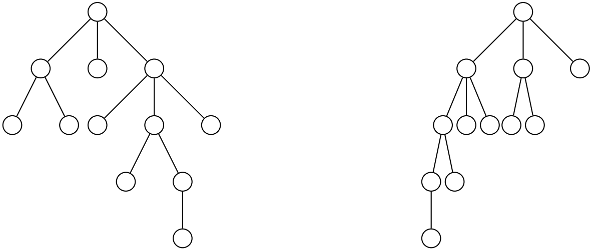
- Rooted trees, order of children is **irrelevant!**





Pólya Trees

- Rooted trees, order of children is **irrelevant!**



Proposition

- $\mathcal{T}\ldots$ *Pólya trees,* $T(z)\ldots$ *generating function*

$$\Rightarrow T(z) = z \exp\left(\sum_{k\geq 1} \frac{1}{k} T(z^k)\right)$$

Recursive Characterization / Functional Equation

Proposition

- $T(z, u)\ldots$ *BGF (z*$\ldots$ *tree size, u*$\ldots$ *removed nodes)*

Recursive Characterization / Functional Equation

Proposition

- $T(z,u)\ldots$ *BGF (z… tree size, u… removed nodes)*
- $T_r(z)\ldots$ *GF of trees of height* $< r$

Recursive Characterization / Functional Equation

Proposition

- $T(z,u)\ldots$ *BGF (z... tree size, u... removed nodes)*
- $T_r(z)\ldots$ *GF of trees of height* $< r$

Then:

$$T(z,u) = z\exp\Big(\sum_{k\geq 1}\frac{1}{k}T(z^k,u^k)\Big) + \Big(1-\frac{1}{u}\Big)T_r(zu)$$

Recursive Characterization / Functional Equation

Proposition

- *$T(z,u)$... BGF (z... tree size, u... removed nodes)*
- *$T_r(z)$... GF of trees of height $< r$*

Then:

$$T(z,u) = z\exp\Big(\sum_{k\geq 1}\frac{1}{k}T(z^k,u^k)\Big) + \Big(1-\frac{1}{u}\Big)T_r(zu)$$

Key Observations.

- $R(z,u) := \sum_{k\geq 2}\frac{1}{k}T(z^k,u^k)$. Then: $R(z) = R(z,1)$ has larger RoC than $T(z)$

Recursive Characterization / Functional Equation

Proposition

- $T(z,u)\dots$ *BGF* ($z\dots$ *tree size*, $u\dots$ *removed nodes*)
- $T_r(z)\dots$ *GF of trees of height* $< r$

Then:

$$T(z,u) = z\exp\left(\sum_{k\geq 1}\frac{1}{k}T(z^k,u^k)\right) + \left(1-\frac{1}{u}\right)T_r(zu)$$

Key Observations.

- $R(z,u) := \sum_{k\geq 2}\frac{1}{k}T(z^k,u^k)$. Then: $R(z) = R(z,1)$ has larger RoC than $T(z)$
- implicit differentiation: $\partial_u T(z,1) = \dfrac{T(z)\partial_u R(z,1) + T_r(z)}{1-T(z)}$

Recursive Characterization / Functional Equation

Proposition

- *$T(z, u)$... BGF (z... tree size, u... removed nodes)*
- *$T_r(z)$... GF of trees of height $< r$*

Then:

$$T(z,u) = z\exp\Big(\sum_{k\geq 1}\frac{1}{k}T(z^k,u^k)\Big) + \Big(1-\frac{1}{u}\Big)T_r(zu)$$

Key Observations.

- $R(z,u) := \sum_{k\geq 2}\frac{1}{k}T(z^k,u^k)$. Then: $R(z) = R(z,1)$ has larger RoC than $T(z)$
- implicit differentiation: $\partial_u T(z,1) = \dfrac{T(z)\partial_u R(z,1) + T_r(z)}{1-T(z)}$
- $T(z) \rightsquigarrow$ dominant singularity for $z \to \rho \approx 0.338322$

UNIVERSITÄT KLAGENFURT

Result – Pólya Trees

Theorem (H.–Heuberger–Wagner, '18+)

- $\rho \approx 0.338322\ldots$ *radius of convergence of* $T(z)$
- $X_{n,r}\ldots$ *# of removed nodes after cutting leaves r times*

Then:

Result – Pólya Trees

Theorem (H.–Heuberger–Wagner, '18+)

- $\rho \approx 0.338322$ . . . *radius of convergence of* $T(z)$
- $X_{n,r}$ . . . # *of removed nodes after cutting leaves r times*

Then:

- *for* $n \to \infty$, $X_{n,r}$ *has* **mean** *and* **variance**

$$\mathbb{E}X_{n,r} = \mu_r \cdot n + O(1), \quad \mathbb{V}X_{n,r} = \sigma_r^2 \cdot n + O(1)$$

Result – Pólya Trees

Theorem (H.–Heuberger–Wagner, '18+)

- $\rho \approx 0.338322\ldots$ *radius of convergence of* $T(z)$
- $X_{n,r}\ldots$ *# of removed nodes after cutting leaves r times*

Then:

- *for* $n \to \infty$, $X_{n,r}$ *has* **mean** *and* **variance**

$$\mathbb{E}X_{n,r} = \mu_r \cdot n + O(1), \quad \mathbb{V}X_{n,r} = \sigma_r^2 \cdot n + O(1)$$

- *and for* $r \to \infty$ *we have*

$$\mu_r = 1 - \frac{2}{(1 + \rho R'(\rho))} r^{-1} + o(r^{-1}), \quad \sigma_r^2 = \frac{1}{3(1 + \rho R'(\rho))} + o(1).$$

Result – Pólya Trees

Theorem (H.–Heuberger–Wagner, '18+)

- $\rho \approx 0.338322 \ldots$ *radius of convergence of* $T(z)$
- $X_{n,r} \ldots$ *# of removed nodes after cutting leaves r times*

Then:

- *for* $n \to \infty$, $X_{n,r}$ *has* **mean** *and* **variance**

$$\mathbb{E}X_{n,r} = \mu_r \cdot n + O(1), \quad \mathbb{V}X_{n,r} = \sigma_r^2 \cdot n + O(1)$$

- *and for* $r \to \infty$ *we have*

$$\mu_r = 1 - \frac{2}{(1 + \rho R'(\rho))} r^{-1} + o(r^{-1}), \quad \sigma_r^2 = \frac{1}{3(1 + \rho R'(\rho))} + o(1).$$

Note. $\frac{2}{1+\rho R'(\rho)} \approx 1.644731$, $\frac{1}{3(1+\rho R'(\rho))} \approx 0.274122$