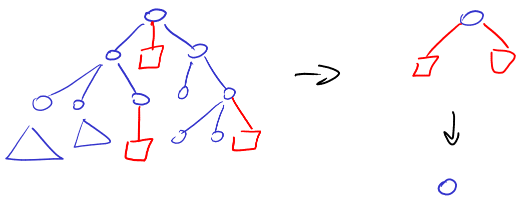
Catalan-Stanley trees + Reductions (Part II: Parameter Analysis)

↳ What we know by now:

→ Catalan-Stanley trees: plane trees where rightmost leaves in branches of root have odd distance to root. (Origin: growth process: add 2 plane trees to special nodes ...)



→ CS-Trees are enumerated by Catalan numbers: C_{n-2} CS-trees of size n.

→ Bivariate Generating function:

$$C(z,t) = z + \frac{tz}{1 - t - T(z)^2}.$$

($z \mathrel{\hat=} \circ$, i.e. any "ordinary" node; $t \mathrel{\hat=} \square$, i.e. special rightmost leaves)

Parameters of Interest

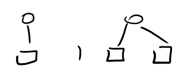
- D_n ... Age (i.e., # generations req. to grow some tree) of a unif. random tree of size n.
- $X_{n,r}$ ... Size of the r-th ancestor of a unif. random CS-tree of size n.

We consider the growth operator Φ.

→ $f(z,t)$ .. GF enum. some family of CS-trees $\mathcal{F}$. ($f(z,t) = z$, $\{\circ\}$)

→ Want: $\Phi(f(z,t))$ should enumerate the family of trees that can be grown from $\mathcal{F}$ after 1 Generation.

Ex.: Consider $f(z,t) = z$, $\mathcal{F} = \{\circ\}$ ⇒ Family grown from $\mathcal{F}$

$= \{\circ,$  $, \ldots\}$

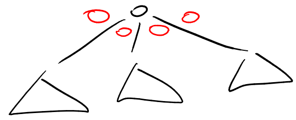
z (Age 0), zt, zt^2 (Age 1)

$$\Rightarrow \Phi(z) \overset{!}{=} \frac{z}{1-t}.$$

- Φ is linear (combinatorially obvious)
- we just need $\Phi(z^n t^k)$!

$$\to \quad \phi(z^n t^k) = z^n \cdot (t \cdot T(z)^2)^k \left(\frac{1}{1-t}\right)^{k+1}$$

$$= z^n \left(\frac{t\, T(z)^2}{1-t}\right)^k \cdot \frac{1}{1-t}.$$



k branches $\Rightarrow$ $k+1$ options

<u>Thus:</u> $\phi(f(z,t)) = \phi\left(\sum_{n,k} z^n t^k \cdot f_{n,k}\right) =$

$$= \sum_{n,k} f_{n,k} \cdot \phi(z^n t^k) = \sum_{n,k} f_{n,k} \cdot z^n \cdot \left(\frac{t\,T(z)^2}{1-t}\right)^k \cdot \frac{1}{1-t}$$

$$= \frac{1}{1-t} \cdot \sum_{n,k} f_{n,k}\, z^n \left(\frac{t\,T(z)^2}{1-t}\right)^k = \frac{1}{1-t} \cdot f\left(z, \frac{t\,T(z)^2}{1-t}\right).$$

We were able to prove:

$$\phi^r(f(z,t)) = \frac{1}{1 - t \cdot \frac{1-T(z)^{2r}}{1-T(z)^2}} \cdot f\left(z, \frac{t\ T(z)^{2r}}{1 - t\, \frac{1-T(z)^{2r}}{1-T(z)^2}}\right).$$

<u>Sketch:</u> Consider $\psi(f(z,t)) = (1-t)\cdot \phi(f(z,t)) = f\left(z, \frac{t\,T(z)^2}{1-t}\right)$

Recurrences for $\psi^r(t)$, $\psi^r(z)$ ... translate back to ϕ.

<u>Observation:</u> growing $\circ$ r times $(= \phi^r(z))$ yields all trees of age $\leq r$.

$$\Rightarrow \quad F_r^{\leq}(z,t) = \phi^r(z) = \frac{1}{1 - t\,\frac{1-T(z)^{2r}}{1-T(z)^2}} \cdot z.$$

<u>Strategy:</u> $\mathbb{E}(D_n) = \sum_{r \geq 1} \mathbb{P}(D_n \geq r)$.

$[z^n]\, F_r^{\geq}(z) \leq C_{n-2}$

We construct $F_r^{\geq}(z) = C(z,z) - F_{r-1}^{\leq}(z,z)$.

$$= \dots = z \cdot (1 + T(z)) \cdot \frac{T(z)^{2r-1}}{1 + T(z)^{2r-1}}.$$

$$= z \cdot (1+T) \cdot \frac{1}{1 + T(z)^{1-2r}}.$$

<u>Then:</u> $$\mathbb{P}(D_n \geq r) = \frac{[z^n]\ z \cdot (1+T)\frac{1}{1+T^{1-2r}}}{C_{n-2}}.$$

Via our SageMath-module, we found: $\mathbb{E}(D_n) = 2{,}71825\ldots - 4.22209\ldots \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right).$