Iterative Cutting of Non-Catalan Trees
Example: Deterministic Plane Tree Reduction

- Remove all leaves!
Example: Deterministic Plane Tree Reduction

- Remove all leaves!
Example: Deterministic Plane Tree Reduction

- Remove all leaves!
Example: Deterministic Plane Tree Reduction

- Remove all leaves!

\[
\begin{align*}
\text{Initial Tree:} & \quad \Rightarrow \\
\text{After first reduction:} & \quad \Rightarrow \\
\text{After second reduction:} & \quad \Rightarrow \\
\text{Final Tree:} & \quad \Rightarrow 
\end{align*}
\]
Example: Deterministic Plane Tree Reduction

- Remove all leaves!

Parameter of Interest:

Size of $r$th reduction $\leftrightarrow$ # of removed nodes
Example: Deterministic Plane Tree Reduction

- Remove all leaves!

Parameter of Interest:
- Size of $r$th reduction $\leftrightarrow$ # of removed nodes
Reduction $\rightarrow$ Expansion

- modelling reduction directly: not suitable
Reduction $\rightarrow$ Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, **growing trees**
Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, *growing trees*
Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, growing trees

\[ \text{...} \]

\[ \text{...} \]
Expansion operators

- $F$... family of plane trees; bivariate generating function $f$
- expansion operator $\Phi \implies \Phi(f)$ counts expanded trees
Expansion operators

- $F$ family of plane trees; bivariate generating function $f$
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

Leaf expansion $\Phi$

- inverse operation to leaf reduction
Expansion operators

- $F \ldots$ family of plane trees; bivariate generating function $f$
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

Leaf expansion $\Phi$

- inverse operation to leaf reduction
  - attach leaves to all current leaves (required)
  - attach leaves to inner nodes (optional)

\[
\begin{array}{c}
\square \xrightarrow{\Phi} \quad \begin{array}{c}
\square \\
\square \ \square
\end{array} + \begin{array}{c}
\square \\
\square \ \square \ \square
\end{array} + \begin{array}{c}
\square \\
\square \ \square \ \square \ \square
\end{array} + \cdots
\end{array}
\]
Expansion operators

- $F\ldots$ family of plane trees; bivariate generating function $f$
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

Leaf expansion $\Phi$

- inverse operation to leaf reduction
  - attach leaves to all current leaves (required)
  - attach leaves to inner nodes (optional)

\[
\begin{align*}
\square & \xrightarrow{\Phi} \square + \square \quad + \quad \square \quad + \ldots \\
\square & \triangleq t, \quad \circ \triangleq z \quad \Rightarrow \quad \Phi(t) = zt + zt^2 + zt^3 + \ldots
\end{align*}
\]
Reductions on Plane Trees

Leaves

\[ E \sim \frac{n}{r+1} \]
\[ V \sim \frac{r(r+2)}{6(r+1)^2} n \]

limit law: ✓
Reductions on Plane Trees

**Leaves**
\[
E \sim \frac{n}{r+1} \\
V \sim \frac{r(r+2)}{6(r+1)^2} n
\]
limit law: ✓

**Paths**
\[
E \sim \frac{n}{2^{r+1}-1} \\
V \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n
\]
limit law: ✓
Reductions on Plane Trees

**Leaves**

\[ E \sim \frac{n}{r+1} \]
\[ V \sim \frac{r(r+2)}{6(r+1)^2} n \]

limit law: ✓

**Paths**

\[ E \sim \frac{n}{2^{r+1}-1} \]
\[ V \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n \]

limit law: ✓

**Old leaves**

\[ E \sim (2 - B_{r-1}(1/4))n \]
\[ V = \Theta(n) \]

limit law: ✓
Reductions on Plane Trees

**Leaves**

\[
E \sim \frac{n}{r+1} \\
V \sim \frac{r(r+2)}{6(r+1)^2} n
\]

limit law: ✓

**Paths**

\[
E \sim \frac{n}{2^{r+1}-1} \\
V \sim \frac{2^{r+1}(2^r - 1)}{3(2^{r+1}-1)^2} n
\]

limit law: ✓

**Old leaves**

\[
E \sim (2 - B_{r-1}(1/4))n \\
V = \Theta(n)
\]

limit law: ✓

**Old paths**

\[
E \sim \frac{2n}{r+2} \\
V \sim \frac{2r(r+1)}{3(r+2)^2} n
\]

limit law: ✓
 reductions on plane trees

leaves

\[ E \sim \frac{n}{r+1} \]
\[ V \sim \frac{r(r+2)}{6(r+1)^2} n \]

limit law: ✓

paths

\[ E \sim \frac{n}{2^{r+1}-1} \]
\[ V \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n \]

limit law: ✓

old leaves

\[ E \sim (2 - B_{r-1}(1/4)) n \]
\[ V = \Theta(n) \]

limit law: ✓

old paths

\[ E \sim \frac{2n}{r+2} \]
\[ V \sim \frac{2r(r+1)}{3(r+2)^2} n \]

limit law: ✓

question

what can be done for other tree classes?
Additive Tree Parameters

\[ \mathcal{T} \ldots \text{rooted trees.} \quad F : \mathcal{T} \rightarrow \mathbb{R} \ldots \text{additive tree parameter, if} \]

\[ \tau \in \mathcal{T} \quad \text{tree;} \quad \tau_1, \tau_2, \ldots, \tau_k \quad \text{branches of} \quad \tau \]
Additive Tree Parameters

\[ \mathcal{T} \ldots \text{rooted trees. } F: \mathcal{T} \rightarrow \mathbb{R} \ldots \text{additive tree parameter, if} \]

\begin{itemize}
  \item \( \tau \in \mathcal{T} \) tree; \( \tau_1, \tau_2, \ldots, \tau_k \) branches of \( \tau \)
  \item \( F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau) \),
  \item with toll function \( f: \mathcal{T} \rightarrow \mathbb{R} \)
\end{itemize}
Additive Tree Parameters

\( \mathcal{T} \) ... rooted trees. \( F : \mathcal{T} \rightarrow \mathbb{R} \) ... additive tree parameter, if

- \( \tau \in \mathcal{T} \) tree; \( \tau_1, \tau_2, \ldots, \tau_k \) branches of \( \tau \)
- \( F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau) \),
  - with toll function \( f : \mathcal{T} \rightarrow \mathbb{R} \)
Additive Tree Parameters

$\mathcal{T}$ ... rooted trees. $F : \mathcal{T} \to \mathbb{R}$ ... additive tree parameter, if

- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of $\tau$
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
  - with toll function $f : \mathcal{T} \to \mathbb{R}$
Additive Tree Parameters

$\mathcal{T}$ ... rooted trees. $F: \mathcal{T} \rightarrow \mathbb{R}$ ... additive tree parameter, if

- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of $\tau$
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
  - with toll function $f: \mathcal{T} \rightarrow \mathbb{R}$
Additive Tree Parameters

$\mathcal{T}$... rooted trees. $F : \mathcal{T} \rightarrow \mathbb{R}$ ... additive tree parameter, if

- $\tau \in \mathcal{T}$ tree; $\tau_1, \tau_2, \ldots, \tau_k$ branches of $\tau$
- $F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau)$,
  - with toll function $f : \mathcal{T} \rightarrow \mathbb{R}$
Additive Tree Parameters

\( \mathcal{T} \) ... rooted trees. \( F : \mathcal{T} \to \mathbb{R} \) ... additive tree parameter, if

- \( \tau \in \mathcal{T} \) tree; \( \tau_1, \tau_2, \ldots, \tau_k \) branches of \( \tau \)
- \( F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau) \),
  - with toll function \( f : \mathcal{T} \to \mathbb{R} \)
Additive Tree Parameters

\[ \mathcal{T} \] ... rooted trees. \( F : \mathcal{T} \rightarrow \mathbb{R} \) ... additive tree parameter, if

- \( \tau \in \mathcal{T} \) tree; \( \tau_1, \tau_2, \ldots, \tau_k \) branches of \( \tau \)
- \( F(\tau) = F(\tau_1) + F(\tau_2) + \cdots + F(\tau_k) + f(\tau) \),
  with toll function \( f : \mathcal{T} \rightarrow \mathbb{R} \)

- Wagner (2015), Janson (2016), Wagner et al. (2018)...:
  - \( \tau_n \) random tree, size \( n \); \( f \) suitable
    \( \rightsquigarrow F(\tau_n) \) asymptotically Gaussian
Example: Removed Leaves

- $a_r(\tau) \mapsto$ # of removed nodes when cutting leaves $r$ times
Example: Removed Leaves

- $a_r(\tau) \rightarrow \# \text{ of removed nodes when cutting leaves } r \text{ times}$
- **Example**: $r = 3$

![Diagram showing a tree with labeled nodes](image)
Example: Removed Leaves

- $a_r(\tau) \sim \#$ of removed nodes when cutting leaves $r$ times
- **Example:** $r = 3$
Example: Removed Leaves

- $a_r(\tau) \sim \#$ of removed nodes when cutting leaves $r$ times
- **Example:** $r = 3$
Example: Removed Leaves

- \( a_r(\tau) \sim \# \) of removed nodes when cutting leaves \( r \) times
- **Example**: \( r = 3 \)
Example: Removed Leaves

- $a_r(\tau) \rightsquigarrow \# \text{ of removed nodes when cutting leaves } r \text{ times}$
- **Example:** $r = 3$

```
"Cutting Leaves" – Toll Function

\[
f(\tau) = \begin{cases} 
0 & \text{if } \tau \text{ has height } > r, \\
1 & \text{if } \tau \text{ has height } \leq r.
\end{cases}
\]
```
Simply Generated Trees

Plane Trees:

- **rooted**: designated root node
- **ordered**: order of children matters

\[
\begin{align*}
\text{rooted: designated root node} \\
\text{ordered: order of children matters}
\end{align*}
\]
Simply Generated Trees

Plane Trees:

- *rooted*: designated root node
- *ordered*: order of children matters

Generalization:

- *weight sequence*: \( w = (w_k)_{k \geq 0} \subseteq \mathbb{R}_{\geq 0} \)
Simply Generated Trees

Plane Trees:
- rooted: designated root node
- ordered: order of children matters

Generalization:
- weight sequence: $w = (w_k)_{k \geq 0} \subseteq \mathbb{R}_{\geq 0}$
- weight of tree $\tau$:
  $$w(\tau) := \prod_{\nu \text{ node in } \tau} w(\# \text{ children of } \nu)$$
Simply Generated Trees

Plane Trees:

- *rooted*: designated root node
- *ordered*: order of children matters

Generalization:

- *weight sequence*: \( w = (w_k)_{k \geq 0} \subseteq \mathbb{R}_{\geq 0} \)
- *weight of tree* \( \tau \):

\[
\mathcal{W}(\tau) : = \prod_{\nu \text{ node in } \tau} w(# \text{ children of } \nu)
\]

\[
\mathcal{W}(\tau) = w_0^4 w_1 w_2 w_3
\]
**Simply Generated Trees**

**Plane Trees:**
- *rooted*: designated root node
- *ordered*: order of children matters

**Generalization:**
- *weight sequence*: \( w = (w_k)_{k \geq 0} \subseteq \mathbb{R}_{\geq 0} \)
- *weight of tree* \( \tau \):

\[
    w(\tau) := \prod_{\nu \text{ node in } \tau} w(\# \text{ children of } \nu)
\]

\[
    w(\tau) = w_0^4 w_1 w_2 w_3
\]

- *partition function and probability distribution*:

\[
    Z_n = Z_n(w) := \sum_{\tau \in T_n} w(\tau) \quad \Rightarrow \quad \mathbb{P}(\tau_n = \tau) = \frac{w(\tau)}{Z_n}
\]
Simply Generated Trees: Examples

**Plane Trees**

\[ w_0 = w_1 = \cdots = 1 \]
Simply Generated Trees: Examples

**Plane Trees**

\[ w_0 = w_1 = \cdots = 1 \]

**Unary-Binary Trees**

\[ w_0 = w_1 = w_2 = 1 \]
Simply Generated Trees: Examples

**Plane Trees**

```
    w_0 = w_1 = \cdots = 1
```

**Unary-Binary Trees**

```
    w_0 = w_1 = w_2 = 1
```

**d-ary Trees**

```
    w_0 = w_d = 1
```
Simply Generated Trees: Examples

**Plane Trees**

\[ w_0 = w_1 = \cdots = 1 \]

**Unary-Binary Trees**

\[ w_0 = w_1 = w_2 = 1 \]

**d-ary Trees**

\[ w_0 = w_d = 1 \]

**k-colored leaves**

\[ w_0 = k, \ w_1 = \cdots = 1 \]
Recursive Characterization

**Proposition**

\[ T(z, u) \ldots BWGF (z\ldots tree size, u\ldots removed nodes) \]
Recursive Characterization

**Proposition**

- $T(z, u)\ldots$ BWGF ($z\ldots$ tree size, $u\ldots$ removed nodes)
- $\Phi(t)\ldots$ GF of $w$, $\Phi(t) = \sum_{k \geq 0} w_k t^k$
Recursive Characterization

Proposition

- \( T(z, u) \) \( \ldots \) BWGF (\( z \ldots \) tree size, \( u \ldots \) removed nodes)
- \( \Phi(t) \) \( \ldots \) GF of \( w \), \( \Phi(t) = \sum_{k \geq 0} w_k t^k \)
- \( T_r(z) \) \( \ldots \) WGF of trees of height \( < r \)
Recursive Characterization

Proposition

- \( T(z, u) \ldots \) BWGF (\( z \ldots \) tree size, \( u \ldots \) removed nodes)
- \( \Phi(t) \ldots \) GF of \( w \), \( \Phi(t) = \sum_{k \geq 0} w_k t^k \)
- \( T_r(z) \ldots \) WGF of trees of height \( < r \)

Then:

\[
T(z, u) = z\Phi(T(z, u)) + \left(1 - \frac{1}{u}\right)T_r(zu)
\]
Recursive Characterization

**Proposition**

- \( T(z, u) \) \( BWGF \) \( (z \ldots \text{tree size}, u \ldots \text{removed nodes}) \)
- \( \Phi(t) \) \( GF \) \( of \ w \), \( \Phi(t) = \sum_{k \geq 0} w_k t^k \)
- \( T_r(z) \) \( WGF \) \( of \ trees of height < r \)

Then:

\[
T(z, u) = z\Phi(T(z, u)) + \left(1 - \frac{1}{u}\right)T_r(zu)
\]

**Observation.** \( \mathcal{T} \ldots w \)-simply generated trees, \( GF \ T(z) = T(z, 1) \)
Recursive Characterization

Proposition

▶ $T(z, u)$ . . . BWGF (z . . . tree size, u . . . removed nodes)
▶ $\Phi(t)$ . . . GF of $w$, $\Phi(t) = \sum_{k \geq 0} w_k t^k$
▶ $T_r(z)$ . . . WGF of trees of height $< r$

Then:

$$T(z, u) = z\Phi(T(z, u)) + \left(1 - \frac{1}{u}\right) T_r(zu)$$

Observation. $\mathcal{T}$ . . . $w$-simply generated trees, GF $T(z) = T(z, 1)$

$$\mathcal{T} = \sum_{k \geq 0 \atop w_k \neq 0} \mathcal{T} \underbrace{\mathcal{T} \mathcal{T} \cdots \mathcal{T}}_{k \text{ branches}}$$
Recursive Characterization

**Proposition**

- \( T(z, u) \) \( \ldots \) BWGF (\( z \ldots \) tree size, \( u \ldots \) removed nodes)
- \( \Phi(t) \) \( \ldots \) GF of \( w \), \( \Phi(t) = \sum_{k \geq 0} w_k t^k \)
- \( T_r(z) \) \( \ldots \) WGF of trees of height \(< r\)

Then:

\[
T(z, u) = z\Phi(T(z, u)) + \left(1 - \frac{1}{u}\right) T_r(zu)
\]

**Observation.** \( T \ldots w \)-simply generated trees, GF \( T(z) = T(z, 1) \)

\[
T = \sum_{k \geq 0} \sum_{w_k \neq 0} T \ldots T \ldots T \\
\text{k branches}
\Rightarrow \quad T(z) = z\Phi(T(z))
\]
Recursive Characterization – Proof

**Functional Equation**

\[ T(z, u) = z\Phi(T(z, u)) + \left(1 - \frac{1}{u}\right)T_r(zu) \]

**Proof.** \(T_r\) ... trees with height less than \(r\)
Recursive Characterization – Proof

**Functional Equation**

\[ T(z, u) = z\Phi(T(z, u)) + \left( 1 - \frac{1}{u} \right) T_r(zu) \]

**Proof.** \( T_r \ldots \) trees with height less than \( r \)

\[ T(z, u) = \sum_{\tau \in T} w(\tau)z^{\mid \tau \mid}u^{a_r(\tau)} \]
Recursive Characterization – Proof

Functional Equation

\[ T(z, u) = z\Phi(T(z, u)) + \left(1 - \frac{1}{u}\right)T_r(zu) \]

**Proof.** \( T_r \ldots \) trees with height less than \( r \)

\[ T(z, u) = \sum_{\tau \in \mathcal{T}} w(\tau)z^{\mid\tau\mid}u^{ar(\tau)} \]

\[ = \sum_{k \geq 0} w_k \sum_{\tau_1} \cdots \sum_{\tau_k} \left(\prod_{j=1}^{k} w(\tau_j)\right)z^{1+\sum_{j=1}^{k} |\tau_j|}u^{\sum a_r(\tau_j)} \]

\[ + \sum_{\tau \in \mathcal{T}_r} w(\tau)z^{\mid\tau\mid}(u^{\mid\tau\mid} - u^{\mid\tau\mid}^{-1}) \]
Recursive Characterization – Proof

Functional Equation

\[ T(z, u) = z\Phi(T(z, u)) + \left(1 - \frac{1}{u}\right)T_r(zu) \]

**Proof.** \( T_r \ldots \) trees with height less than \( r \)

\[
T(z, u) = \sum_{\tau \in T} w(\tau)z^{|\tau|}u^{a_r(\tau)}
\]

\[
= \sum_{k \geq 0} w_k \sum_{\tau_1} \cdots \sum_{\tau_k} \left( \prod_{j=1}^{k} w(\tau_j) \right) z^{1+\sum|\tau_j|}u^{\sum a_r(\tau_j)}
\]

\[
+ \sum_{\tau \in T_r} w(\tau)z^{|\tau|}(u^{|\tau|} - u^{|\tau|-1})
\]

\[
= \cdots = z\Phi(T(z, u)) + \left(1 - \frac{1}{u}\right)T_r(zu).
\]
Singular Expansion

- **Goal:** extract information from functional equation
Singular Expansion

- **Goal:** extract information from functional equation
- In particular: partial derivatives $\partial_u T(z, 1)$, $\partial_{uu} T(z, 1)$
Singular Expansion

- **Goal**: extract information from functional equation
- In particular: partial derivatives $\partial_u T(z, 1), \partial_{uu} T(z, 1)$
- Via implicit differentiation:

$$\partial_u T(z, 1) = \frac{z T'(z) T_r(z)}{T(z)}$$
Singular Expansion

- **Goal:** extract information from functional equation
- In particular: partial derivatives $\partial_u T(z, 1)$, $\partial_{uu} T(z, 1)$
- Via implicit differentiation:

$$\partial_u T(z, 1) = \frac{z T'(z) T_r(z)}{T(z)}$$

- $T(z)$ satisfies $T(z) = z \Phi(T(z)) \leadsto$ singular inversion!
Singular Expansion

► **Goal:** extract information from functional equation
► In particular: partial derivatives $\partial_u T(z, 1), \partial_{uu} T(z, 1)$
► Via implicit differentiation:

$$\partial_u T(z, 1) = \frac{z T'(z) T_r(z)}{T(z)}$$

► $T(z)$ satisfies $T(z) = z\Phi(T(z)) \leadsto$ singular inversion!
  ► *fundamental constant* $\tau > 0$: solution of $\tau \Phi'(\tau) - \Phi(\tau) = 0$
Singular Expansion

- **Goal:** extract information from functional equation
- In particular: partial derivatives $\partial_u T(z, 1), \partial_{uu} T(z, 1)$
- Via implicit differentiation:

  $$\partial_u T(z, 1) = \frac{zT'(z)T_r(z)}{T(z)}$$

- $T(z)$ satisfies $T(z) = z\Phi(T(z)) \sim\sim$ singular inversion!
  - *fundamental constant* $\tau > 0$: solution of $\tau \Phi'(\tau) - \Phi(\tau) = 0$
  - $T$ has square root singularity at $\rho = \tau/\Phi(\tau)$
Singular Expansion

- **Goal**: extract information from functional equation
- In particular: partial derivatives $\partial_u T(z, 1), \partial_{uu} T(z, 1)$
- Via implicit differentiation:

$$
\partial_u T(z, 1) = \frac{z T'(z) T_r(z)}{T(z)}
$$

- $T(z)$ satisfies $T(z) = z \Phi(T(z)) \leadsto$ singular inversion!
  - *fundamental constant* $\tau > 0$: solution of $\tau \Phi'(\tau) - \Phi(\tau) = 0$
  - $T$ has square root singularity at $\rho = \tau / \Phi(\tau)$

$$
T(z) \xrightarrow{z \to \rho} \tau - \sqrt{\frac{2\tau}{\rho \Phi''(\tau)}} \sqrt{1 - z/\rho} + O(1 - z/\rho)
$$
Result – Simply Generated Trees

Theorem (H.–Heuberger–Prodinger–Wagner, ’18+)

- weight sequence $w$, fundamental constant $\tau$, $\rho = \tau / \Phi(\tau)$
- $X_{n,r} \ldots$ # of removed nodes after cutting leaves $r$ times

Then:

- $X_{n,r} \ldots$ is asymptotically normally distributed for $n \to \infty$
- $E[X_{n,r}] = \tau \cdot n + O(1)$
- $V[X_{n,r}] = \sigma^2 r \cdot n + O(1)$

and for $r \to \infty$

- $F_r(\rho) = 1 - 2 \rho \tau \Phi''(\tau) - 1 + o(r - 1)$
- $\sigma^2 r = \frac{1}{3} \rho \tau \Phi''(\tau) + o(1)$
Result – Simply Generated Trees

Theorem (H.–Heuberger–Prodinger–Wagner, ’18+)

- weight sequence $w$, fundamental constant $\tau$, $\rho = \tau / \Phi(\tau)$
- $X_{n,r} \ldots$ # of removed nodes after cutting leaves $r$ times

Then:

- $X_{n,r}$ is asymptotically normally distributed for $n \to \infty$
Result – Simply Generated Trees

Theorem (H.–Heuberger–Prodinger–Wagner, ’18+)

- weight sequence $w$, fundamental constant $\tau$, $\rho = \tau / \Phi(\tau)$
- $X_{n,r}\ldots$ # of removed nodes after cutting leaves $r$ times

Then:

- $X_{n,r}$ is asymptotically normally distributed for $n \to \infty$ with mean and variance

$$\mathbb{E} X_{n,r} = \frac{T_r(\rho)}{\tau} \cdot n + O(1), \quad \forall X_{n,r} = \sigma_r^2 \cdot n + O(1)$$
Result – Simply Generated Trees

Theorem (H.–Heuberger–Prodinger–Wagner, ’18+)

- weight sequence $w$, fundamental constant $\tau$, $\rho = \tau / \Phi(\tau)$
- $X_{n,r} \ldots \#$ of removed nodes after cutting leaves $r$ times

Then:

- $X_{n,r}$ is asymptotically normally distributed for $n \to \infty$ with mean and variance
  $$E X_{n,r} = \frac{T_r(\rho)}{\tau} \cdot n + O(1), \quad \forall X_{n,r} = \sigma_r^2 \cdot n + O(1)$$

- and for $r \to \infty$ we have
  $$F_r(\rho) = 1 - \frac{2}{\rho \tau \Phi''(\tau)} r^{-1} + o(r^{-1}), \quad \sigma_r^2 = \frac{1}{3\rho \tau \Phi''(\tau)} + o(1).$$
Result – Simply Generated Trees

Theorem (H.-Heuberger–Prodinger–Wagner, ’18+)

- weight sequence $w$, fundamental constant $\tau$, $\rho = \tau / \Phi(\tau)$
- $X_{n,r} \ldots$ # of removed nodes after cutting leaves $r$ times

Then:

- $X_{n,r}$ is asymptotically normally distributed for $n \to \infty$ with mean and variance
  \[ \mathbb{E}X_{n,r} = \frac{T_r(\rho)}{\tau} \cdot n + O(1), \quad \forall X_{n,r} = \sigma_r^2 \cdot n + O(1) \]
- and for $r \to \infty$ we have
  \[ \frac{F_r(\rho)}{\tau} = 1 - \frac{2}{\rho \tau \Phi''(\tau)} r^{-1} + o(r^{-1}), \quad \sigma_r^2 = \frac{1}{3 \rho \tau \Phi''(\tau)} + o(1). \]

Technical Detail. $w$ periodic $\Rightarrow$ parity!
Pólya Trees

- Rooted trees, order of children is **irrelevant!**
Pólya Trees

- Rooted trees, order of children is irrelevant!

\[ T(z) = z \exp \left( \sum_{k \geq 1} \frac{1}{k} T(z^k) \right) \]
Recursive Characterization / Functional Equation

Proposition

\[ T(z, u) \ldots \text{BGF (} z \ldots \text{tree size, } u \ldots \text{removed nodes} \) \]
Recursive Characterization / Functional Equation

**Proposition**

- \( T(z, u) \ldots \text{BGF (}z\ldots \text{tree size, } u\ldots \text{removed nodes)} \)
- \( T_r(z) \ldots \text{GF of trees of height < } r \)
### Proposition

- \( T(z, u) \) \( \ldots \) BGF \( (z \ldots \text{tree size}, u \ldots \text{removed nodes}) \)
- \( T_r(z) \) \( \ldots \) GF of trees of height \( < r \)

Then:

\[
T(z, u) = z \exp \left( \sum_{k \geq 1} \frac{1}{k} T(z^k, u^k) \right) + \left( 1 - \frac{1}{u} \right) T_r(zu)
\]
Recursive Characterization / Functional Equation

Proposition

- $T(z, u)$... BGF ($z$... tree size, $u$... removed nodes)
- $T_r(z)$... GF of trees of height $< r$

Then:

$$T(z, u) = z \exp \left( \sum_{k \geq 1} \frac{1}{k} T(z^k, u^k) \right) + \left(1 - \frac{1}{u}\right) T_r(zu)$$

Key Observations.

- $R(z, u) := \sum_{k \geq 2} \frac{1}{k} T(z^k, u^k)$. Then: $R(z) = R(z, 1)$ has larger RoC than $T(z)$
Recursive Characterization / Functional Equation

**Proposition**

- \( T(z, u) \) is BGF (tree size, removed nodes)
- \( T_r(z) \) is GF of trees of height \(< r\)

Then:

\[
T(z, u) = z \exp \left( \sum_{k \geq 1} \frac{1}{k} T(z^k, u^k) \right) + \left( 1 - \frac{1}{u} \right) T_r(z u)
\]

**Key Observations.**

- \( R(z, u) := \sum_{k \geq 2} \frac{1}{k} T(z^k, u^k) \). Then: \( R(z) = R(z, 1) \) has larger RoC than \( T(z) \)
- implicit differentiation: \( \partial_u T(z, 1) = \frac{T(z) \partial_u R(z, 1) + T_r(z)}{1 - T(z)} \)
Recursive Characterization / Functional Equation

Proposition

- \( T(z, u) \) BGF (\( z \) tree size, \( u \) removed nodes)
- \( T_r(z) \) GF of trees of height \(< r\)

Then:

\[
T(z, u) = z \exp \left( \sum_{k \geq 1}^{\infty} \frac{1}{k} T(z^k, u^k) \right) + \left( 1 - \frac{1}{u} \right) T_r(zu)
\]

Key Observations.

- \( R(z, u) := \sum_{k \geq 2}^{\infty} \frac{1}{k} T(z^k, u^k) \). Then: \( R(z) = R(z, 1) \) has larger RoC than \( T(z) \)
- implicit differentiation: \( \partial_u T(z, 1) = \frac{T(z) \partial_u R(z, 1) + T_r(z)}{1 - T(z)} \)
- \( T(z) \approx \) dominant singularity for \( z \to \rho \approx 0.338322\)
Result – Pólya Trees

Theorem (H.–Heuberger–Prodinger–Wagner, ’18+)

- $\rho \approx 0.338322$ ... radius of convergence of $T(z)$
- $X_{n,r}$ ... # of removed nodes after cutting leaves $r$ times

Then:

Note. $\frac{2}{1 + \rho R'(\rho)} \approx 1.644731$,
$\frac{1}{3}(1 + \rho R'(\rho)) \approx 0.274122$.
Result – Pólya Trees

Theorem (H.–Heuberger–Prodinger–Wagner, ’18+)

- $\rho \approx 0.338322 \ldots$ radius of convergence of $T(z)$
- $X_{n,r} \ldots$ # of removed nodes after cutting leaves $r$ times

Then:

- for $n \to \infty$, $X_{n,r}$ has mean and variance

$$\mathbb{E}X_{n,r} = \mu_r \cdot n + O(1), \quad \mathbb{V}X_{n,r} = \sigma^2_r \cdot n + O(1)$$

Note. $\frac{1}{1 + \rho R'_{\rho}} \approx 1.644731$, $\frac{1}{3(1 + \rho R'_{\rho})} \approx 0.274122$
Result – Pólya Trees

Theorem (H.–Heuberger–Prodinger–Wagner, ’18+)

\( \rho \approx 0.338322 \ldots \) radius of convergence of \( T(z) \)

\( X_{n,r} \ldots \) # of removed nodes after cutting leaves \( r \) times

Then:

\( \mu_r = 1 - \frac{2}{(1 + \rho R'(\rho))} r^{-1} + o(r^{-1}), \quad \sigma_r^2 = \frac{1}{3(1 + \rho R'(\rho))} + o(1). \)
Result – Pólya Trees

Theorem (H.–Heuberger–Prodinger–Wagner, ’18+)

- \( \rho \approx 0.338322 \ldots \) radius of convergence of \( T(z) \)
- \( X_{n,r} \ldots \# \) of removed nodes after cutting leaves \( r \) times

Then:

- for \( n \to \infty \), \( X_{n,r} \) has mean and variance

\[
\mathbb{E} X_{n,r} = \mu_r \cdot n + O(1), \quad \mathbb{V} X_{n,r} = \sigma_r^2 \cdot n + O(1)
\]

- and for \( r \to \infty \) we have

\[
\mu_r = 1 - \frac{2}{(1 + \rho R'(\rho))} r^{-1} + o(r^{-1}), \quad \sigma_r^2 = \frac{1}{3(1 + \rho R'(\rho))} + o(1).
\]

Note.

\[
\frac{2}{1 + \rho R'(\rho)} \approx 1.644731, \quad \frac{1}{3(1 + \rho R'(\rho))} \approx 0.274122
\]