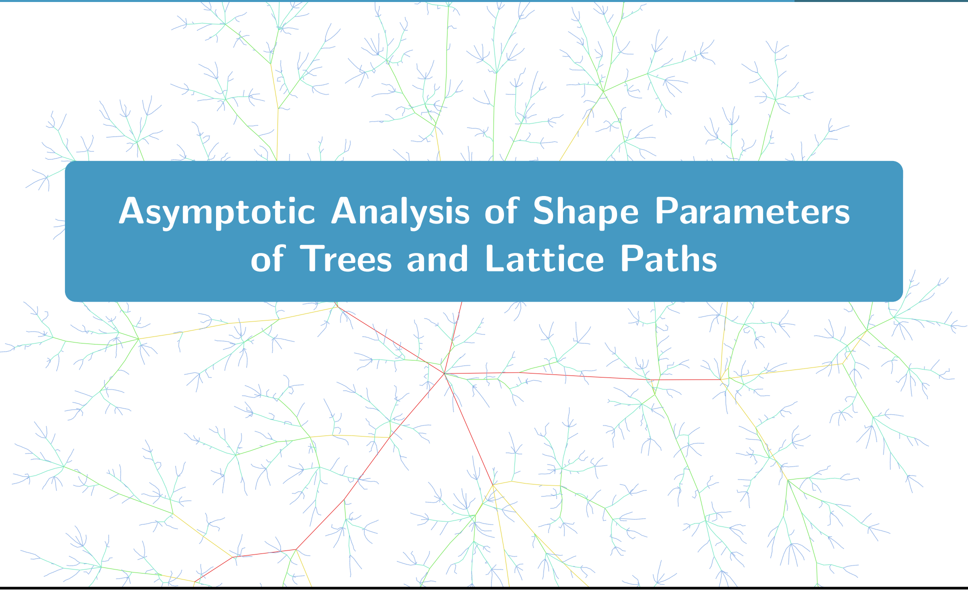
Benjamin Hackl

May 30, 2018

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT | WIEN GRAZ



Asymptotic Analysis of Shape Parameters of Trees and Lattice Paths

DK mso modeling simulation optimization

KARL POPPER KOLLEG

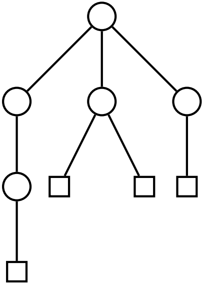
FWF Der Wissenschaftsfonds.

Thesis Overview

1. **Reductions of Binary Trees and Lattice Paths induced by the Register Function**
   - Joint work with Clemens Heuberger, Helmut Prodinger
   - Published: Theoret. Comput. Sci. **705** (2018), 31–57.
2. **Fringe Analysis of Plane Trees Related to Cutting and Pruning**
   - Joint work with Clemens Heuberger, Sara Kropf, Helmut Prodinger
   - Published: Aequationes Math. **92** (2018), 311–353.
3. **Growing and Destroying Catalan–Stanley Trees**
   - Joint work with Helmut Prodinger
   - Published: Discrete Math. Theor. Comput. Sci. **20** (2018).
4. **Ascents in Non-Negative Lattice Paths**
   - Joint work with Clemens Heuberger, Helmut Prodinger
   - arXiv:1801.02996 [math.CO]

Example: Deterministic Tree Reduction

- Remove all leaves!



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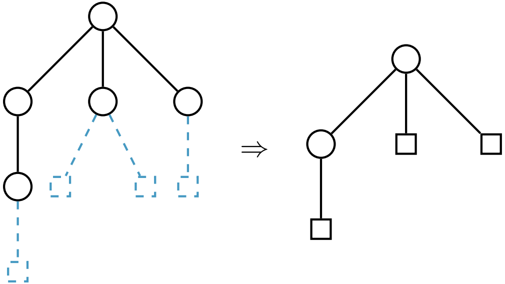

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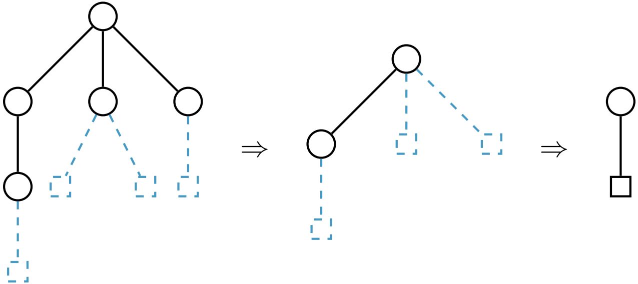

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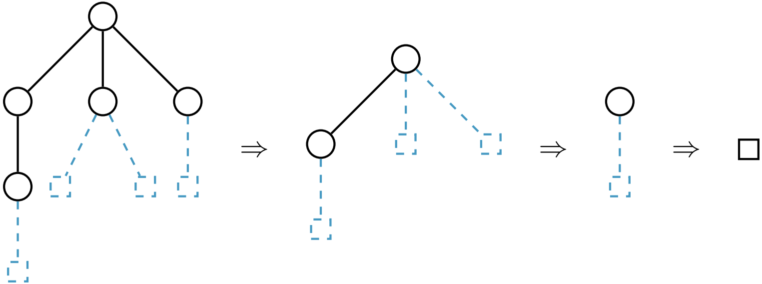

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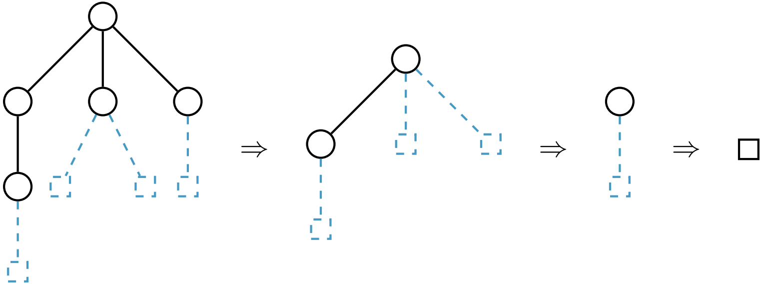

Parameters of Interest:

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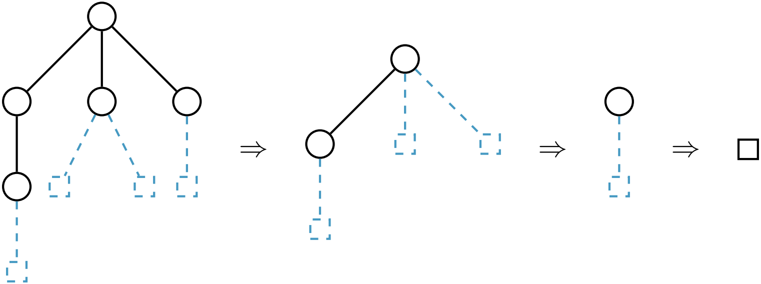

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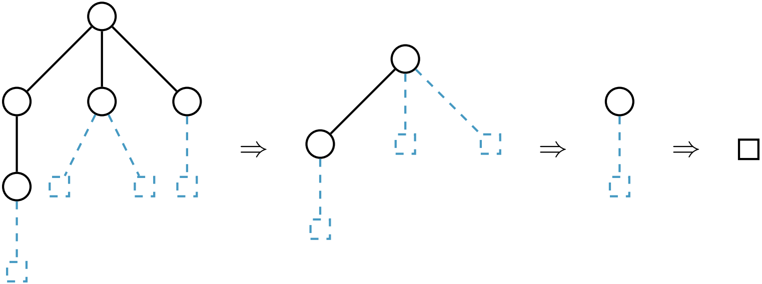

Parameters of Interest:

- Size of rth reduction
- Age: # of possible reductions

Reduction → Expansion

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□

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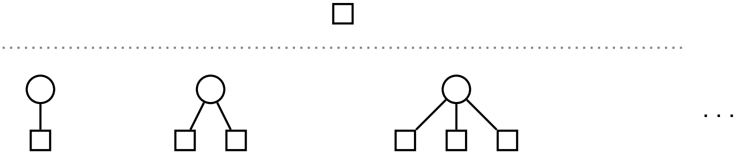

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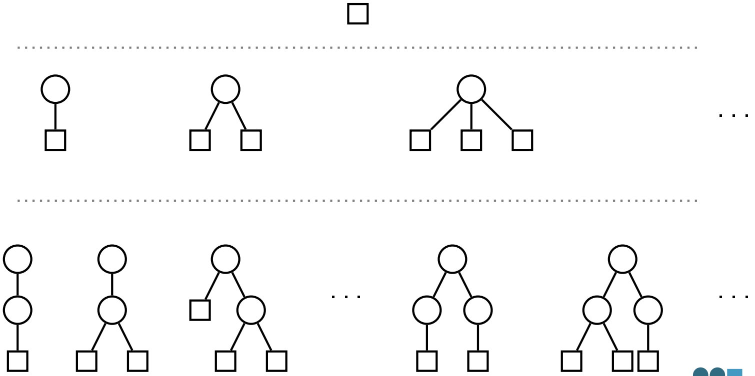

Expansion operators

- $F \ldots$ family of plane trees; bivariate generating function f
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees

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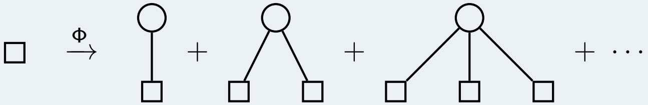
  - attach leaves to all current leaves (required)
  - attach leaves to inner nodes (optional)



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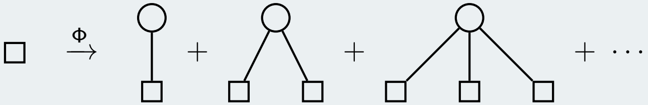
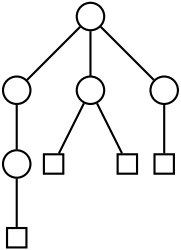


$$\square \triangleq t, \ \bigcirc \triangleq z \quad \Rightarrow \quad \Phi(t) = zt + zt^2 + zt^3 + \cdots$$

Reductions on Plane Trees

Leaves



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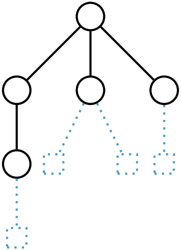

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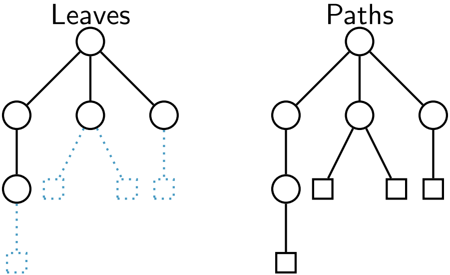

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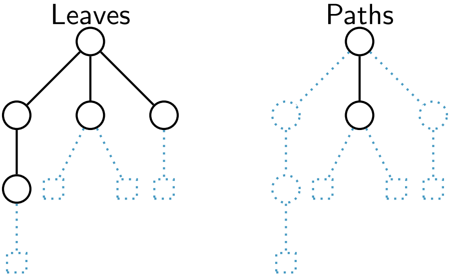

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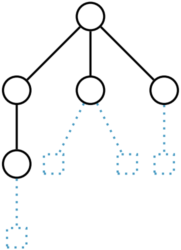
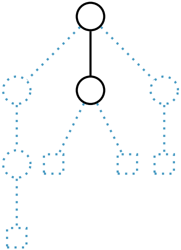
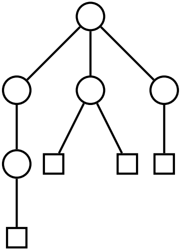


Paths



Old leaves



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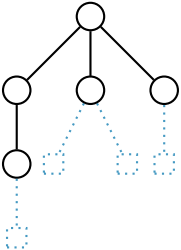

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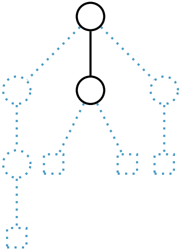

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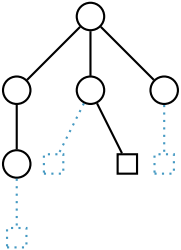

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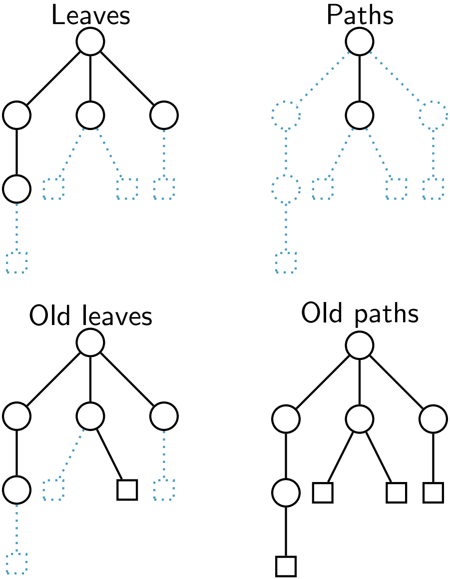

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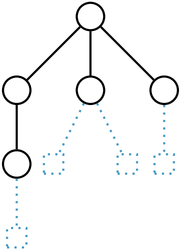

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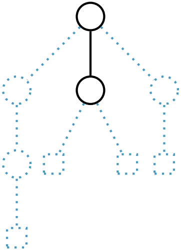

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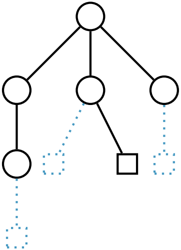

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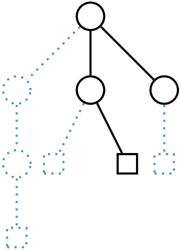

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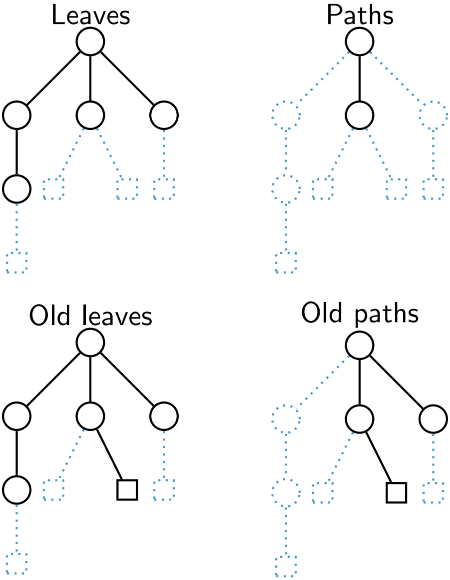

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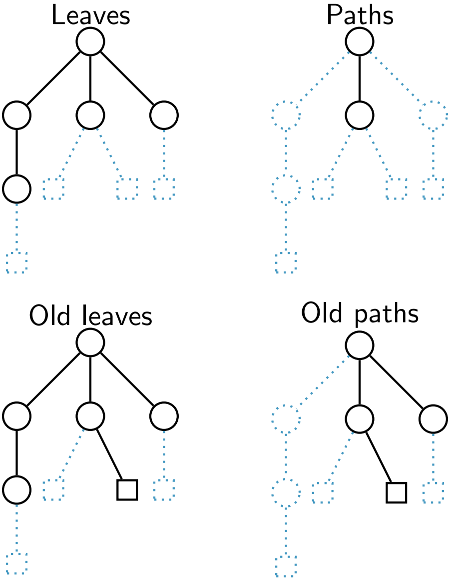

Parameters of Interest:

- tree size after r reductions
- cumulative reduction size

Bivariate Generating Function

Proposition

- $\mathcal{T}\ldots$ *rooted plane trees*

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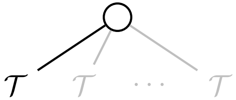
$$\Rightarrow T(z,t) = \frac{1-(z-t)-\sqrt{1-2(z+t)+(z-t)^2}}{2}$$

Proof. Symbolic equation

$$\mathcal{T} \quad = \quad \square \quad +$$



translates into

$$T(z,t) = t + z \cdot \frac{T(z,t)}{1-T(z,t)}$$

which can be solved explicitly.

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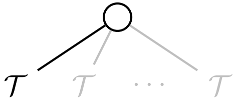

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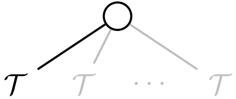

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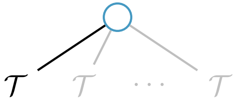

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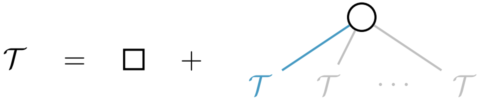

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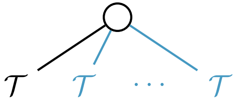

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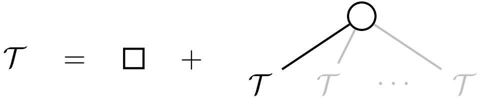

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Leaf expansion operator Φ

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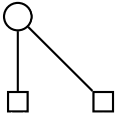

- In total:

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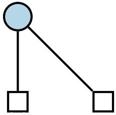

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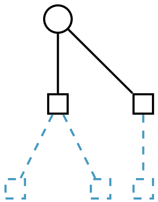

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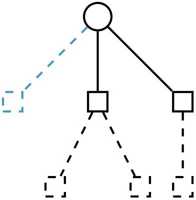

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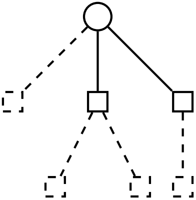

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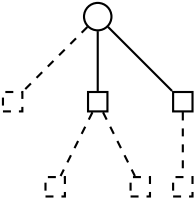

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- As Φ is linear, this proves the proposition.

Properties of Φ

- Functional equation: $T(z,t) = \Phi(T(z,t)) + t$

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- With $z = u/(1+u)^2$ and by some manipulations

$$\Phi^r(T(z,t))|_{t=z} = \frac{1-u^{r+2}}{(1-u^{r+1})(1+u)} T\Big(\frac{u(1-u^{r+1})^2}{(1-u^{r+2})^2}, \frac{u^{r+1}(1-u)^2}{(1-u^{r+2})^2}\Big)$$

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- BGF $G_r(z,v)$ for size comparison: z tracks original size, v size of r-fold reduced tree

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- With $z = u/(1+u)^2$ and by some manipulations

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Cutting leaves

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After r reductions of a random tree of size n, the remaining size $X_{n,r}$ has **mean** *and* **variance**

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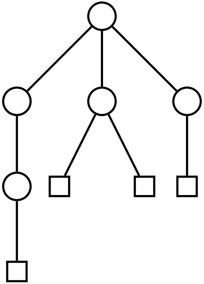
- Asymptotic normality: $n - X_{n,r}$ is a tree parameter with small toll function, limit law by Wagner (2015)

Pruning

- Remove all paths that end in a leaf!



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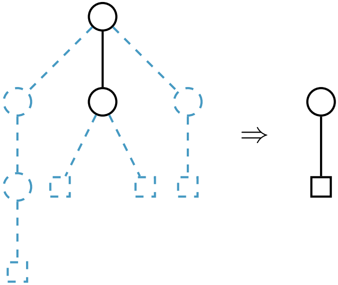

Branches in a Tree

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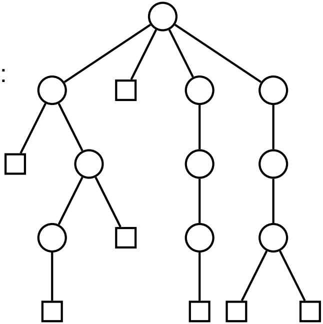

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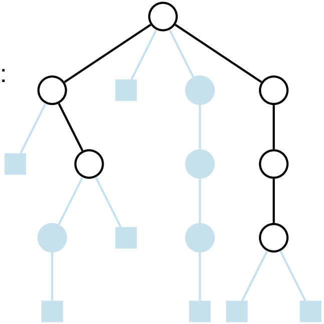

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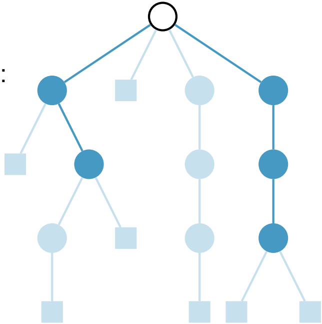

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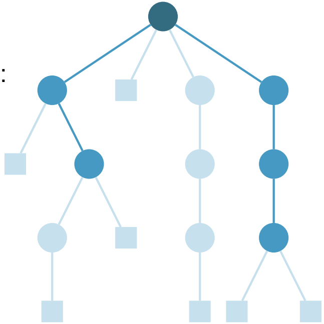

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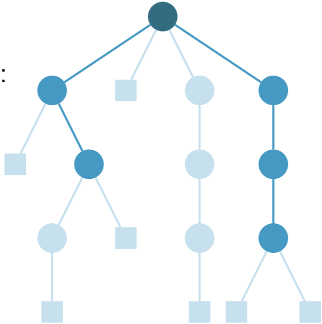

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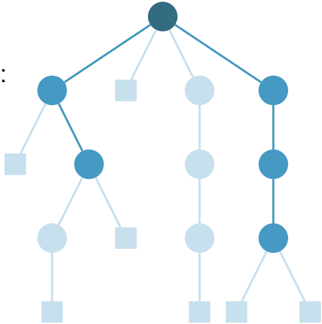

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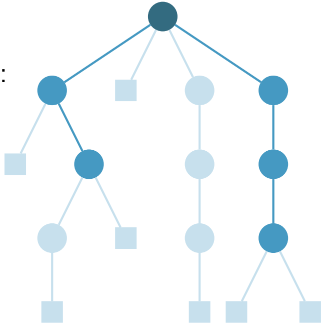

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Total # of branches $\triangleq$ # of leaves in all reduction stages

Proof: all branches end in exactly one leaf (at some point). □

Branches in a Tree – Result

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Average # of branches in a random plane tree of size n is

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- $\delta\ldots$ *periodic fluctuation:*

$$\delta(x) := \frac{1}{\log 2}\sum_{k\in\mathbb{Z}\setminus\{0\}} (-1+\chi_k)\Gamma(\chi_k/2)\zeta(-1+\chi_k)e^{2k\pi ix}, \quad \chi_k = \frac{2\pi ik}{\log 2}.$$

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Summary: Reductions on Plane Trees

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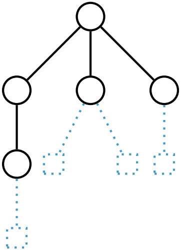

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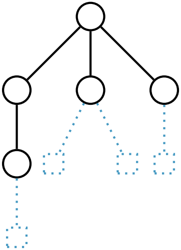

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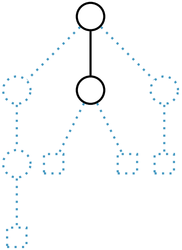

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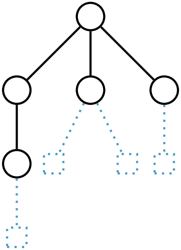

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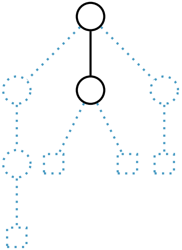

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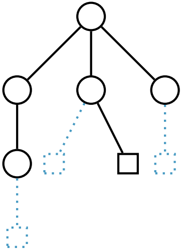

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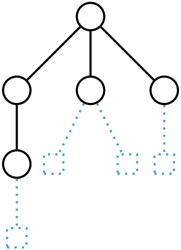

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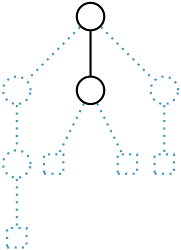

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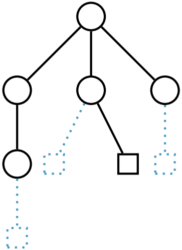

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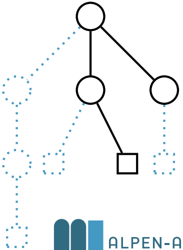

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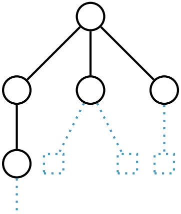

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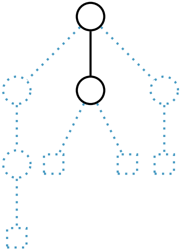

Disclaimer

Results are **not always** that nice!

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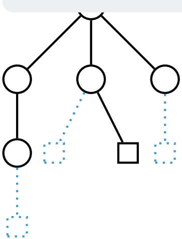

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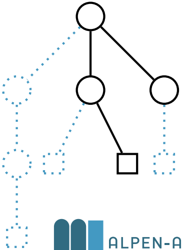

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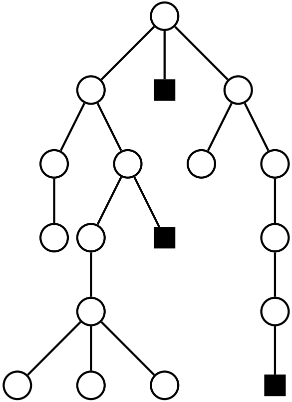
Counterexample: Catalan–Stanley trees

- Motivation: Stanley's Catalan interpretation #26
- Rightmost leaves in all branches of root have odd distance



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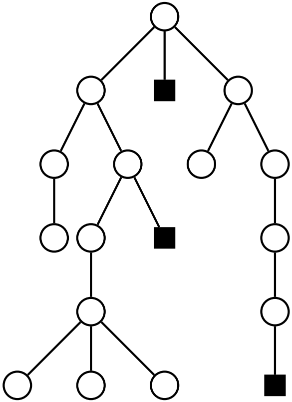

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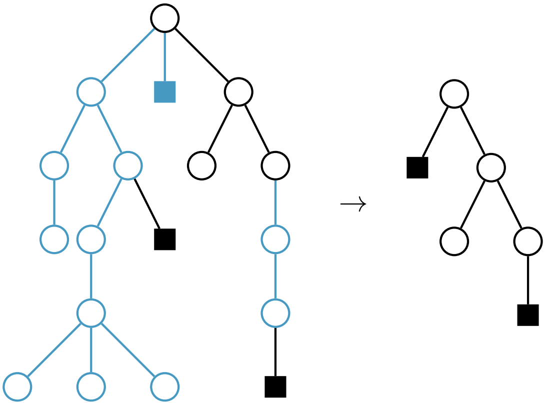

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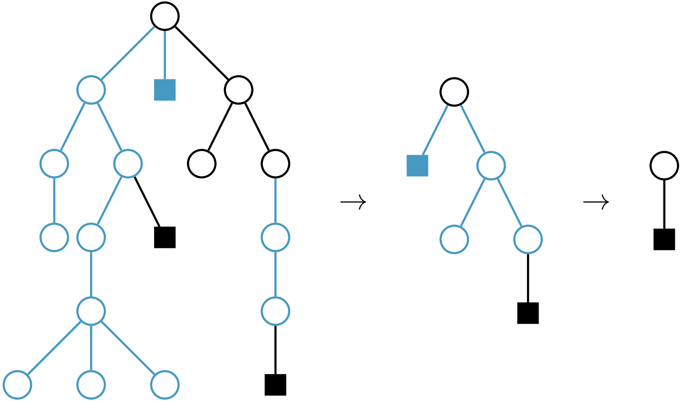

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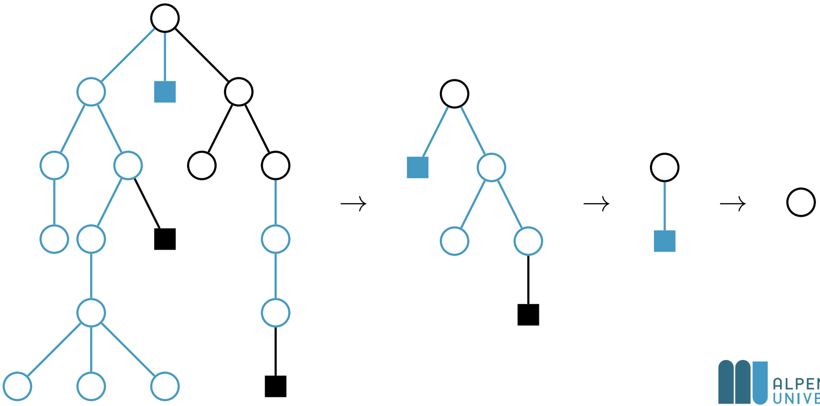

Counterexample: Results

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Size of rth Reduction

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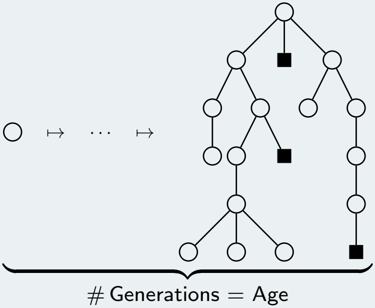

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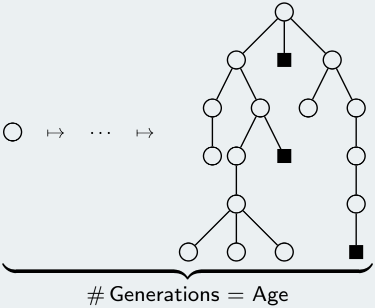

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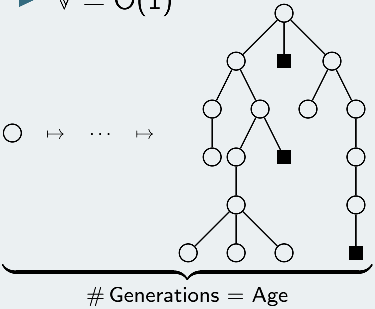

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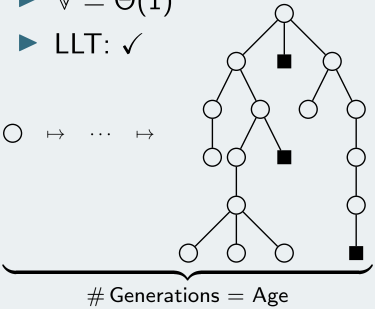

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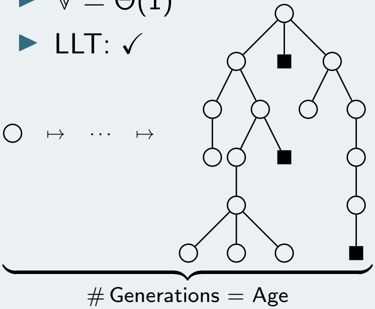

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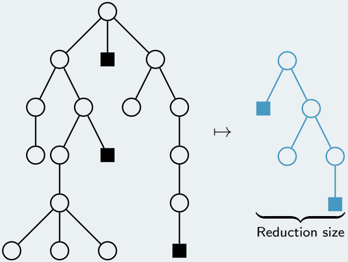

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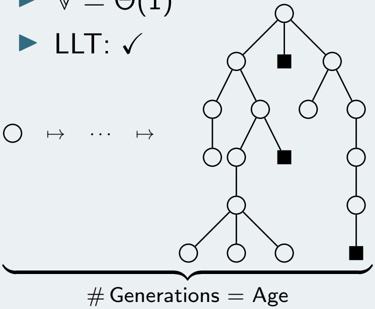

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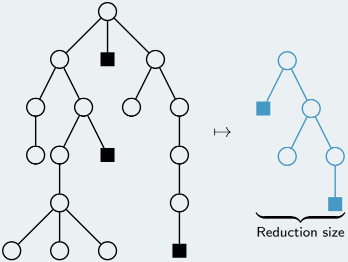

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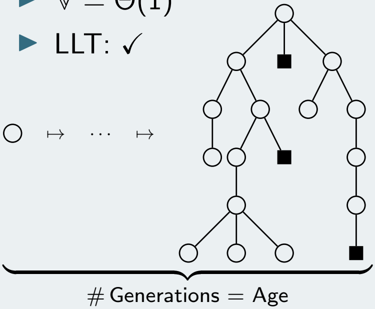

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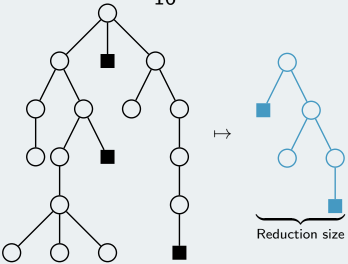
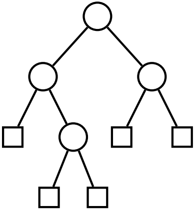


Trimming Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent



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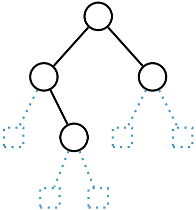

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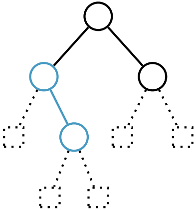

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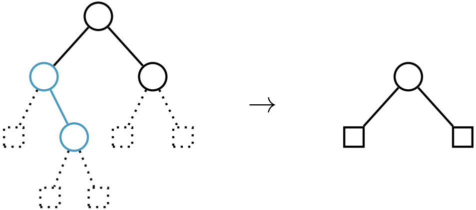

Trimming Binary Trees

Cutting strategy:

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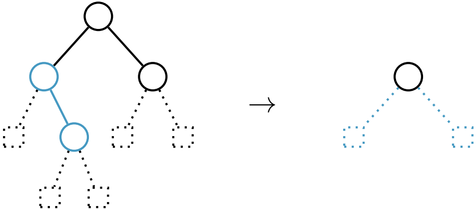

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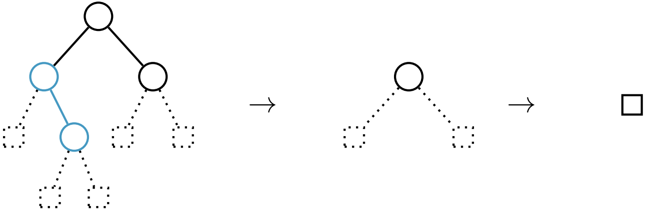

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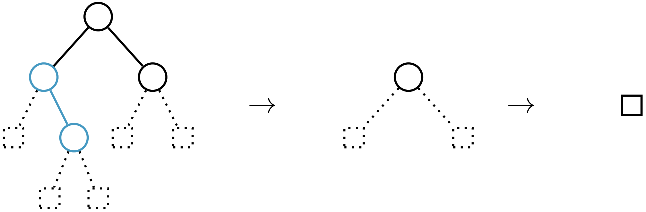

Corollary

$$B(z) = 1 + \frac{z}{1-2z} B\left(\frac{z^2}{(1-2z)^2}\right)$$

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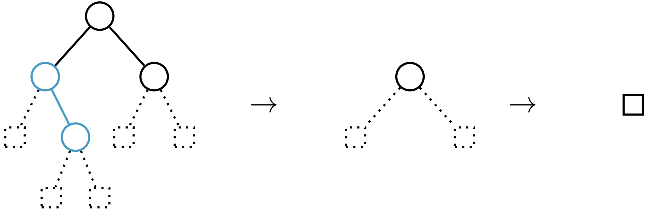

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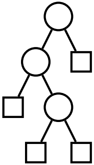
$\frac{z}{1-2z} \rightsquigarrow$



How “old” do the nodes get?

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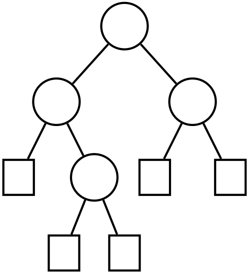

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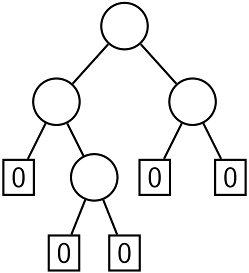

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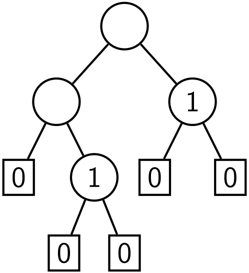

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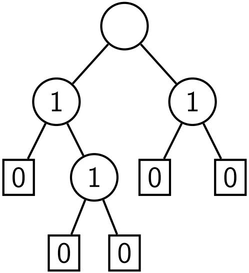

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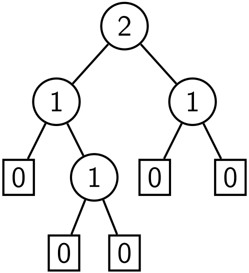

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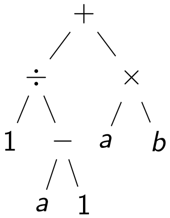

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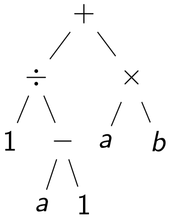

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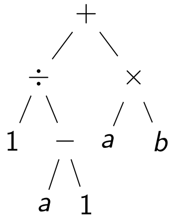

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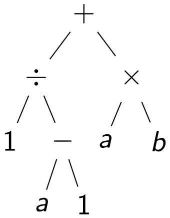

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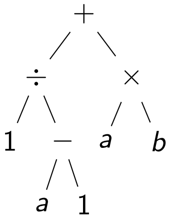

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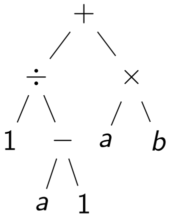

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  - r-branches, Numerics: Yamamoto, Yamazaki (2009)

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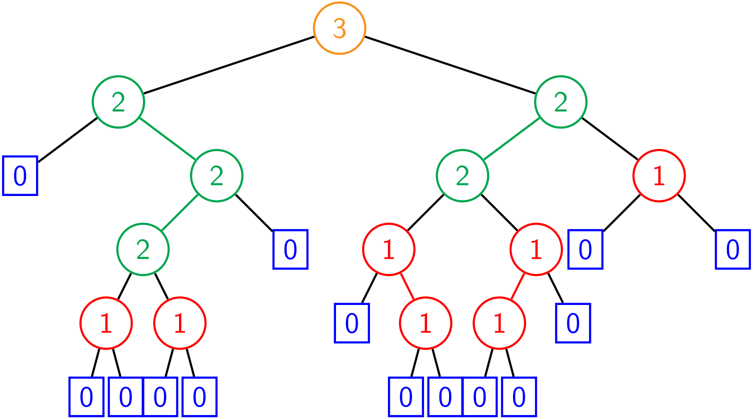
Local Structures – “r-branches”



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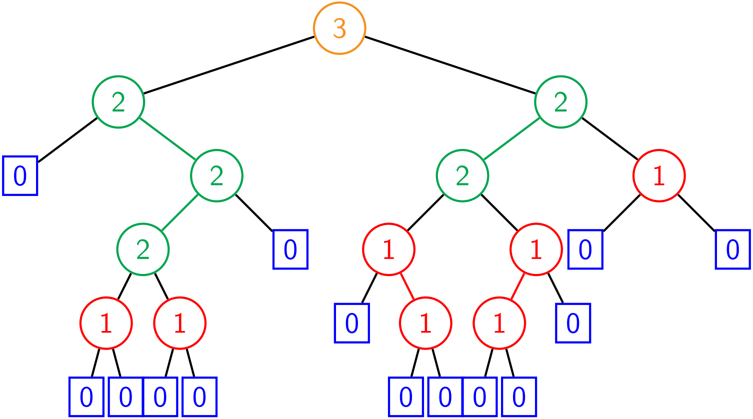

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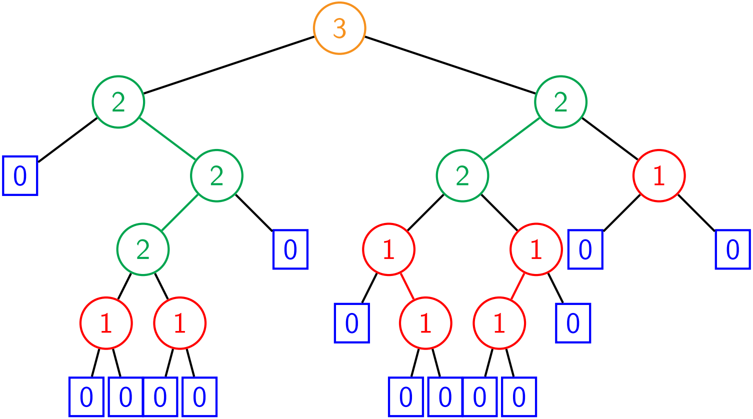

- Number / Distribution of (r-)branches?
- Example:

r	0	1	2	3
# r-branches	14	5	2	1

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"r-branches" – Results

Theorem (H.–Heuberger–Prodinger)

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$$\mathbb{E} = \frac{n}{4^r} + \frac{1}{6}\Big(1 + \frac{5}{4^r}\Big) + O(n^{-1}), \qquad \mathbb{V} = \frac{4^r - 1}{3 \cdot 16^r} n + O(1)$$

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- **expected total #** *of branches is*

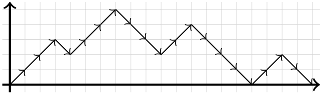
$$\frac{4}{3}n + \frac{1}{6}\log_4 n + C + \delta(\log_4 n) + O(n^{-1}\log n),$$

  - $C \approx 1.36190$, δ... *periodic fluctuation*

Non-Negative Lattice Paths

Dyck Paths:

- Sequences of $\{-1, 1\} \triangleq \{\searrow, \nearrow\}$,
- Never below axis, end on axis.



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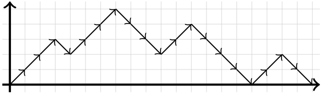

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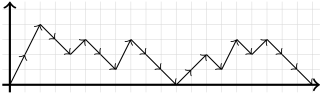
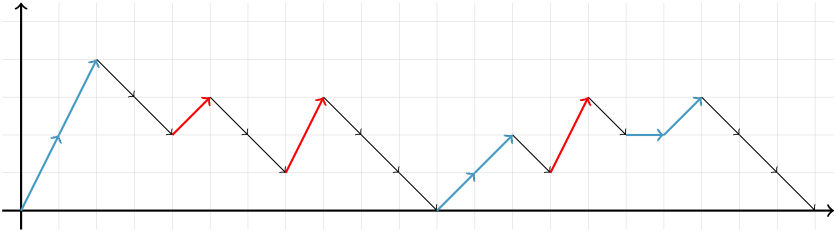
Łukasiewicz Excursions:



- Sequences of $\mathcal{S} = \{-1\} \cup N$, $N \subseteq \mathbb{N}_0$,
- Never below axis, end on axis.

Ascents



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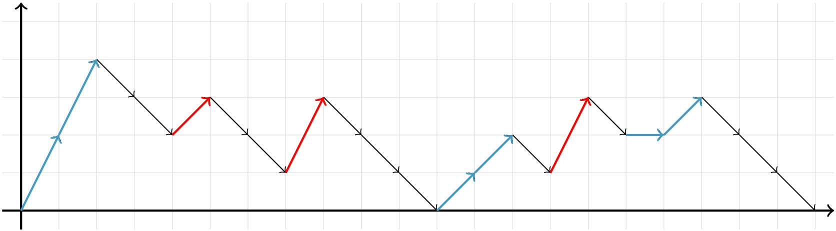

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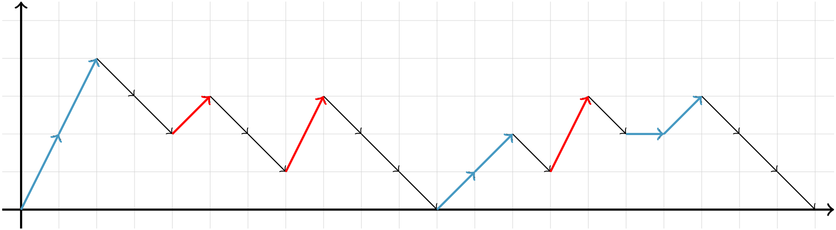
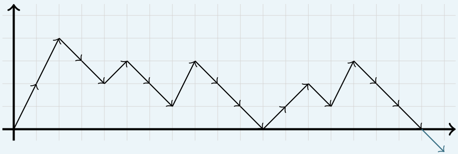


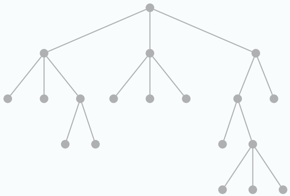
- **Ascent:** maximal sequence of non-negative steps,
- ***r*-Ascent:** ascent of length r.

Bijection: Excursions ⟷ Trees

$\mathcal{S} = \{-1, 1, 2\}$



\# of children $\in \{0, 2, 3\}$



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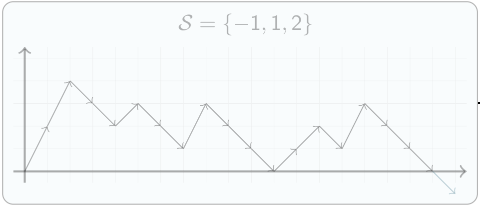

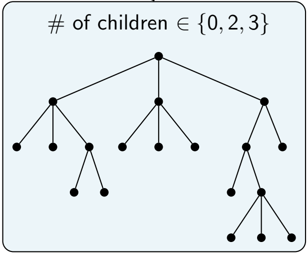

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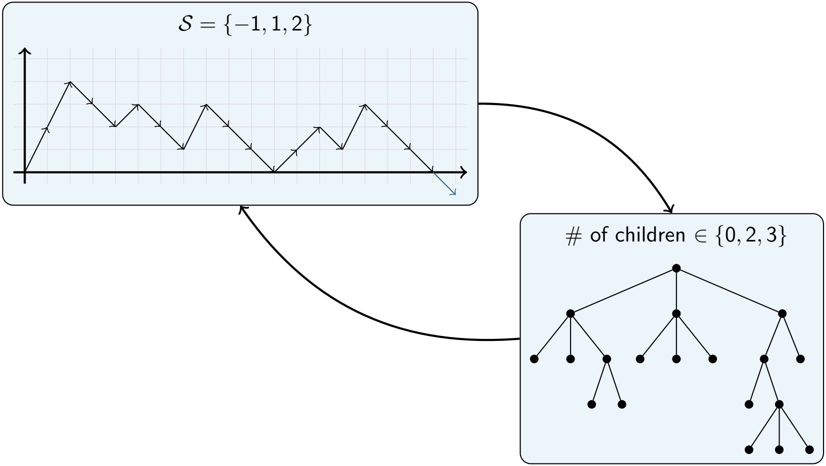

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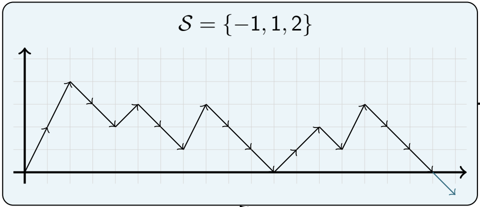

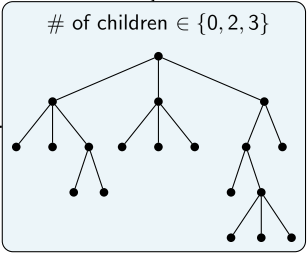

Bijection

Excursions($\mathcal{S}$) $\longleftrightarrow$ Plane Trees($\mathcal{S}+1$)

Ascents and Generating Functions

- $V(z,t)\ldots$ BGF for Plane Trees$(\mathcal{S}+1)$
  - $z \rightsquigarrow$ tree size, $t \rightsquigarrow \#$ of r-ascents

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Proposition

- $\tau\dots$*structural constant, unique* $\tau > 0$ *with* $S'(\tau) = 0$
- $p = \gcd(\mathcal{S}+1)\dots$*period,* $\zeta^p = 1$

$V(z)$ *has radius of convergence* $\rho = 1/S(\tau)$*, singularities at* $\zeta\rho$ *and*

$$V(z) \overset{z\to\zeta\rho}{=} \zeta\tau - \zeta\sqrt{\frac{2S(\tau)}{S''(\tau)}}\Big(1 - \frac{z}{\zeta\rho}\Big)^{1/2} + O\Big(1 - \frac{z}{\zeta\rho}\Big).$$

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Theorem (H.–Heuberger–Prodinger)

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Excursions – Example

Example (r-Ascents in Dyck paths)

- $\mathcal{S} = \{-1, 1\}$, $p = 2$, $\tau = 1$.

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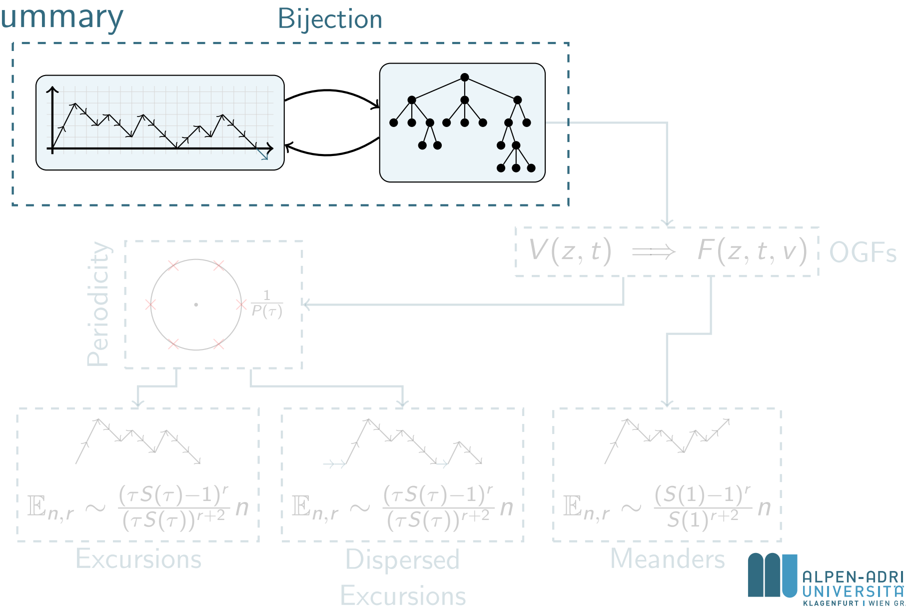
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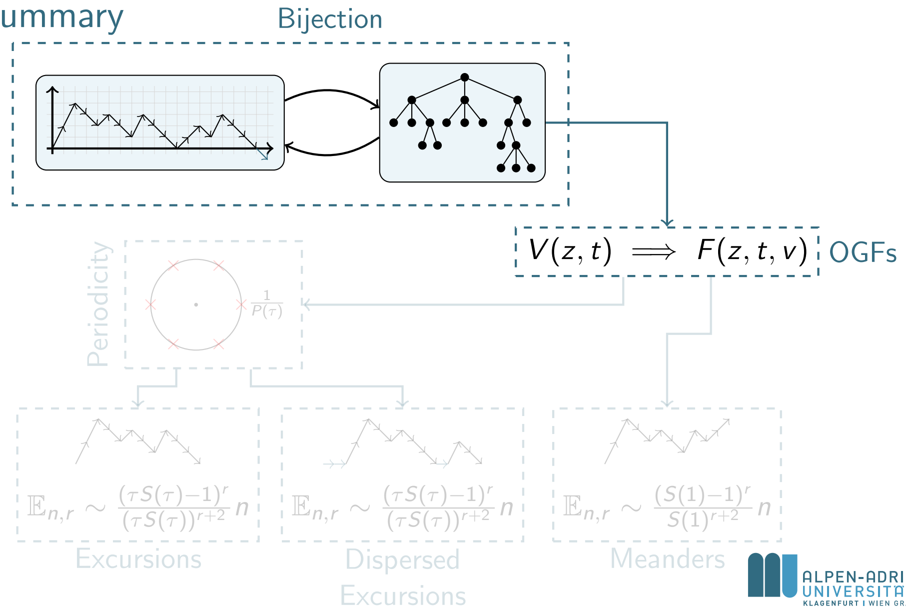
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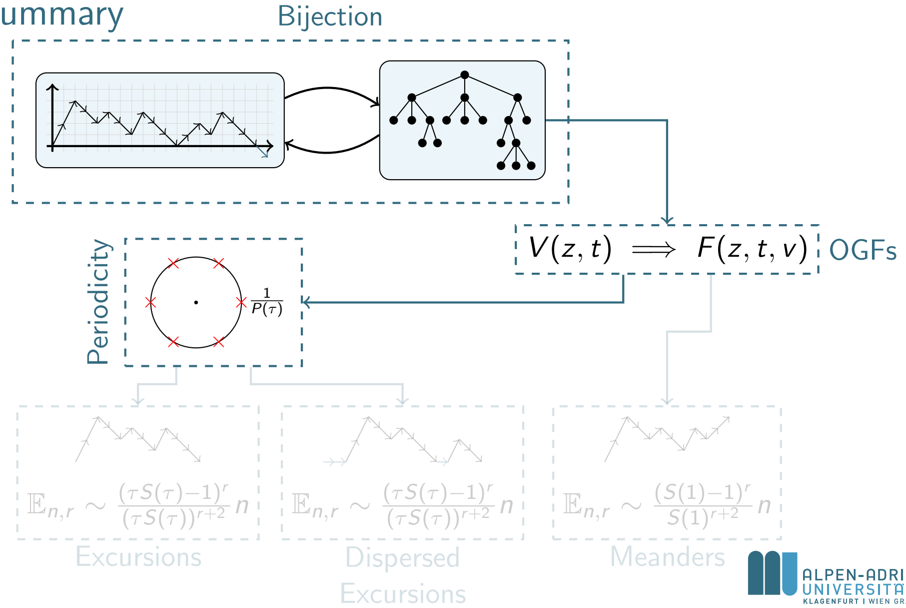
Dispersed Excursions
$\mathbb{E}_{n,r} \sim \frac{(S(1)-1)^r}{S(1)^{r+2}} n$
Meanders

Summary



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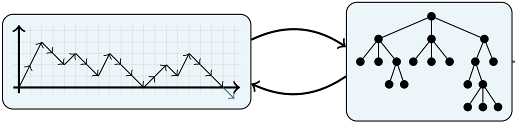
Summary

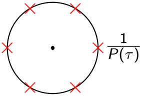
Bijection



OGFs

$$V(z,t) \Longrightarrow F(z,t,v)$$

Periodicity



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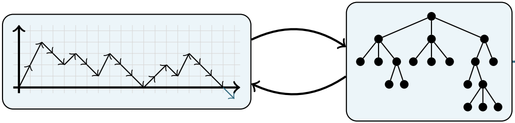

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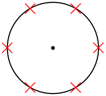

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Lagrange Inversion

- y, Φ, H ... formal power series with $y = x\Phi(y)$ and $\Phi(0) \neq 0$

Then:

$$[x^n]H(y) = \frac{1}{n}[y^{n-1}]H'(y)\Phi(y)^n$$

Singularity Analysis – Standard Scale

- $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$
- $f(z) = (1-z)^{-\alpha}$

Then:

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\Big(1 + \frac{e_1(\alpha)}{n} + \frac{e_2(\alpha)}{n^2} + \dots\Big),$$

where

$$e_1(\alpha) = \frac{\alpha(\alpha-1)}{2},$$
$$e_2(\alpha) = \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24},$$
$$e_3(\alpha) = \frac{\alpha^2(\alpha-1)^2(\alpha-2)(\alpha-3)}{48}.$$

Singularity Analysis – Logarithmic Scale

- $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$
- $f(z) = (1-z)^{-\alpha}\Big(\frac{1}{z}\log\frac{1}{1-z}\Big)^{\beta}$

Then:

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\Big(1 + \frac{C_1}{\log n} + \frac{C_2}{\log^2 n} + \dots\Big),$$

where

$$C_k = \binom{\beta}{k}\Gamma(\alpha)\frac{d^k}{ds^k}\frac{1}{\Gamma(s)}\Big|_{s=\alpha}.$$

Mellin Transform – Properties

$$f^*(s) = \int_0^\infty f(x) x^{s-1}\ dx \qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s}\ ds$$

$f(x)$	$f^*(s)$	$\langle\alpha, \beta\rangle$
lin. comb.	lin. comb.	
$x^k f(x)$	$f^*(s+k)$	$\langle\alpha - k, \beta - k\rangle$
$f(x^\rho)$	$\frac{1}{\rho} f^*(\frac{s}{\rho})$	$\langle\rho\alpha, \rho\beta\rangle$
$f(\mu x)$	$\mu^{-s} f^*(s)$	$\langle\alpha, \beta\rangle$ $(\mu > 0)$
$f(x)\log x$	$\frac{d}{ds} f^*(s)$	$\langle\alpha, \beta\rangle$
$x\frac{d}{dx} f(x)$	$-sf^*(s)$	

$$\mathcal{M}(e^{-x})(s) = \Gamma(s), \quad \mathcal{M}(e^{-x^2})(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right), \quad \mathcal{M}([\![0 \le s \le 1]\!])(s) = \frac{1}{s}$$

$$\mathcal{M}(\log(1+x))(s) = \frac{\pi}{s\sin(\pi s)}, \quad \mathcal{M}\left(\frac{1}{e^x - 1}\right) = \zeta(s)\Gamma(s)$$

Mellin Transform – Correspondence

$f(x)$	$f^*(s)$
$f(x) = O(x^\alpha)$, $x \to 0$	$-\alpha$ left border of FS
$f(x) = O(x^\beta)$, $x \to \infty$	$-\beta$ right border of FS
Expansion up to $O(x^\gamma)$, $x \to 0$	Meromorph. Cont. up to $\operatorname{Re} s > -\gamma$
Expansion up to $O(x^\delta)$, $x \to \infty$	Meromorph. Cont. up to $\operatorname{Re} s < -\delta$
Growth $x^k \log^\ell x$, $x \to 0$	Pole with expansion $\frac{(-1)^\ell \ell!}{(s+k)^{\ell+1}}$
Growth $x^k \log^\ell x$, $x \to \infty$	Pole with expansion $-\frac{(-1)^\ell \ell!}{(s+k)^{\ell+1}}$

Hurwitz Zeta Function – Estimate

$$\zeta(s,\alpha) := \sum_{n>-\alpha} \frac{1}{(n+\alpha)^s}, \quad \operatorname{Re} s > 1$$

- $s = \sigma + it$, $\sigma_0 \leq \sigma \leq \sigma_1$
- $|t| \to \infty$

$$|\zeta(s,\alpha)| = O(|t|^{\tau(\sigma)} \log |t|), \quad \tau(\sigma) = \begin{cases} \frac{1}{2} - \sigma, & \sigma \leq 0 \\ 1/2, & 0 \leq \sigma \leq \frac{1}{2} \\ 1 - \sigma, & \frac{1}{2} \leq \sigma \leq 1 \\ 0, & \sigma \geq 1 \end{cases}$$

The logarithmic factor is only necessary in a neighborhood of $\sigma = 0$, $\sigma = 1$.

Gamma Function – Estimate

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} \, dt$$

- $z \to \infty$ with $|\arg z| \leq \pi - \delta$, $\delta > 0$

$$\Gamma(z) \sim e^{-z} z^z \Big(\frac{2\pi}{z}\Big)^{1/2} \Big(1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots\Big)$$