Ascents in Non-Negative Lattice Paths

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February 14, 2018

msO DK modeling simulation optimization

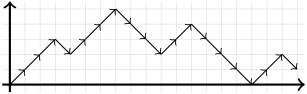
KARL POPPER KOLLEG

FWF Der Wissenschaftsfonds.

Łukasiewicz Paths

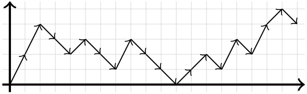
Dyck Meanders:

- Sequences of $\{-1, 1\} \triangleq \{\searrow, \nearrow\}$,
- Never below axis.



Łukasiewicz Paths:

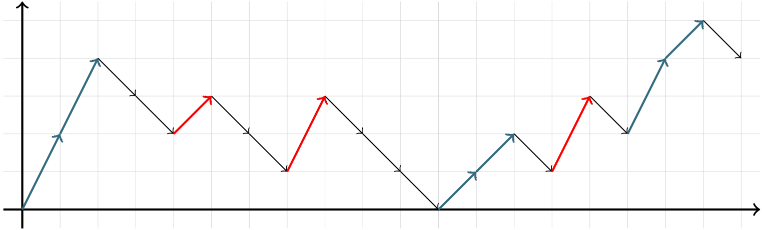
- Sequences of $\mathcal{S} = \{-1\} \cup N$, $N \subseteq \mathbb{N}_0$,
- Never below axis.



Notation. $S(u)$ … GF of $\mathcal{S}$, $S_+(u) = S(u) - u^{-1}$.

$$\mathcal{S} = \{-1, 0, 2\} \leftrightsquigarrow S(u) = u^{-1} + 1 + u^2,\ S_+(u) = 1 + u^2$$

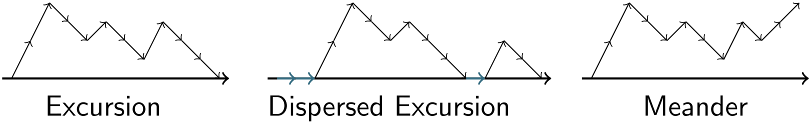
Ascents



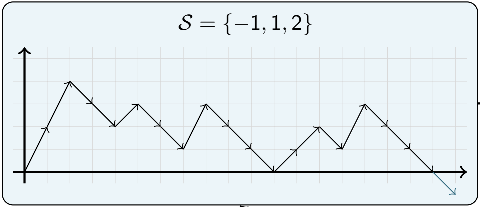
- **Ascent:** maximal sequence of non-negative steps,
- r-**Ascent:** ascent of length r.

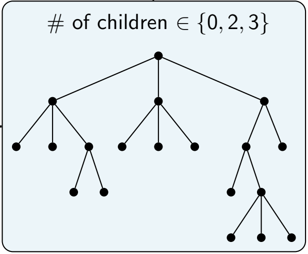
Path Classes of Interest

- **Excursions**: end on axis
- **Dispersed Excursions**: additional step ($\rightarrow$) only on axis
- **Meanders**: all non-negative paths



Bijection: Excursions $\longleftrightarrow$ Trees





Result

Excursions($\mathcal{S}$) $\longleftrightarrow$ Plane Trees($\mathcal{S} + 1$)

Exploiting the Tree Structure

Theorem

- *$V(z,t)$ ... OGF of $\mathcal{V}$: plane trees, # of children $\in \mathcal{S}+1$*
  - *z ... size of tree*
  - *t ... r-ascents in corresponding Łukasiewicz excursion*

Then:

1. *$V(z,t)/z$ ... Łukasiewicz excursions w.r.t. $\mathcal{S}$*
2. *$V(0,t) = 0$ and $V(z,t) = z\,L(z,t,V(z,t))$ with*

$$L(z,t,v) = \frac{1}{1 - zS_+(v)} + (t-1)(zS_+(v))^r,$$

which enumerates seq. of non-negative steps. v ... height

1. Consequence of bijection.
2. Decompose w.r.t. leftmost path:

$$\mathcal{V} = \circ \times \mathrm{SEQ}(\circ \times \sum\nolimits_{s\in\mathcal{S},\ s\geq 0} \mathcal{V}^s)$$

From Excursions to Arbitrary Paths

Observation:

- $L(z,t,v)z/v \rightsquigarrow$ sequence of non-negative steps followed by $\searrow$
- $\frac{1}{1-L(z,t,v)z/v} \rightsquigarrow$ **all** paths (also crossing axis) ending on $\searrow$
- subtract "bad paths": excursion $\times \searrow \times$ arbitrary, i.e.,

$$\frac{V(z,t)}{z}\frac{z}{v}\frac{1}{1-L(z,t,v)z/v}.$$

Theorem

- $F(z,t,v)$ ... *OGF counting Łukasiewicz paths w.r.t.* $\mathcal{S}$
  - z ... *length,* t ... r*-ascents,* v ... *ending altitude*

Then:

$$F(z,t,v) = \frac{v - V(z,t)}{v - z\,L(z,t,v)} L(z,t,v).$$

- $v = 0 \rightsquigarrow$ excursions, $v = 1 \rightsquigarrow$ meanders

OGF for Dispersed Excursions

Theorem

- $D(z,t)$ ... *OGF for dispersed excursions*

$$D(z,t) = \frac{1}{z}\frac{V(z,t)}{1-V(z,t)}$$

Proof. Decomposition: (excursion $\times \rightarrow)^* \times$ excursion.

$$\Rightarrow \quad D(z,t) = \frac{1}{1 - z\,V(z,t)/z}\,\frac{V(z,t)}{z}.$$

$\square$

Expressing Partial Derivatives

Proposition

Partial derivatives of the form $\partial_t^j V(z,t)|_{t=1}$ can be expressed in terms of $V(z,1)$, e.g.,

$$\partial_t V(z,t)|_{t=1} = -z \frac{(V(z,1)-z)^r}{V(z,1)^{r+2} S'(V(z,1))}.$$

Sketch of Proof. Implicit differentiation of defining equation

$$V(z,t) = z\, L(z,t,V(z,t)).$$

□

Remark.

- $V(z,1)$ satisfies $V(z,1) = zV(z,1)S(V(z,1))$,
- $\rightsquigarrow$ i.e., it has type $y = z\varphi(y)$
- analytic details (singular expansion, ...) via **singular inversion**!

Periodic Walks

Observation:

- Dyck excursions ($\mathcal{S} = \{-1, 1\}$) only touch axis after even number of steps $\rightsquigarrow$ $\mathcal{S}$ is 2-periodic

Proposition

$$p = \gcd_{s\in\mathcal{S}}(s+1) \implies \mathcal{S} \textit{ is } p\textit{-periodic}$$

Proof. $S(u)^n$ …GF of unrestricted paths of length n. u …height.

$$[u^0]S(u)^n = [u^n](uS(u))^n = [u^n]Q(u^p)^n,$$

with $Q(u^p) = uS(u)$. □

- Technical Detail: $V(z, 1)$ has p square root singularities on its radius of convergence!

Singular Expansions of V

Proposition

- $\mathcal{S}$ *has period* p,
- $\tau > 0$ ... *"structural constant", unique positive solution of* $S'(\tau) = 0$.

Then:

1. $V(z, 1)$ *has radius of convergence* $\rho = 1/S(\tau)$,
2. *dominant singularities: square-root singularities at* $\zeta\rho$
   - ζ ... *pth root of unity*
3. *Singular expansion* $z \to \zeta\rho$:

$$V(z, 1) = \zeta\tau - \zeta\sqrt{\frac{2S(\tau)}{S''(\tau)}}\Big(1 - \frac{z}{\zeta\rho}\Big)^{1/2} + O\Big(1 - \frac{z}{\zeta\rho}\Big).$$

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Excursions – Result

Theorem (r-Ascents in Excursions)

- $r \in \mathbb{N}$, $n \in \mathbb{N}_0$, $p \geq 1$ *… period of* $\mathcal{S}$,
- $\tau > 0$ *… structural constant; define* $c := \tau S(\tau)$,
- $E_{n,r}$ *… RV counting r-ascents in (unif. random) Łukasiewicz excursions of length n.*

1. $E_{n,r} = 0$ *if* $n \not\equiv 0 \pmod p$,
2. *For* $n \equiv 0 \pmod p$ *and* $n \to \infty$*:*

$$\mathbb{E}E_{n,r} = \frac{(c-1)^r}{c^{r+2}} n + O(1),$$

$$\mathbb{V}E_{n,r} = \Big(\frac{(c-1)^r}{c^{r+2}} + \frac{(2c-2r-3)(c-1)^{2r}}{c^{2r+4}} - \frac{(c-1)^{2r-2}(2c-r-2)^2}{c^{2r+3}\tau^3 S''(\tau)}\Big) n + O(n^{1/2}).$$

Excursions – Examples 1/2

Example (r-Ascents in Dyck paths)

- $\mathcal{S} = \{-1, 1\}$, $p = 2$, $\tau = 1$.
- Explicit $V(z,1) = \frac{1-\sqrt{1-4z^2}}{2z} \Rightarrow$ higher precision!

$$\mathbb{E}D_{2n,r} = \frac{n}{2^{r+1}} - \frac{(r+1)(r-4)}{2^{r+3}} + \frac{(r^2-11r+22)(r+1)r}{2^{r+6}} n^{-1} + O(n^{-2})$$

$$\mathbb{V}D_{2n,r} = \Big(\frac{1}{2^{r+1}} - \frac{r^2-2r+3}{2^{2r+3}}\Big) n + O(1)$$

Excursions – Examples 2/2

Example (2018–02–14)

- $\mathcal{S} = \{-1, 2, 14, 2018\}$, $S(u) = u^{-1} + u^2 + u^{14} + u^{2018}$, $p = 3$,
- $\tau > 0 : S'(\tau) = 0 \to \tau \approx 0.74556$

1. If $n \not\equiv 0 \pmod 3$, then $E_{n,r} = 0$.
2. Else:

$$\mathbb{E}E_{n,r} \sim 0.49132\ldots \cdot (0.29906\ldots)^r n$$

$$\mathbb{V}E_{n,r} \sim (0.49132\ldots \cdot (0.29906\ldots)^r - 0.24139\ldots \cdot (0.14669\ldots + 2r)(0.29906\ldots)^{2r} - 0.03348\ldots \cdot (0.85330\ldots - r)^2(0.29906\ldots)^{2r-2})n$$

Dispersed Excursions – Result

Theorem (r-Ascents in Dispersed Excursions, $\tau \neq 1$)

- $r \in \mathbb{N}$, $n \in \mathbb{N}_0$, $p \geq 1$ ... *period of* $\mathcal{S}$,
- $\tau > 0$ ... *structural constant*, $\tau \neq 1$,
- d_n ... *number of dispersed excursions of length* n,
- $D_{n,r}$ ... *RV counting* r*-ascents in dispersed excursions of length* n

1. *For* $n \to \infty$ *and* $n \equiv k \pmod{p}$, $0 \leq k < p$

$$d_n = \frac{1}{\sqrt{2\pi}} \frac{p\tau^k(\tau^p(p-k-1)+k+1)}{(1-\tau^p)^2} \sqrt{\frac{S(\tau)^3}{S''(\tau)}} S(\tau)^n n^{-3/2} + O(S(\tau)^n n^{-5/2}).$$

2. $$\mathbb{E}D_{n,r} = \frac{(\tau S(\tau)-1)^r}{(\tau S(\tau))^{r+2}} n + O(1).$$

Dispersed Excursions – Special Case

Proposition (r-Ascents in Dispersed Dyck Paths)

- $\mathcal{S} = \{-1, 1\}$, $p = 2$, $\tau = 1$.

$$d_n = \binom{n}{\lfloor n/2 \rfloor} = \sqrt{\frac{2}{\pi}} 2^n n^{-1/2} - \frac{2 - (-1)^n}{2\sqrt{2\pi}} 2^n n^{-3/2} + O(2^n n^{-5/2}),$$

$$\mathbb{E} D_{n,r} = \frac{n}{2^{r+2}} - \sqrt{\frac{\pi}{2}} \frac{r-2}{2^{r+2}} n^{1/2} + \frac{(r-1)(r-4)}{2^{r+3}} + O(n^{-1/2}).$$

Meanders – Result

Theorem (r-Ascents in Meanders, $\tau \neq 1$)

- $\tau > 0$ ... *structural constant,* $\tau \neq 1$,
- $M_{n,r}$ ... *RV counting r-ascents in meanders of length n*

Then:

$$\mathbb{E}M_{n,r} = \mu n + c_{\mathcal{S}} + O\Big(\Big(\frac{S(\tau)}{S(1)}\Big)^n n^{5/2}\Big), \quad \mathbb{V}M_{n,r} = \sigma^2 n + O(1),$$

with

$$\mu = \frac{(S(1)-1)^r}{S(1)^{r+2}}, \qquad \sigma^2 = \mu + \frac{(S(1)-1)^{2r}(2S(1)-3-2r)}{S(1)^{2r+4}}.$$

Also, $M_{n,r}$ is asymptotically normally distributed for $n \to \infty$.

Meanders – Special Cases

Proposition (r-Ascents in Dyck Meanders)

- $\mathcal{S} = \{-1, 1\}$, $p = 2$, $\tau = 1$.

$$\mathbb{E}M_{n,r} = \frac{n}{2^{r+2}} + \frac{\sqrt{2\pi}(r-2)}{2^{r+3}} n^{1/2} - \frac{r^2 - r - 8}{2^{r+3}} + O(n^{-1/2}),$$

$$\mathbb{V}M_{n,r} = \frac{2^{r+3} - r^2(\pi - 2) + 4r(\pi - 3) - 4\pi + 10}{2^{2r+5}} n + O(n^{1/2}).$$

Proposition (r-Ascents in Motzkin Meanders)

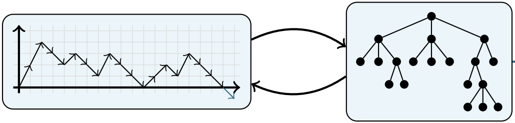
- $\mathcal{S} = \{-1, 0, 1\}$, $p = 1$, $\tau = 1$.

$$\mathbb{E}M_{n,r} = \frac{2^r}{3^{r+2}} n + \frac{\sqrt{3\pi}(r-4)2^{r-2}}{3^{r+2}} n^{1/2} + O(1),$$

$$\mathbb{V}M_{n,r} = \frac{3^{r+2}2^{r+4} - 2^{2r}(3r^2(\pi - 2) - 8r(3\pi - 10) + 48\pi - 144)}{16 \cdot 3^{2r+4}} n + O(n^{1/2}).$$

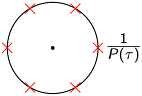
Summary

Bijection



$V(z,t) \Longrightarrow F(z,t,v)$ OGFs

Periodicity



$$\mathbb{E}_{n,r} \sim \frac{(\tau S(\tau)-1)^r}{(\tau S(\tau))^{r+2}} n$$

Excursions

$$\mathbb{E}_{n,r} \sim \frac{(\tau S(\tau)-1)^r}{(\tau S(\tau))^{r+2}} n$$

Dispersed Excursions

$$\mathbb{E}_{n,r} \sim \frac{(S(1)-1)^r}{S(1)^{r+2}} n$$

Meanders