Asymptotic Analysis of Shape Parameters of Trees and Lattice Paths
Thesis Overview

1. **Reductions of Binary Trees and Lattice Paths induced by the Register Function**
   - Joint work with Clemens Heuberger, Helmut Prodinger

2. **Fringe Analysis of Plane Trees Related to Cutting and Pruning**
   - Joint work with Clemens Heuberger, Sara Kropf, Helmut Prodinger

3. **Growing and Destroying Catalan–Stanley Trees**
   - Joint work with Helmut Prodinger

4. **Ascents in Non-Negative Lattice Paths**
   - Joint work with Clemens Heuberger, Helmut Prodinger
   - arXiv:1801.02996 [math.CO]
Example: Deterministic Tree Reduction

- Remove all leaves!
Example: Deterministic Tree Reduction

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Example: Deterministic Tree Reduction

- Remove all leaves!

![Diagram of tree reduction process]
Example: Deterministic Tree Reduction

- Remove all leaves!

\[
\begin{array}{c}
\text{Original Tree} \\
\Rightarrow \\
\text{Reduced Tree} \\
\Rightarrow \\
\Rightarrow \\
\end{array}
\]
Example: Deterministic Tree Reduction

▶ Remove all leaves!

Parameters of Interest:
Example: Deterministic Tree Reduction

- Remove all leaves!

Parameters of Interest:
- Size of $r$th reduction
Example: Deterministic Tree Reduction

- Remove all leaves!

Parameters of Interest:
- Size of $r$th reduction
- Age: $\#$ of possible reductions
Reduction $\rightarrow$ Expansion

- modelling reduction directly: not suitable
Reduction → Expansion

- modelling reduction directly: not suitable
- instead: see inverse operation, growing trees
Reduction → Expansion

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- instead: see inverse operation, growing trees

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\ldots
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Reduction → Expansion

▶ modelling reduction directly: not suitable
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\[
\begin{align*}
\text{Reduction} \quad &\quad \text{Expansion} \\
\text{modelling reduction directly: not suitable} \\
\text{instead: see inverse operation, growing trees}
\end{align*}
\]
Expansion operators

- $F$ . . . family of plane trees; bivariate generating function $f$
- expansion operator $\Phi \Rightarrow \Phi(f)$ counts expanded trees
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Leaf expansion $\Phi$
- inverse operation to leaf reduction
Expansion operators

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Leaf expansion $\Phi$

- inverse operation to leaf reduction
  - attach leaves to all current leaves (required)
  - attach leaves to inner nodes (optional)

\[
\begin{array}{ccccc}
\square & \Phi & \Rightarrow & \square & + \\
 & & & \square & + \\
 & & & \square & + \\
\end{array}
\]

\[\Phi(t) = zt + zt^2 + zt^3 + \cdots\]
Expansion operators

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$\square \xrightarrow{\Phi} \bigcirc + \bigcirc + \bigcirc + \cdots$

$\square \triangleq t$, $\bigcirc \triangleq z \Rightarrow \Phi(t) = zt + zt^2 + zt^3 + \cdots$
Reductions on Plane Trees

Leaves

Analysis of Shape Parameters – Benjamin Hackl
Reductions on Plane Trees

Leaves

Parameters of Interest:
- tree size after $r$ reductions
- cumulative reduction size
Reductions on Plane Trees

Leaves

Paths

Parameters of Interest:

▶ tree size after \( r \) reductions

▶ cumulative reduction size
Reductions on Plane Trees

Leaves

Paths
Reductions on Plane Trees

Leaves

Paths

Old leaves
Reductions on Plane Trees

Leaves

Paths

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Reductions on Plane Trees

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Reductions on Plane Trees

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Reductions on Plane Trees

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Bivariate Generating Function

Proposition

\[ T \text{... rooted plane trees} \]
Bivariate Generating Function

Proposition

- $\mathcal{T}$... rooted plane trees
- $T(z, t)$... BGF for $\mathcal{T}$ ($z \sim$ inner nodes, $t \sim$ leaves)
Bivariate Generating Function

**Proposition**

- $\mathcal{T}$ \ldots rooted plane trees
- $T(z, t)$ \ldots BGF for $\mathcal{T}$ (z $\rightsquigarrow$ inner nodes, t $\rightsquigarrow$ leaves)

\[ T(z, t) = \frac{1 - (z - t) - \sqrt{1 - 2(z + t) + (z - t)^2}}{2} \]

**Proof.** Symbolic equation

\[ \mathcal{T} = \square + \frac{T(z, t)}{1 - T(z, t)} \]

translates into

\[ T(z, t) = t + z \cdot \frac{T(z, t)}{1 - T(z, t)} \]

which can be solved explicitly.
Bivariate Generating Function

Proposition

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- $T(z, t)$ ... BGF for $T$ (z $\sim$ inner nodes, t $\sim$ leaves)

$$
\Rightarrow T(z, t) = \frac{1 - (z - t) - \sqrt{1 - 2(z + t) + (z - t)^2}}{2}
$$

Proof. Symbolic equation

$$
T = \square + \frac{T(z, t)}{1 - T(z, t)}
$$

translates into

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Proof. Symbolic equation

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**Proof.** Symbolic equation

\[ \mathcal{T} = \square + \mathcal{T} \cdot \mathcal{T} \cdots \mathcal{T} \]

translates into

\[ T(z, t) = t + z \cdot \frac{T(z, t)}{1 - T(z, t)} \]

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\Rightarrow T(z, t) = \frac{1 - (z - t) - \sqrt{1 - 2(z + t) + (z - t)^2}}{2}
\]

**Proof.** Symbolic equation

\[
\mathcal{T} = \square + \sum \mathcal{T}
\]

translates into

\[
T(z, t) = t + z \cdot \frac{T(z, t)}{1 - T(z, t)}
\]

which can be solved explicitly.
Leaf expansion operator $\Phi$

**Proposition**

$$\Phi(f(z, t)) = (1 - t)f\left(\frac{z}{(1 - t)^2}, \frac{zt}{(1 - t)^2}\right)$$
Leaf expansion operator $\Phi$

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- Tree with $n$ inner nodes and $k$ leaves $\leadsto z^n t^k$
- **Expansion:**

- In total:

$$\Phi(z^n t^k) =$$
Leaf expansion operator $\Phi$

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- Tree with $n$ inner nodes and $k$ leaves $\leadsto z^n t^k$
- **Expansion:**
  - inner nodes stay inner nodes

- In total:
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  $$
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- Tree with $n$ inner nodes and $k$ leaves $\leadsto z^n t^k$
- **Expansion:**
  - inner nodes stay inner nodes
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- In total:
  $$\Phi(z^n t^k) = z^n \cdot \left(\frac{zt}{1-t}\right)^k.$$
Leaf expansion operator $\Phi$

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- **Expansion:**
  - inner nodes stay inner nodes
  - attach a non-empty sequence of leaves to all current leaves
  - there are $2n + k - 1$ positions where sequences of leaves can be inserted
- In total:
  $$\Phi(z^n t^k) = z^n \cdot \left(\frac{zt}{1 - t}\right)^k \cdot \frac{1}{(1 - t)^{2n+k-1}}$$
Leaf expansion operator $\Phi$

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$$\Phi(z^n t^k) = z^n \cdot \left(\frac{zt}{1 - t}\right)^k \cdot \frac{1}{(1 - t)^{2n+k-1}} = (1 - t)\left(\frac{z}{(1 - t)^2}\right)^n \left(\frac{zt}{(1 - t)^2}\right)^k$$
Leaf expansion operator $\Phi$

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  $$\Phi(z^n t^k) = z^n \cdot \left(\frac{zt}{1 - t}\right)^k \cdot \frac{1}{(1 - t)^{2n+k-1}} = (1-t)\left(\frac{z}{(1 - t)^2}\right)^n \left(\frac{zt}{(1 - t)^2}\right)^k$$
- As $\Phi$ is linear, this proves the proposition.
Properties of $\Phi$

- Functional equation: $T(z, t) = \Phi(T(z, t)) + t$
Properties of $\Phi$

- Functional equation: $T(z, t) = \Phi(T(z, t)) + t$
- With $z = u/(1 + u)^2$ and by some manipulations

$$\Phi^r(T(z, t))|_{t=z} = \frac{1 - u^{r+2}}{(1 - u^{r+1})(1 + u)} T\left(\frac{u(1 - u^{r+1})^2}{(1 - u^{r+2})^2}, \frac{u^{r+1}(1 - u)^2}{(1 - u^{r+2})^2}\right)$$
Properties of $\Phi$

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$$

- BGF $G_r(z, \nu)$ for size comparison: $z$ tracks original size, $\nu$ size of $r$-fold reduced tree
Properties of $\Phi$

- Functional equation: $T(z, t) = \Phi(T(z, t)) + t$
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- Intuition: $\nu$ “remembers” size while tree family is expanded
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- Functional equation: $T(z, t) = \Phi(T(z, t)) + t$
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Cutting leaves

Theorem (H.–Heuberger–Kropf–Prodinger)

After $r$ reductions of a random tree of size $n$, the remaining size $X_{n,r}$ has mean and variance

$$\mathbb{E} X_{n,r} = \frac{n}{r + 1} - \frac{r(r - 1)}{6(r + 1)} + O(n^{-1}),$$

and $X_{n,r}$ is asymptotically normally distributed.
Cutting leaves

Theorem (H.–Heuberger–Kropf–Prodinger)

After \( r \) reductions of a random tree of size \( n \), the remaining size \( X_{n,r} \) has mean and variance

\[
\mathbb{E} X_{n,r} = \frac{n}{r+1} - \frac{r(r-1)}{6(r+1)} + O(n^{-1}),
\]

\[
\text{Var} X_{n,r} = \frac{r(r+2)}{6(r+1)^2} n + O(1),
\]
Cutting leaves

Theorem (H.–Heuberger–Kropf–Prodinger)

After $r$ reductions of a random tree of size $n$, the remaining size $X_{n,r}$ has mean and variance

$$
\mathbb{E}X_{n,r} = \frac{n}{r+1} - \frac{r(r-1)}{6(r+1)} + O(n^{-1}),
$$

$$
\mathbb{V}X_{n,r} = \frac{r(r+2)}{6(r+1)^2} n + O(1),
$$

and $X_{n,r}$ is asymptotically normally distributed.

Proof insights:

- $\mathbb{E}X_{n,r}$ and $\mathbb{V}X_{n,r}$ follow via singularity analysis.
Cutting leaves

**Theorem (H.–Heuberger–Kropf–Prodinger)**

After $r$ reductions of a random tree of size $n$, the remaining size $X_{n,r}$ has **mean** and **variance**

\[
E X_{n,r} = \frac{n}{r+1} - \frac{r(r-1)}{6(r+1)} + O(n^{-1}),
\]

\[
\nabla X_{n,r} = \frac{r(r+2)}{6(r+1)^2} n + O(1),
\]

and $X_{n,r}$ is **asymptotically normally distributed**.

**Proof insights:**

- $E X_{n,r}$ and $\nabla X_{n,r}$ follow via singularity analysis
- Asymptotic normality: $n - X_{n,r}$ is a tree parameter with small toll function, limit law by Wagner (2015)
Pruning

- Remove all paths that end in a leaf!
Pruning

- Remove all paths that end in a leaf!
Branches in a Tree

- Trees can be partitioned into branches:

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- **Q**: How many branches are there?
Branches in a Tree

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- **Q**: How many branches are there?

**Observation**

Total # of branches $\triangleq$ # of leaves in all reduction stages
Branches in a Tree

- Trees can be partitioned into branches:
- **Q:** How many branches are there?

**Observation**

Total # of branches $\triangleq$ # of leaves in all reduction stages

**Proof:** all branches end in exactly one leaf (at some point).
Branches in a Tree – Result

Theorem (H.–Heuberger–Kropf–Prodinger)

Average # of branches in a random plane tree of size $n$ is

$$\alpha n + 1 + \frac{C}{\log^4 n} + O\left(\frac{n}{\log n}\right),$$

$\alpha \approx 0.60669$, $C \approx -0.11811$, $\delta$ periodic fluctuation:

$$\delta(x) := \log 2 \sum_{k \in \mathbb{Z} \setminus \{0\}} \left(1 - \chi_k \right) \Gamma\left(\frac{\chi_k}{2}\right) \zeta\left(\frac{\chi_k}{2}\right) e^{2k\pi ix}, \chi_k = \frac{2\pi i k}{\log 2}.$$

Analysis of Shape Parameters – Benjamin Hackl
**Branches in a Tree – Result**

**Theorem (H.–Heuberger–Kropf–Prodinger)**

*Average # of branches in a random plane tree of size $n$ is*

$$\alpha n + \frac{1}{6} \log_4 n + C + \delta(\log_4 n) + O(n^{-1/4}),$$
Branches in a Tree – Result

Theorem (H.–Heuberger–Kropf–Prodinger)

Average # of branches in a random plane tree of size $n$ is

$$\alpha n + \frac{1}{6} \log_4 n + C + \delta(\log_4 n) + O(n^{-1/4}),$$

where

$$\alpha = \sum_{k \geq 2} \frac{1}{2^k - 1} \approx 0.60669,$$

with

- $\alpha$ being a constant in the theorem,
- $C$ is a constant,
- $\delta$ is a periodic fluctuation,
- $\log_4 n$ representing the logarithm of $n$ to the base 4,
Branches in a Tree – Result

**Theorem (H.–Heuberger–Kropf–Prodinger)**

Average # of branches in a random plane tree of size $n$ is

$$\alpha n + \frac{1}{6} \log_4 n + C + \delta(\log_4 n) + O(n^{-1/4}),$$

- $\alpha = \sum_{k \geq 2} \frac{1}{2^k - 1} \approx 0.60669,$

- $C = -\frac{\gamma + 4\alpha \log 2 + \log 2 + 24\zeta'(-1) + 2}{12 \log 2} \approx -0.11811,$
### Theorem (H.‐Heuberger‐Kropf‐Prodinger)

Average # of branches in a random plane tree of size \( n \) is

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- \( \alpha = \sum_{k \geq 2} \frac{1}{2^k - 1} \approx 0.60669, \)
- \( C = -\gamma + 4\alpha \log 2 + \log 2 + 24\zeta'(-1) + 2 \frac{12 \log 2}{12 \log 2} \approx -0.11811, \)
- \( \delta \ldots \text{periodic fluctuation:} \)

\[
\delta(x) := \frac{1}{\log 2} \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1 + \chi_k) \Gamma(\chi_k/2) \zeta(-1 + \chi_k) e^{2k\pi i x}, \quad \chi_k = \frac{2\pi ik}{\log 2}.
\]
Summary: Reductions on Plane Trees

Leaves

\[ E \sim \frac{n}{r+1} \]
\[ V \sim \frac{r(r+2)}{6(r+1)^2} n \]

limit law: ✓
Summary: Reductions on Plane Trees

Leaves

\[ E \sim \frac{n}{r+1} \]
\[ V \sim \frac{r(r+2)}{6(r+1)^2} n \]

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Paths

\[ E \sim \frac{n}{2r+1-1} \]
\[ V \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n \]

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Summary: Reductions on Plane Trees

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\[ E \sim \frac{n}{r+1} \]
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limit law: √

Old leaves
\[ E \sim (2 - B_{r-1}(1/4)) n \]
\[ V = \Theta(n) \]
limit law: √
Summary: Reductions on Plane Trees

**Leaves**

\[ E \sim \frac{n}{r+1}, \quad V \sim \frac{r(r+2)}{6(r+1)^2} n \]

Limit law: ✓

**Paths**

\[ E \sim \frac{n}{2^{r+1}-1}, \quad V \sim \frac{2^{r+1}(2^r - 1)}{3(2^{r+1}-1)^2} n \]

Limit law: ✓

**Old leaves**

\[ E \sim (2 - B_{r-1}(1/4)) n, \quad V = \Theta(n) \]

Limit law: ✓

**Old paths**

\[ E \sim \frac{2n}{r+2}, \quad V \sim \frac{2r(r+1)}{3(r+2)^2} n \]

Limit law: ✓
Summary: Reductions on Plane Trees

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\[ E \sim \frac{n}{r+1}, \quad V \sim \frac{r(r+2)}{6(r+1)^2} n \]
limit law: ✓

Old leaves
\[ E \sim (2 - B_{r-1}(1/4))n, \quad V = \Theta(n) \]
limit law: ✓

Paths
\[ E \sim \frac{n}{2^{r+1}-1}, \quad V \sim \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n \]
limit law: ✓

Old paths
\[ E \sim \frac{2n}{r+2}, \quad V \sim \frac{2r(r+1)}{3(r+2)^2} n \]
limit law: ✓

Disclaimer
Results are not always that nice!
Counterexample: Catalan–Stanley trees

- Motivation: Stanley’s Catalan interpretation #26
- Rightmost leaves in all branches of root have odd distance
Counterexample: Catalan–Stanley trees

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- **Reduction:** remove parent/grandparent (except root) of □
Counterexample: Catalan–Stanley trees

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Cutting Down & Growing

Plane Trees

Register Function

Ascents

Counterexample: Catalan–Stanley trees

- Motivation: Stanley’s Catalan interpretation #26
- Rightmost leaves in all branches of root have odd distance
- **Reduction**: remove parent/grandparent (except root) of ■

![Diagram showing reduction process]
Counterexample: Catalan–Stanley trees

- Motivation: Stanley’s Catalan interpretation #26
- Rightmost leaves in all branches of root have odd distance
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### Counterexample: Results

- Reduction with different parameter behavior ✓

<table>
<thead>
<tr>
<th>Age</th>
<th>Size of $r$th Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Counterexample: Results

- Reduction with different parameter behavior ✓

### Age

- \( \# \text{ Generations} = \text{Age} \)

### Size of \( r \)th Reduction

- \( E \sim \frac{1}{4^n} \)
- \( V \sim \frac{(2^r+1)(2^r-1)}{16r^2} \)
Counterexample: Results

- Reduction with different parameter behavior ✓

### Age

- \( E = \Theta(1) \)

### Size of \( r \)th Reduction

\[ E \sim 1^{\frac{r}{n}} \]

\[ V \sim \left(2^r + 1\right)\left(2^r - 1\right)\frac{1}{16r^2} \]
## Counterexample: Results

- Reduction with different parameter behavior ✓

### Age
- \( E = \Theta(1) \)
- \( V = \Theta(1) \)

### Size of \( r \)th Reduction

\[
E \sim \frac{1}{4^r}
\]
\[
V \sim \left(2^r + 1\right)\left(2^r - 1\right) - \frac{1}{16^r}
\]

# Generations = Age
Counterexample: Results

- Reduction with different parameter behavior ✓

**Age**
- $E = \Theta(1)$
- $V = \Theta(1)$
- LLT: ✓

**Size of $r$th Reduction**

# Generations = Age

$\# \text{ Generations} = \text{Age}$
Counterexample: Results

- Reduction with different parameter behavior ✓

### Age

- $E = \Theta(1)$
- $V = \Theta(1)$
- LLT: ✓

# Generations = Age

### Size of $r$th Reduction

- $E \sim 1/4^r$
- $V \sim (2^r + 1)(2^r - 1)/16^r$

Reduction size
Counterexample: Results

- Reduction with different parameter behavior

<table>
<thead>
<tr>
<th>Age</th>
<th>Size of $r$th Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{E} = \Theta(1)$</td>
<td>$\mathbb{E} \sim \frac{1}{4^r} n$</td>
</tr>
<tr>
<td>$\mathbb{V} = \Theta(1)$</td>
<td></td>
</tr>
<tr>
<td>LLT: ✓</td>
<td></td>
</tr>
</tbody>
</table>

### Age

- # Generations = Age

### Size of $r$th Reduction

- Reduction size

Analysis of Shape Parameters – Benjamin Hackl
Counterexample: Results

- Reduction with different parameter behavior ✓

<table>
<thead>
<tr>
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<th>Size of rth Reduction</th>
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<tr>
<td>▶ $E = \Theta(1)$</td>
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</tr>
<tr>
<td>▶ $V = \Theta(1)$</td>
<td>▶ $V \sim \frac{(2^r+1)(2^r-1)}{16^r} n^2$</td>
</tr>
<tr>
<td>▶ LLT: ✓</td>
<td></td>
</tr>
</tbody>
</table>
Trimming Binary Trees

Cutting strategy:

► Remove Leaves
► Merge single children with their corresponding parent
Trimming Binary Trees

Cutting strategy:

▶ Remove Leaves
▶ Merge single children with their corresponding parent

\[ B(z) = 1 + z - 2zB(z^2(1 - 2z^2)) \]
Trimming Binary Trees

Cutting strategy:

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Trimming Binary Trees

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Analysis of Shape Parameters – Benjamin Hackl
Trimming Binary Trees

Cutting strategy:

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Analysis of Shape Parameters – Benjamin Hackl
Trimming Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent

\[
B(z) = 1 + z - 2zB(z^2(1 - 2z^2))z^{1 - 2z}
\]
Trimming Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent

$$B(z) = 1 + \frac{z}{1 - 2z} B\left(\frac{z^2}{(1 - 2z)^2}\right)$$
Trimming Binary Trees

Cutting strategy:

- Remove Leaves
- Merge single children with their corresponding parent

\[ B(z) = 1 + \frac{z}{1 - 2z} B \left( \frac{z^2}{(1 - 2z)^2} \right) \]
How “old” do the nodes get?

We label the nodes according to the following rules:

- Leaves → 0
- age(left child) = age(right child) → increase by 1
- Otherwise: maximum of children
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```
  1
 / \  \
0   1
 / \ / \  \
0 1 0 0 0
```
How “old” do the nodes get?

We label the nodes according to the following rules:

- Leaves → 0
- age(left child) = age(right child) → increase by 1
- Otherwise: maximum of children

```
      2
     / \
    1   1
   / \ / \ 
  0  1 0  0
 / \ / \ / \ 
0  0 0 0 0
```
The Register Function

Age $\mapsto$ Register function (Horton-Strahler-Index)
The Register Function

Age $\sim$ Register function (Horton-Strahler-Index)

- Applications:
The Register Function

Age $\leadsto$ Register function (Horton-Strahler-Index)

- Applications:
  - Required stack size for evaluating arithmetic expressions

```
        +
       / \   /  /
      /   /  /
     /    /  a
    /    /  b
   /    /  
  /    /   
 /    /    
 a    1     
```

- Asymptotic analysis:
  - Flajolet, Raoult, Vuillemin (1979)
  - Flajolet, Prodinger (1986)
  - $r$-branches, Numerics: Yamamoto, Yamazaki (2009)
The Register Function

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Applications:
- Required stack size for evaluating arithmetic expressions
- Branching complexity of river networks (e.g. Danube: 9)
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Age ⇝ Register function (Horton-Strahler-Index)

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The Register Function

**Age** \(\sim\) **Register function (Horton-Strahler-Index)**

- **Applications:**
  - Required stack size for evaluating arithmetic expressions
  - Branching complexity of river networks (e.g. Danube: 9)

\[
\begin{array}{c}
+ \\
\div \\
\times \\
/ \\
/ \\
/ \\
1 - a b \\
/ \\
/ \\
a 1
\end{array}
\]

- **Asymptotic analysis:**
  - Flajolet, Raoult, Vuillemin (1979)
  - Flajolet, Prodinger (1986)
The Register Function

Age $\leadsto$ Register function (Horton-Strahler-Index)

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  ► Required stack size for evaluating arithmetic expressions
  ► Branching complexity of river networks (e.g. Danube: 9)

+ 
  \[ \div \times \]
  \[ 1 \quad a \quad b \]

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  ► Flajolet, Raoult, Vuillemin (1979)
  ► Flajolet, Prodinger (1986)
  ► $r$-branches, Numerics: Yamamoto, Yamazaki (2009)
Local Structures – “$r$-branches”

Number / Distribution of ($r$-)branches?

Example:

$\begin{align*}
r & \quad \#r\text{-branches} \\
0 & \quad 14 \\
1 & \quad 5 \\
2 & \quad 2 \\
3 & \quad 1 \\
\end{align*}$

Analysis of Shape Parameters – Benjamin Hackl
Local Structures – “r-branches”

Number / Distribution of (r-)branches?
Local Structures – “$r$-branches”

- Number / Distribution of $(r\,)$branches?
- Example:

<table>
<thead>
<tr>
<th>$r$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td># $r$-branches</td>
<td>14</td>
<td>5</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
“r-branches” – Results

**Theorem (H.–Heuberger–Prodinger)**

*In a random binary tree of size n...*
Theorem (H.–Heuberger–Prodinger)

*In a random binary tree of size* \( n \) ... 

- # of \( r \)-branches is *asymptotically normally distributed*
“r-branches” – Results

Theorem (H.–Heuberger–Prodinger)

In a random binary tree of size $n$...

- The number of $r$-branches is asymptotically normally distributed.
- With mean and variance

$$E = \frac{n}{4^r} + \frac{1}{6} \left(1 + \frac{5}{4^r}\right) + O(n^{-1}), \quad \mathbb{V} = \frac{4^r - 1}{3 \cdot 16^r} n + O(1)$$
"r-branches" – Results

**Theorem (H.–Heuberger–Prodinger)**

*In a random binary tree of size* $n$ . . .

- # of *r-branches* is asymptotically normally distributed
- with mean and variance

$$
\mathbb{E} = \frac{n}{4^r} + \frac{1}{6} \left(1 + \frac{5}{4^r}\right) + O(n^{-1}), \quad \mathbb{V} = \frac{4^r - 1}{3 \cdot 16^r} n + O(1)
$$

- expected total # of branches is

$$
\frac{4}{3} n + \frac{1}{6} \log_4 n + C + \delta(\log_4 n) + O(n^{-1} \log n),
$$

- $C \approx 1.36190$, $\delta$ . . . periodic fluctuation
Non-Negative Lattice Paths

Dyck Paths:
- Sequences of \{-1, 1\} \triangleq \{\downarrow, \uparrow\},
- Never below axis, end on axis.
Non-Negative Lattice Paths

Dyck Paths:
- Sequences of $\{-1, 1\} \triangleq \{\downarrow, \uparrow\}$,
- Never below axis, end on axis.

Łukasiewicz Excursions:
- Sequences of $S = \{-1\} \cup N$, $N \subseteq \mathbb{N}_0$,
- Never below axis, end on axis.
Ascents
Ascents

**Ascent**: maximal sequence of non-negative steps,
Ascents

- **Ascent**: maximal sequence of non-negative steps,
- **r-Ascent**: ascent of length $r$. 

Analysis of Shape Parameters – Benjamin Hackl
Bijection: Excursions $\leftrightarrow$ Trees

$S = \{-1, 1, 2\}$

# of children $\in \{0, 2, 3\}$
Bijection: Excursions $\leftrightarrow$ Trees

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Bijection: Excursions \leftrightarrow Trees

\[ S = \{-1, 1, 2\} \]

# of children \( \in \{0, 2, 3\} \)
Ascents and Generating Functions

- $V(z, t) \ldots$ BGF for Plane Trees($S + 1$)
- $z \rightsquigarrow$ tree size, $t \rightsquigarrow$ # of $r$-ascents
Ascents and Generating Functions

- $V(z, t)$... BGF for Plane Trees ($S + 1$)
- $z \rightsquigarrow$ tree size, $t \rightsquigarrow$ # of $r$-ascents

Consequences of recursive tree structure:
Ascents and Generating Functions

- \( V(z, t) \)… BGF for Plane Trees\((S + 1)\)
  - \( z \mapsto \) tree size, \( t \mapsto \) \# of \( r \)-ascents

Consequences of **recursive tree structure**:

- relevant quantities are expressable via \( V(z) := V(z, 1) \)
Ascents and Generating Functions

- \( V(z, t) \ldots \) BGF for Plane Trees\((S + 1)\)
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- \( V(z) \) satisfies \( V(z) = zV(z)S(V(z)) \),
Ascents and Generating Functions

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Ascents and Generating Functions

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  - functional equation type \( V = z\varphi(V) \mapsto \text{singular inversion} \)
Ascents and Generating Functions

- \( V(z, t) \ldots \text{BGF for Plane Trees}(S + 1) \)
  - \( z \leadsto \text{tree size}, \ t \leadsto \# \text{ of } r\text{-ascents} \)

Consequences of \textit{recursive tree structure}:

- relevant quantities are expressable via \( V(z) := V(z, 1) \)
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  - functional equation type \( V = z \varphi(V) \leadsto \text{singular inversion} \)

Proposition

- \( \tau \ldots \text{structural constant, unique } \tau > 0 \text{ with } S'(\tau) = 0 \)
- \( \rho = \gcd(S + 1) \ldots \text{period, } \zeta^\rho = 1 \)

\( V(z) \) has radius of convergence \( \rho = 1/S(\tau) \), singularities at \( \zeta \rho \) and

\[
V(z) \xrightarrow{z \to \zeta \rho} \zeta \tau - \zeta \sqrt{\frac{2S(\tau)}{S''(\tau)}} \left(1 - \frac{z}{\zeta \rho}\right)^{1/2} + O\left(1 - \frac{z}{\zeta \rho}\right).
\]
Analysis of $r$-Ascents in Excursions

**Theorem (H.–Heuberger–Prodinger)**

- $p$ ... *period of* $S$, $\tau$ ... *structural constant*, $c := \tau S(\tau)$
Analysis of $r$-Ascents in Excursions

Theorem (H.–Heuberger–Prodinger)

- $p \ldots$ period of $S$, $\tau \ldots$ structural constant, $c := \tau S(\tau)$

1. If $p \nmid n$, there are no excursions of length $n$,
Theorem (H.–Heuberger–Prodinger)

1. If $p \nmid n$, there are no excursions of length $n$,
2. Otherwise, $\#$ of $r$-ascents has mean and variance

\[ E = (c - 1) r c + 2 n + O(1) \]
\[ V = (c - 1) r c + 2 + (2c - 2r - 3)(c - 1)^2 r + 4 - (c - 1)^2 r - 2(2c - r - 2)c^2 r + 3 \tau^3 S''(\tau) \]
\[ n + O(n^{1/2}) \]
Analysis of \( r \)-Ascents in Excursions

Theorem (H.–Heuberger–Prodinger)

\( p \) \ldots period of \( S \), \( \tau \) \ldots structural constant, \( c := \tau S(\tau) \)

1. If \( p \nmid n \), there are no excursions of length \( n \),
2. Otherwise, \( \# \) of \( r \)-ascents has mean and variance

\[
\mathbb{E} = \frac{(c - 1)r}{c^{r+2}} n + O(1),
\]
Analysis of $r$-Ascents in Excursions

Theorem (H.–Heuberger–Prodinger)

- $p$ ... period of $S$, $\tau$ ... structural constant, $c := \tau S(\tau)$

1. If $p \nmid n$, there are no excursions of length $n$,
2. Otherwise, # of $r$-ascents has mean and variance

$$\mathbb{E} = \frac{(c - 1)^r}{c^{r+2}} n + O(1),$$

$$\mathbb{V} = \left( \frac{(c - 1)^r}{c^{r+2}} + \frac{(2c - 2r - 3)(c - 1)^{2r}}{c^{2r+4}} - \frac{(c - 1)^{2r-2}(2c - r - 2)^2}{c^{2r+3}\tau^3 S''(\tau)} \right) n + O(n^{1/2}).$$
Excursions – Example

Example ($r$-Ascents in Dyck paths)

$S = \{-1, 1\}$, $p = 2$, $\tau = 1$. 
Excursions – Example

Example ($r$-Ascents in Dyck paths)

- $S = \{-1, 1\}$, $p = 2$, $\tau = 1$.
- Explicit $V(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z} \Rightarrow \text{higher precision!}$
Excursions – Example

Example ($r$-Ascents in Dyck paths)

- $S = \{-1, 1\}$, $p = 2$, $\tau = 1$.
- Explicit $V(z) = \frac{1-\sqrt{1-4z^2}}{2z}$ ⇒ higher precision!

$$\mathbb{E}D_{2n,r} = \frac{n}{2r+1} - \frac{(r + 1)(r - 4)}{2r+3}$$
$$+ \frac{(r^2 - 11r + 22)(r + 1)r}{2r+6}n^{-1} + O(n^{-2})$$
Excursions – Example

Example ($r$-Ascents in Dyck paths)

- $S = \{-1, 1\}, \ p = 2, \ \tau = 1$.
- Explicit $V(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z} \Rightarrow$ higher precision!

$$\mathbb{E}D_{2n,r} = \frac{n}{2r+1} - \frac{(r + 1)(r - 4)}{2r+3} \frac{n^{-1}}{2} + O(n^{-2})$$

$$\nabla D_{2n,r} = \left(\frac{1}{2r+1} - \frac{r^2 - 2r + 3}{2^{2r+3}}\right)n + O(1)$$
Summary

Bijection

\[ V(z, t) \mapsto F(z, t, v) \]

OGFs

Periodicity

\[ \frac{1}{P(\tau)} \]

Excursions

\[ E_{n,r} \sim \frac{(\tau S(\tau) - 1)^r}{(\tau S(\tau))^r + 2} n \]

Dispersed Excursions

\[ E_{n,r} \sim \frac{(\tau S(\tau) - 1)^r}{(\tau S(\tau))^r + 2} n \]

Meanders

\[ E_{n,r} \sim \frac{(S(1) - 1)^r}{S(1)^r + 2} n \]
Summary

Bijection

\[ V(z, t) \Rightarrow F(z, t, v) \]

OGFs

Periodicity

\[ \frac{1}{P(\tau)} \]

Excursions

\[ E_{n,r} \sim \frac{\tau S(\tau) - 1}{\tau S(\tau)}^r n \]

Dispersed Excursions

\[ E_{n,r} \sim \frac{\tau S(\tau) - 1}{\tau S(\tau)}^r n \]

Meanders

\[ E_{n,r} \sim \frac{S(1) - 1}{S(1)}^r n \]

Analysis of Shape Parameters – Benjamin Hackl
Summary

Bijection

\[ V(z, t) \implies F(z, t, v) \]

OGFs

Periodicity

Excursions

Dispersed Excursions

Meanders

Analysis of Shape Parameters – Benjamin Hackl
Summary

Bijection

\[ V(z, t) \Rightarrow F(z, t, v) \]

Periodicity

\[ \frac{1}{P(\tau)} \]

Excursions

\[ E_{n,r} \sim \frac{(\tau S(\tau) - 1)^r}{(\tau S(\tau))^r + 2} n \]

Dispersed Excursions

\[ E_{n,r} \sim \frac{(\tau S(\tau) - 1)^r}{(\tau S(\tau))^r + 2} n \]

Meanders

\[ E_{n,r} \sim \frac{S(1) - 1)^r}{S(1)^r + 2} n \]

Analysis of Shape Parameters – Benjamin Hackl
Summary

Bijection

Periodicity

\[ V(z, t) \equiv F(z, t, \nu) \]

Excursions

\[ E_{n,r} \sim \frac{(\tau S(\tau) - 1)^r}{\tau S(\tau)^r} n \]

Dispersed Excursions

\[ E_{n,r} \sim \frac{(\tau S(\tau) - 1)^r}{\tau S(\tau)^r + 2} n \]

Meanders

\[ E_{n,r} \sim \frac{(S(1) - 1)^r}{S(1)^r + 2} n \]

OGFs

Analysis of Shape Parameters – Benjamin Hackl
Lagrange Inversion

\[ y, \Phi, H \ldots \text{formal power series with } y = x\Phi(y) \text{ and } \Phi(0) \neq 0 \]

Then:

\[ [x^n]H(y) = \frac{1}{n} [y^{n-1}]H'(y)\Phi(y)^n \]
Singularity Analysis – Standard Scale

\( \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \)

\( f(z) = (1 - z)^{-\alpha} \)

Then:

\[
[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left( 1 + \frac{e_1(\alpha)}{n} + \frac{e_2(\alpha)}{n^2} + \ldots \right),
\]

where

\[
e_1(\alpha) = \frac{\alpha(\alpha - 1)}{2},
\]

\[
e_2(\alpha) = \frac{\alpha(\alpha - 1)(\alpha - 2)(3\alpha - 1)}{24},
\]

\[
e_3(\alpha) = \frac{\alpha^2(\alpha - 1)^2(\alpha - 2)(\alpha - 3)}{48}.
\]
Singularity Analysis – Logarithmic Scale

\[ \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \]

\[ f(z) = (1 - z)^{-\alpha} \left( \frac{1}{z} \log \frac{1}{1 - z} \right)^\beta \]

Then:

\[ [z^n] f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left( 1 + \frac{C_1}{\log n} + \frac{C_2}{\log^2 n} + \ldots \right), \]

where

\[ C_k = \binom{\beta}{k} \Gamma(\alpha) \frac{d^k}{ds^k} \frac{1}{\Gamma(s)} \bigg|_{s=\alpha}. \]
Mellin Transform – Properties

\[ f^*(s) = \int_0^\infty f(x)x^{s-1} \, dx \quad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s} \, ds \]

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f^*(s) )</th>
<th>( \langle \alpha, \beta \rangle )</th>
</tr>
</thead>
<tbody>
<tr>
<td>lin. comb.</td>
<td>lin. comb.</td>
<td>\langle \alpha - k, \beta - k \rangle</td>
</tr>
<tr>
<td>( x^k f(x) )</td>
<td>( f^*(s + k) )</td>
<td>\langle \alpha - k, \beta - k \rangle</td>
</tr>
<tr>
<td>( f(x^\rho) )</td>
<td>( \frac{1}{\rho} f^*(\frac{s}{\rho}) )</td>
<td>\langle \rho \alpha, \rho \beta \rangle</td>
</tr>
<tr>
<td>( f(\mu x) )</td>
<td>( \mu^{-s} f^*(s) )</td>
<td>\langle \alpha, \beta \rangle (\mu &gt; 0)</td>
</tr>
<tr>
<td>( f(x) \log x )</td>
<td>( \frac{d}{ds} f^*(s) )</td>
<td>\langle \alpha, \beta \rangle</td>
</tr>
<tr>
<td>( x \frac{d}{dx} f(x) )</td>
<td>( -sf^*(s) )</td>
<td></td>
</tr>
</tbody>
</table>

\[ \mathcal{M}(e^{-x}) (s) = \Gamma(s), \quad \mathcal{M}(e^{-x^2}) (s) = \frac{1}{2} \Gamma\left(\frac{s}{2}\right), \quad \mathcal{M}(\lfloor 0 \leq s \leq 1 \rfloor) (s) = \frac{1}{s} \]

\[ \mathcal{M}(\log(1 + x)) (s) = \frac{\pi}{s \sin(\pi s)}, \quad \mathcal{M}\left(\frac{1}{e^x - 1}\right) = \zeta(s) \Gamma(s) \]
### Mellin Transform – Correspondence

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$f^*(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = O(x^\alpha)$, $x \rightarrow 0$</td>
<td>$-\alpha$ left border of FS</td>
</tr>
<tr>
<td>$f(x) = O(x^\beta)$, $x \rightarrow \infty$</td>
<td>$-\beta$ right border of FS</td>
</tr>
<tr>
<td>Expansion up to $O(x^\gamma)$, $x \rightarrow 0$</td>
<td>Meromorph. Cont. up to Re $s &gt; -\gamma$</td>
</tr>
<tr>
<td>Expansion up to $O(x^\delta)$, $x \rightarrow \infty$</td>
<td>Meromorph. Cont. up to Re $s &lt; -\delta$</td>
</tr>
<tr>
<td>Growth $x^k \log^\ell x$, $x \rightarrow 0$</td>
<td>Pole with expansion $\frac{(-1)^{\ell} \ell!}{(s+k)^{\ell+1}}$</td>
</tr>
<tr>
<td>Growth $x^k \log^\ell x$, $x \rightarrow \infty$</td>
<td>Pole with expansion $-\frac{(-1)^{\ell} \ell!}{(s+k)^{\ell+1}}$</td>
</tr>
</tbody>
</table>
Hurwitz Zeta Function – Estimate

\[ \zeta(s, \alpha) := \sum_{n > -\alpha} \frac{1}{(n + \alpha)^s}, \quad \text{Re} s > 1 \]

- \( s = \sigma + it, \sigma_0 \leq \sigma \leq \sigma_1 \)
- \( |t| \to \infty \)

\[ |\zeta(s, \alpha)| = O(|t|^\tau(\sigma) \log |t|), \quad \tau(\sigma) = \begin{cases} 
\frac{1}{2} - \sigma, & \sigma \leq 0 \\
1/2, & 0 \leq \sigma \leq \frac{1}{2} \\
1 - \sigma, & \frac{1}{2} \leq \sigma \leq 1 \\
0, & \sigma \geq 1 
\end{cases} \]

The logarithmic factor is only necessary in a neighborhood of \( \sigma = 0, \sigma = 1 \).
Gamma Function – Estimate

\[ \Gamma(z) := \int_{0}^{\infty} e^{-t} t^{z-1} \, dt \]

\[ z \to \infty \text{ with } |\arg z| \leq \pi - \delta, \; \delta > 0 \]

\[ \Gamma(z) \sim e^{-z} z^{z} \left( \frac{2\pi}{z} \right)^{1/2} \left( 1 + \frac{1}{12z} + \frac{1}{288z^2} + \ldots \right) \]