Benjamin Hackl

July 17, 2018

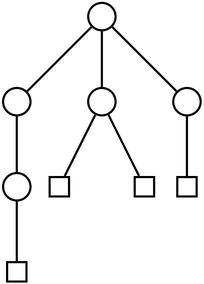
ALPEN-ADRIA UNIVERSITÄT KLAGENFURT | WIEN GRAZ

Iterative Cutting of Non-Catalan Trees

FWF Der Wissenschaftsfonds.

Example: Deterministic Plane Tree Reduction

- Remove all leaves!



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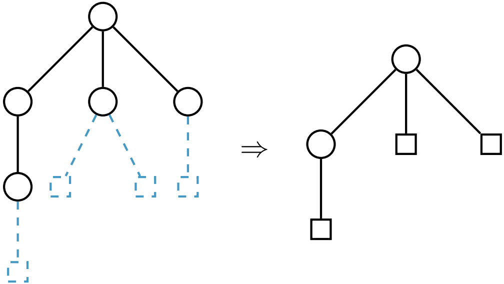

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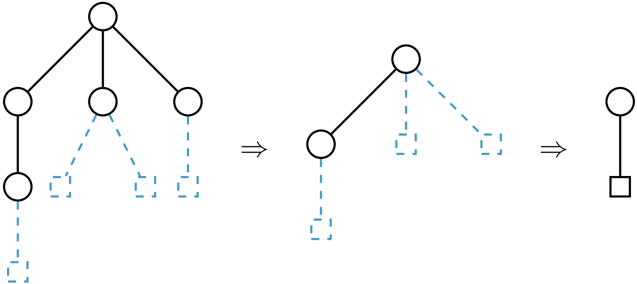

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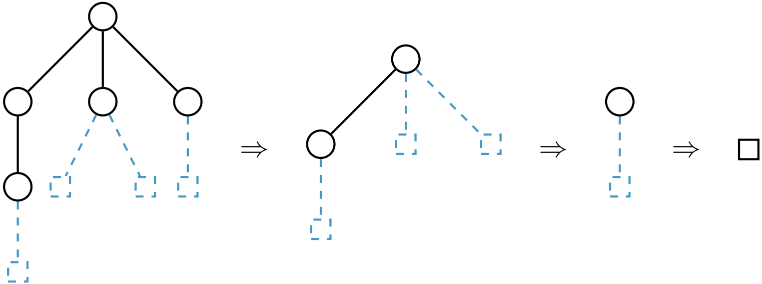

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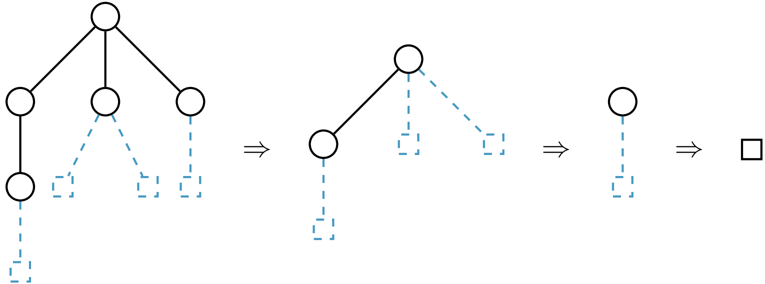

Parameter of Interest:

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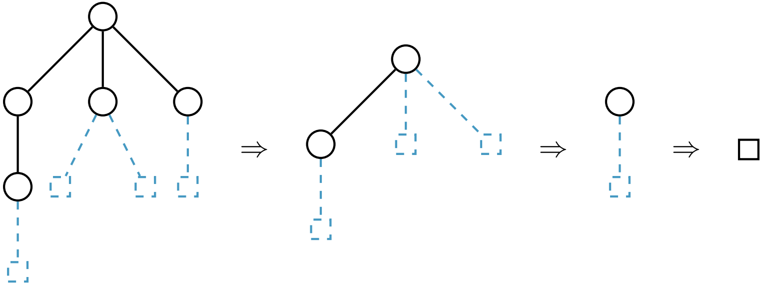

Parameter of Interest:

- Size of rth reduction $\longleftrightarrow$ # of removed nodes

Reduction → Expansion

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$\square$

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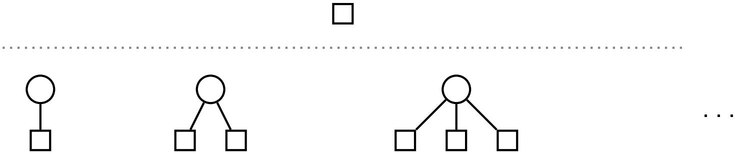

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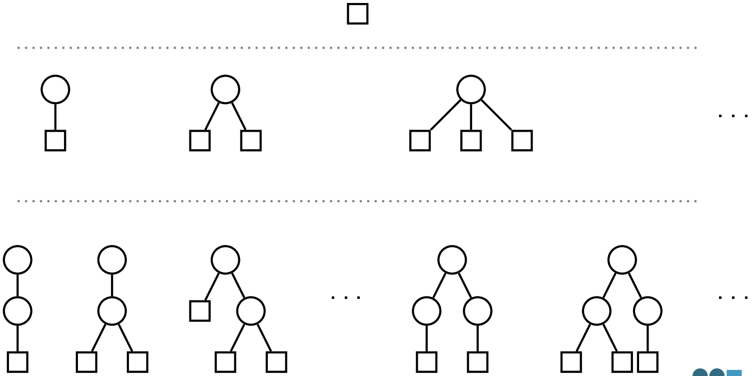

Expansion operators

- F . . . family of plane trees; bivariate generating function f
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

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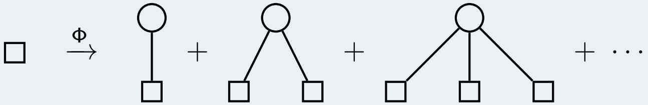

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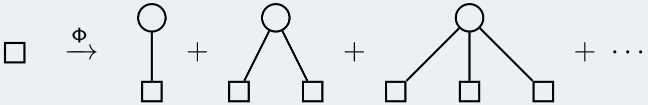


$$\square \triangleq t, \bigcirc \triangleq z \quad \Rightarrow \quad \Phi(t) = zt + zt^2 + zt^3 + \cdots$$

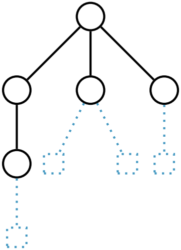
Reductions on Plane Trees

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

$\mathbb{V} \sim \frac{r(r+2)}{6(r+1)^2} n$

limit law: ✓



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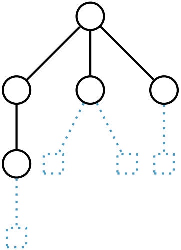

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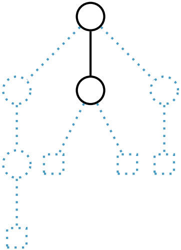

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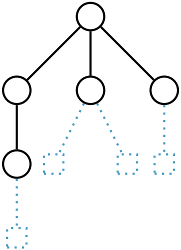

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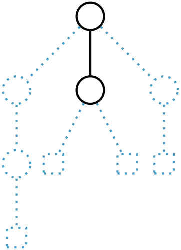

Old leaves

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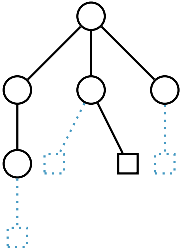

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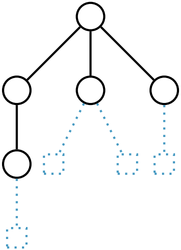

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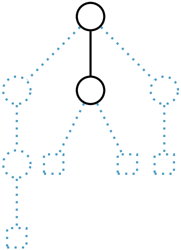

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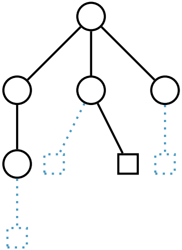

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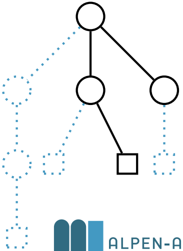

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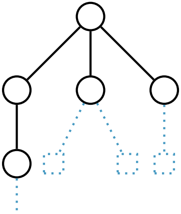

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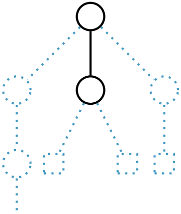

Question

What can be done for other tree classes?

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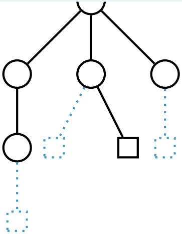

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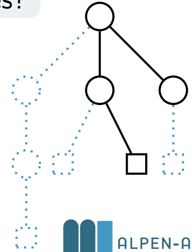
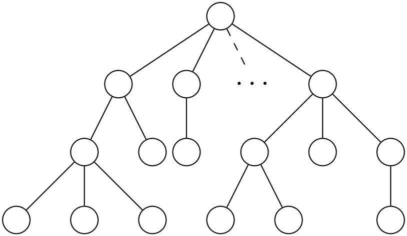


Additive Tree Parameters

$\mathcal{T}$ ... rooted trees. $F\colon \mathcal{T} \to \mathbb{R}$ ... *additive tree parameter*, if

- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of τ



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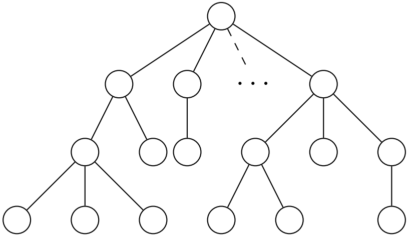

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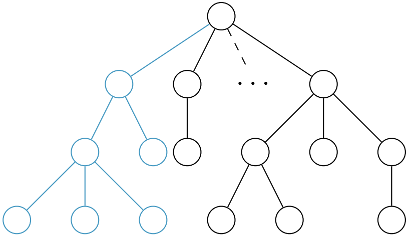

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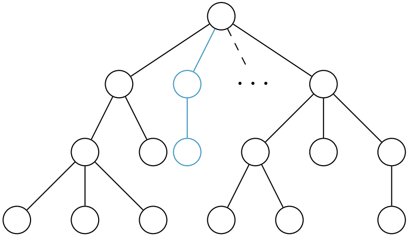

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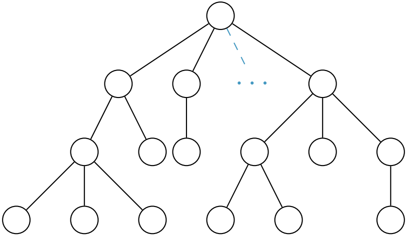

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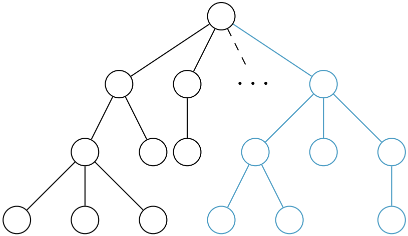

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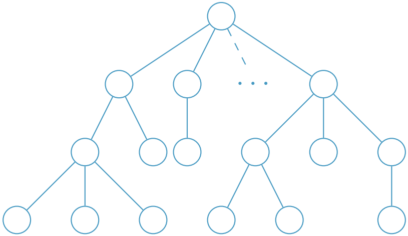

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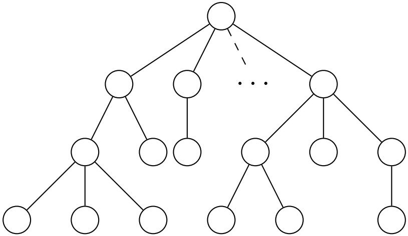

- Wagner (2015), Janson (2016), Wagner et al. (2018)…:
  - τ_n random tree, size n; f *suitable*
    $\rightsquigarrow F(\tau_n)$ asymptotically Gaussian

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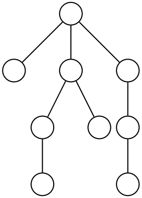

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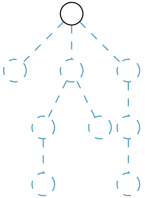

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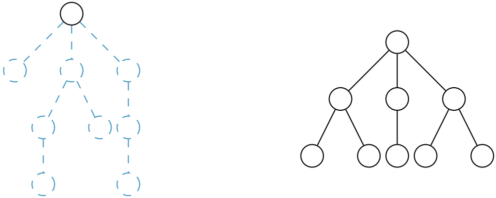

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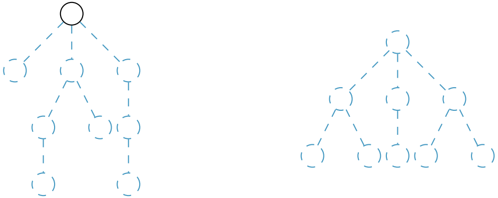

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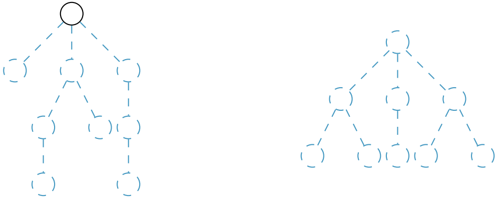


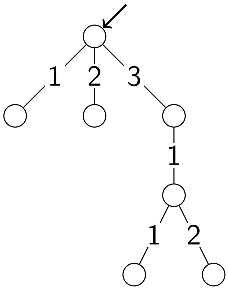
"Cutting Leaves" – Toll Function

$$f(\tau) = \begin{cases} 0 & \text{if } \tau \text{ has height } > r, \\ 1 & \text{if } \tau \text{ has height } \leq r. \end{cases}$$

Simply Generated Trees

Plane Trees:

- *rooted*: designated root node
- *ordered*: order of children matters



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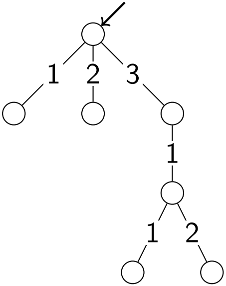

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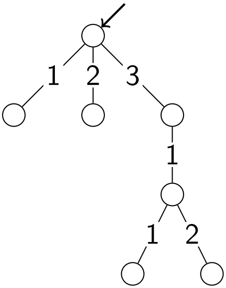

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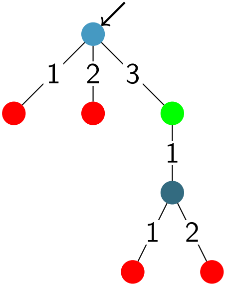

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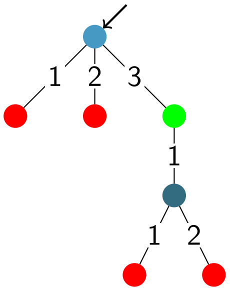

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- *partition function* and *probability distribution*:

$$Z_n = Z_n(\mathbf{w}) := \sum_{\tau \in \mathcal{T}_n} w(\tau) \quad \Rightarrow \quad \mathbb{P}(\tau_n = \tau) = \frac{w(\tau)}{Z_n}$$

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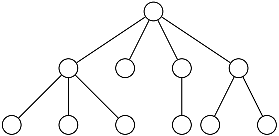

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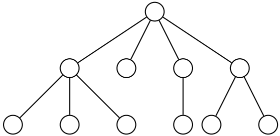

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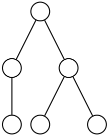
Unary-Binary Trees



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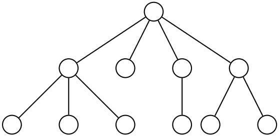
Simply Generated Trees: Examples

Plane Trees



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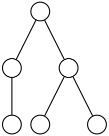

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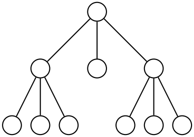

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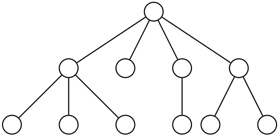

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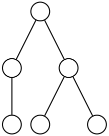

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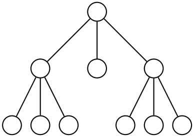

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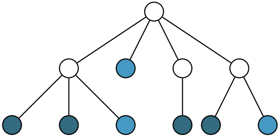
k-colored leaves



$w_0 = k,\ w_1 = \cdots = 1$

Recursive Characterization

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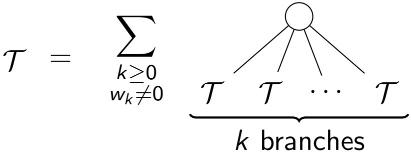

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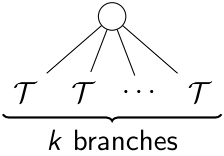

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Recursive Characterization – Proof

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$$T(z) \overset{z\to\rho}{=} \tau - \sqrt{\frac{2\tau}{\rho\Phi''(\tau)}}\sqrt{1 - z/\rho} + O(1 - z/\rho)$$

Result – Simply Generated Trees

Theorem (H.–Heuberger–Prodinger–Wagner, '18+)

- *weight sequence* $\mathbf{w}$, *fundamental constant* τ, $\rho = \tau/\Phi(\tau)$
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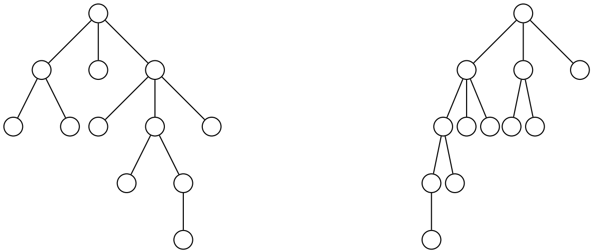
Technical Detail. $\mathbf{w}$ periodic $\rightsquigarrow$ parity!

Pólya Trees

- Rooted trees, order of children is **irrelevant!**



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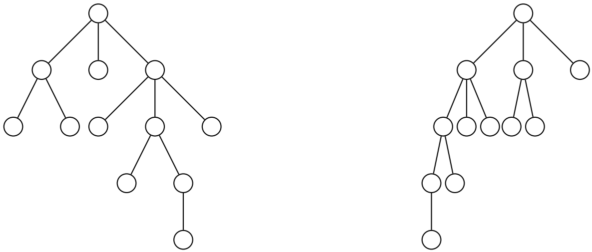

Proposition

- $\mathcal{T}\ldots$ *Pólya trees,* $T(z)\ldots$ *generating function*

$$\Rightarrow T(z) = z \exp\Big(\sum_{k\geq 1} \frac{1}{k} T(z^k)\Big)$$

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- $R(z,u) := \sum_{k\geq 2} \frac{1}{k} T(z^k,u^k)$. Then: $R(z) = R(z,1)$ has larger RoC than $T(z)$

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Note. $\frac{2}{1+\rho R'(\rho)} \approx 1.644731$, $\frac{1}{3(1+\rho R'(\rho))} \approx 0.274122$

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