Benjamin Hackl

May 28, 2018

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT | WIEN GRAZ

Cutting Down and Growing Trees

mso DK modeling simulation optimization

KARL POPPER KOLLEG

FWF Der Wissenschaftsfonds.

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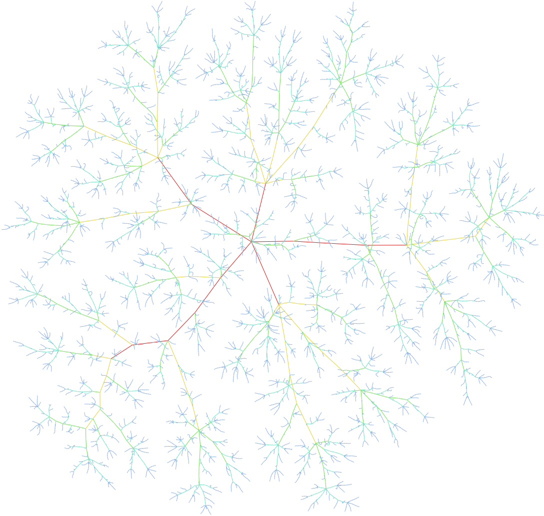

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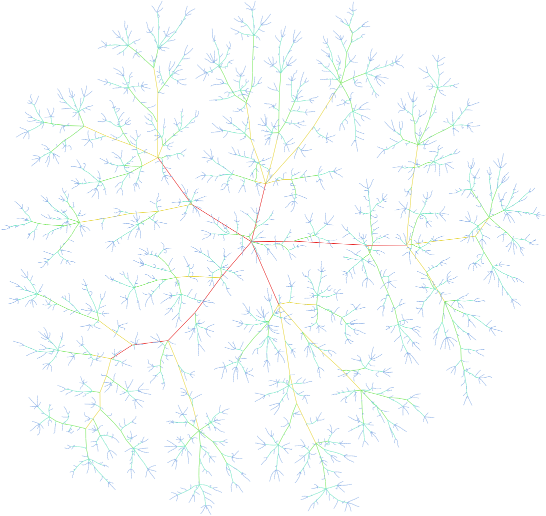

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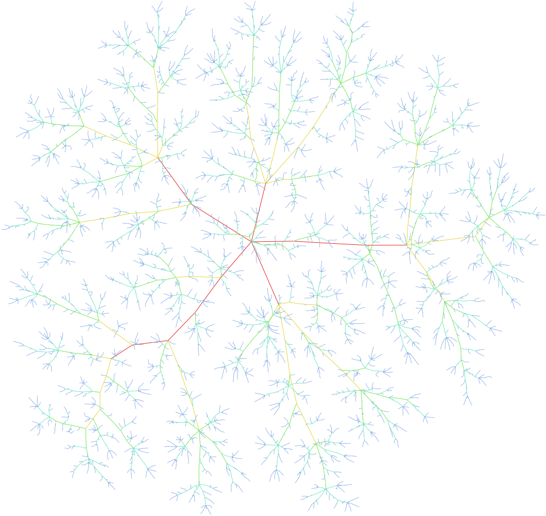

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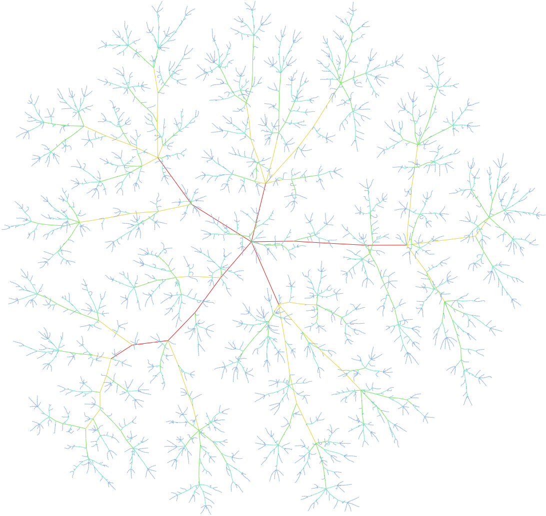

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"Asymptotic Analysis of Shape Parameters of Trees and Lattice Paths"

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Analytic Combinatorics. . .

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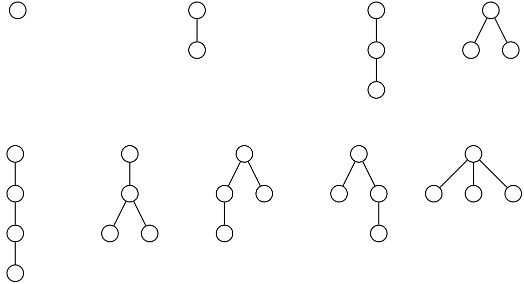

Analytic Combinatorics. . .

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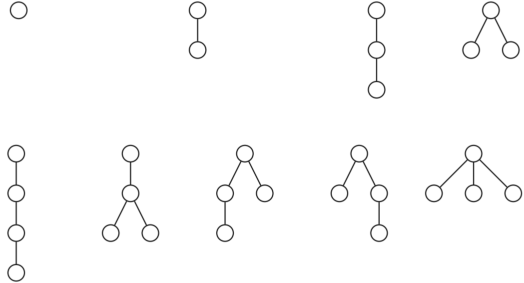
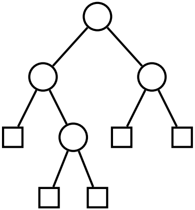


$$\frac{1-\sqrt{1-4z}}{2} = 1z + 1z^2 + 2z^3 + 5z^4 + 14z^5 + 42z^6 + \dots$$

Trimming Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent



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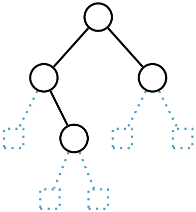

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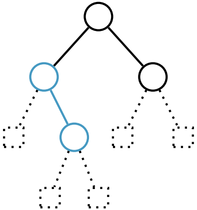

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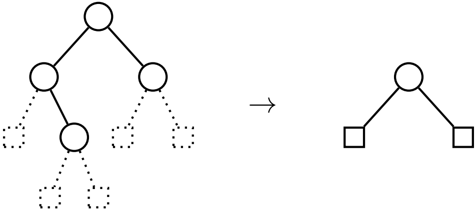

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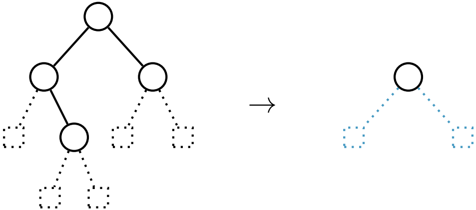

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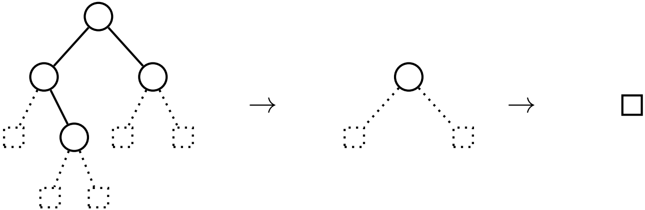

Which nodes „survive“?

We label the nodes according to the following rules:

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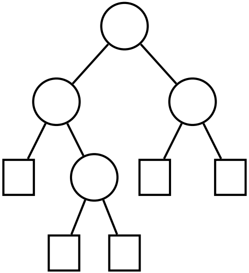

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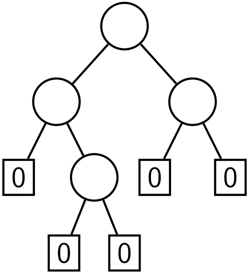

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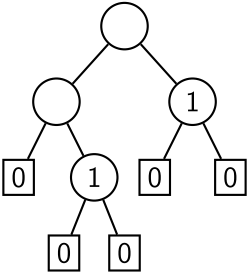

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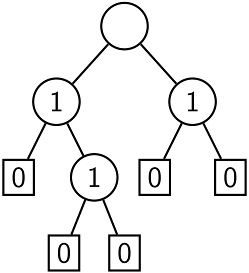

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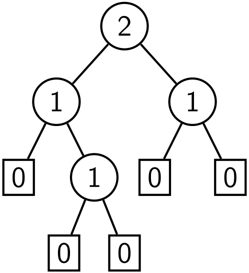

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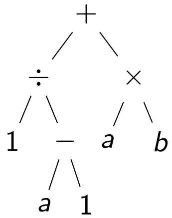

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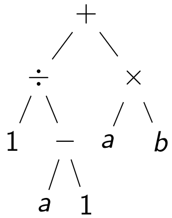

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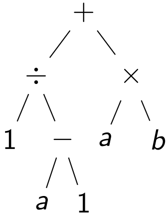

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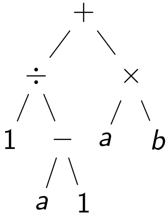

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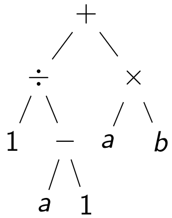

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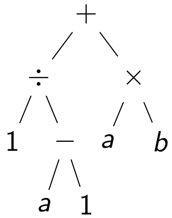

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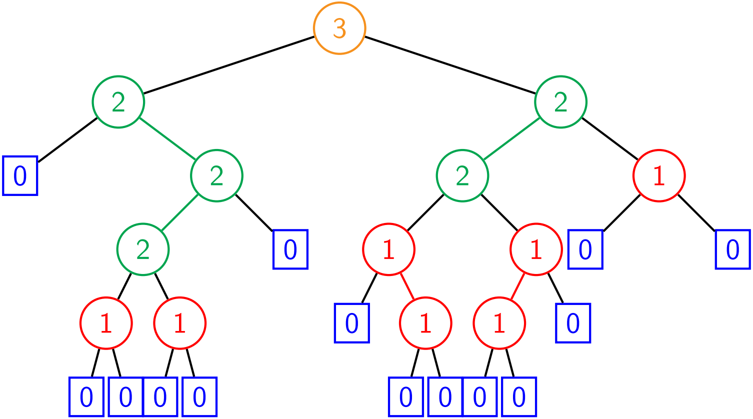
  - r-Branches, Numerics: Yamamoto, Yamazaki (2009)

Local Structures – “r-branches”: chains with same label



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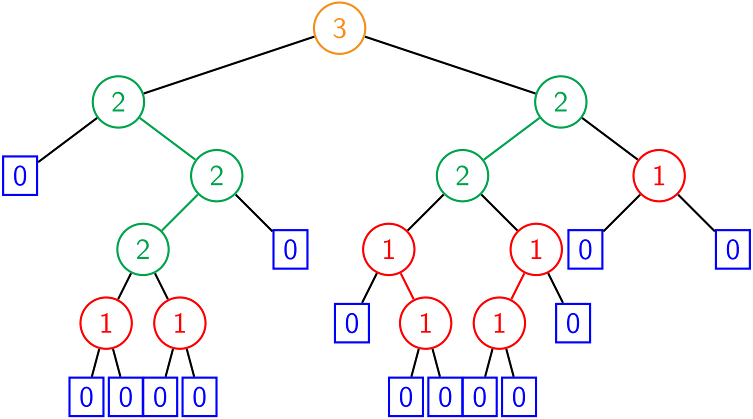

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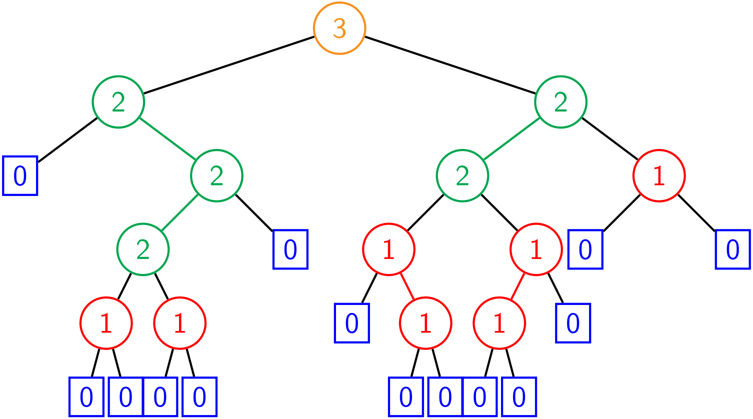

- Number / Distribution of (r-)branches?
- Example:

r	0	1	2	3
# r-branches	14	5	2	1

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“r-branches” – Results

Theorem (H.–Heuberger–Prodinger)

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  - $C \approx 1.36190$, δ... *periodic fluctuation*

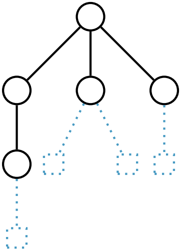
Size of r-fold reduced plane trees

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

$\mathbb{V} \sim \frac{r(r+2)}{6(r+1)^2} n$

limit law: ✓



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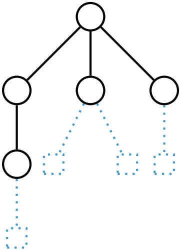

Paths

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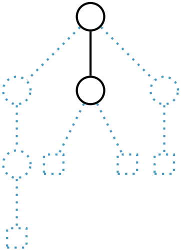

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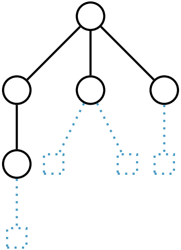

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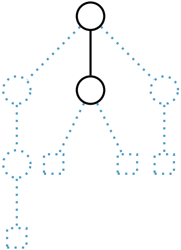

Old leaves

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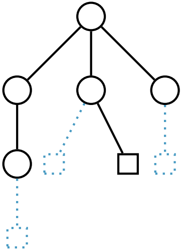

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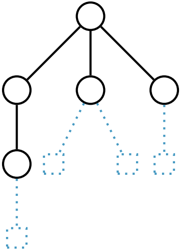

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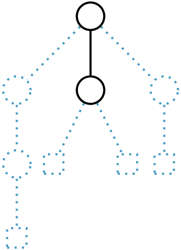

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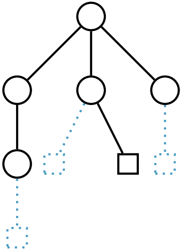

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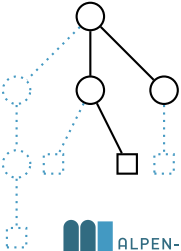

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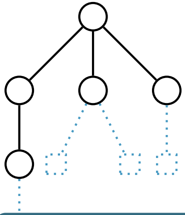

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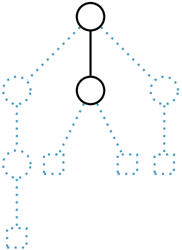

Disclaimer

Results are **not always** that nice!

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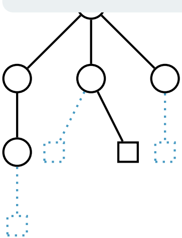

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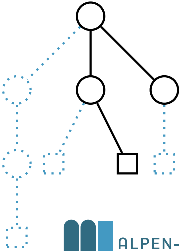
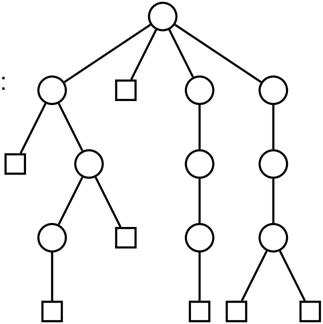


Branches in a Tree

- Trees can be partitioned into branches:



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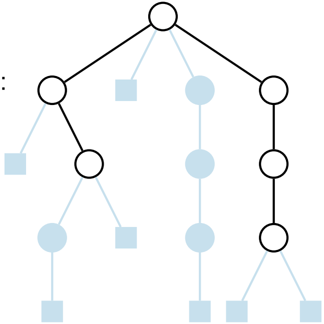

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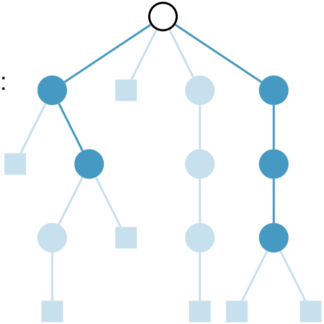

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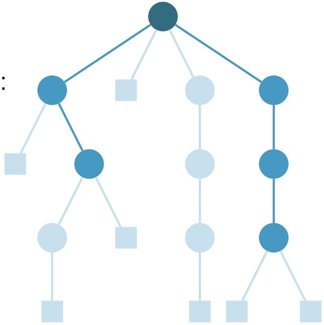

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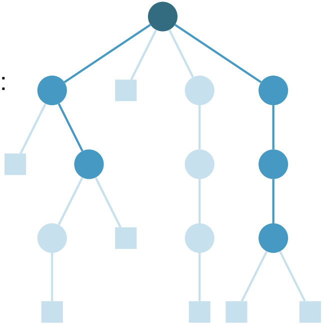
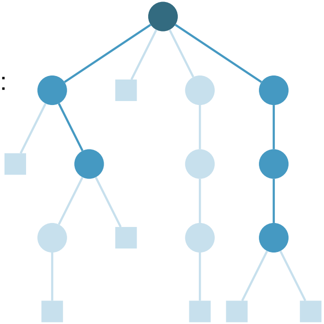


Observation

Total # of branches $\triangleq$ # of leaves in all reduction stages

Branches in a Tree

- Trees can be partitioned into branches:



Observation

Total # of branches $\triangleq$ # of leaves in all reduction stages

Proof: all branches end in exactly one leaf (at some point). □

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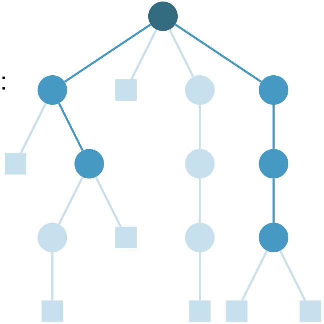

Observation

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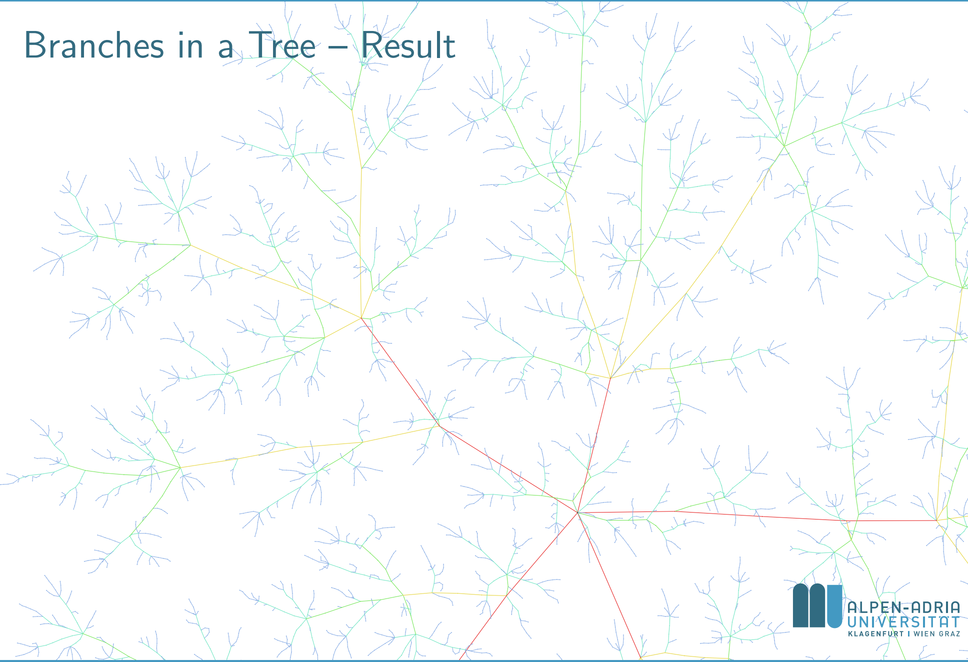
Proof: all branches end in exactly one leaf (at some point). □

- **Q:** How many branches are there?

Branches in a Tree – Result



Branches in a Tree – Result

Theorem (H.–Heuberger–Kropf–Prodinger)

Average # of branches in a random plane tree of size n is

$$\alpha n + \frac{1}{6}\log_4 n + C + \delta(\log_4 n) + O(n^{-1/4}),$$

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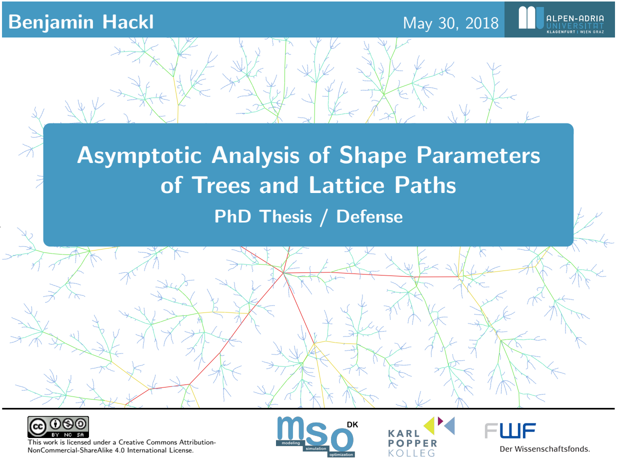
- $\delta \ldots$ *periodic fluctuation:*

$$\delta(x) := \frac{1}{\log 2}\sum_{k\in\mathbb{Z}\setminus\{0\}} (-1+\chi_k)\Gamma(\chi_k/2)\zeta(-1+\chi_k)e^{2k\pi i x}, \quad \chi_k = \frac{2\pi i k}{\log 2}$$

More about my thesis? ☺

- **Defense** @ 30.5., 11:00 / E.1.05

→



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