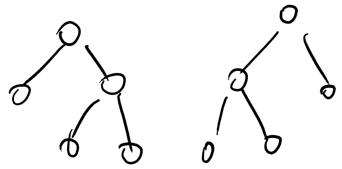
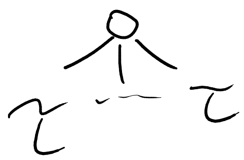
plane trees:



$\mathcal{T}$ ... comb. class of plane trees , $\circ \hat{=} z$

$\mathcal{T} =$  , $\mathcal{T} = \{\circ\} \times \mathrm{SEQ}(\mathcal{T})$

$$T(z) = z \cdot \frac{1}{1 - T(z)} \implies T(z) = \frac{1 \pm \sqrt{1-4z}}{2}.$$

$$T(z) = \sum_{n \geq 0} t_n \cdot z^n, \quad T(0) = t_0 \overset{!}{=} 0$$

$$\implies T(z) = \frac{1 - \sqrt{1-4z}}{2}.$$

Asymptotic expansion: $\sqrt{\pi}42 \cdot 4^n \cdot \sqrt{n} + O(4^n \cdot \frac{1}{n})$

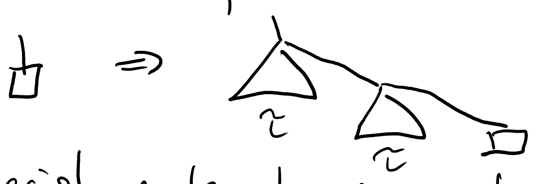
$(\text{Rational})^n \cdot n^{(\text{Rational})}$

$$\frac{1}{n-1} \overset{n \to \infty}{=} \frac{1}{n(1-\frac{1}{n})} = \frac{1}{n} \cdot \frac{1}{1-\frac{1}{n}}$$

$$= \frac{1}{n} \cdot \sum_{j \geq 0} \left(\frac{1}{n}\right)^j = \sum_{j \geq 1} \frac{1}{n^j}.$$

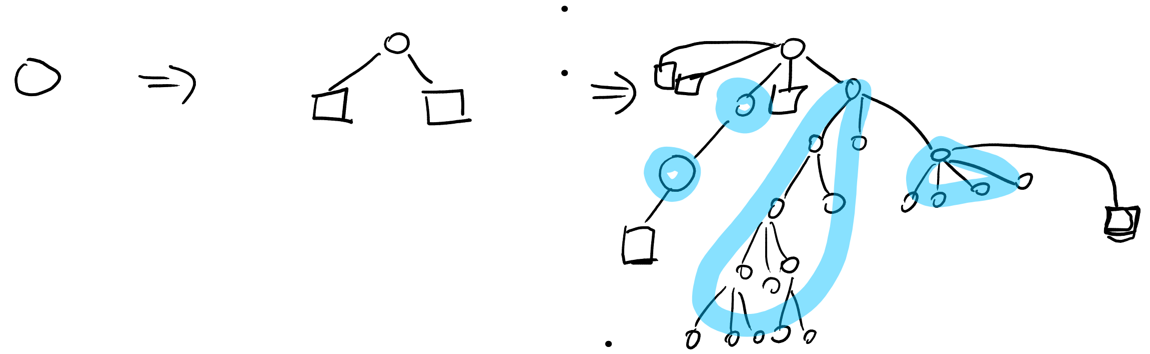
Consider the following growth process:

- Start with $\circ$.

① Before any special node $\square$, add two arbitrary plane trees: 

② (optional): add special nodes to the root of the tree.

Ex.:



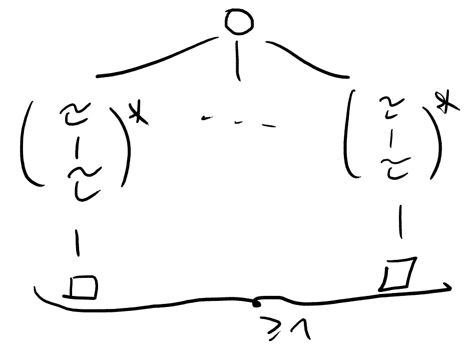
Q: What trees can be grown?

Conj.: All trees where the rightmost leaves in the branches of the root have odd distance to the root.

We call those trees Catalan-Stanley trees, (CS-trees).

Q: How many CS-trees of size n are there?

$\mathcal{C} = \circ \; +$



$$\mathcal{C} = \{\circ\} + \{\circ\} \times SEQ_{\geq 1}\left(SEQ(\mathcal{T}^2) \times \{\square\}\right)$$

$$\Rightarrow \underset{\substack{\uparrow \\ \circ}}{C(z}, \underset{\substack{\uparrow \\ \square}}{t)} = z + z \cdot \frac{\frac{1}{1-T^2} \cdot t}{1 - \frac{1}{1-T^2} \cdot t}$$

$$= z + \frac{zt}{1-t-T^2} = \frac{z(1-T^2)}{1-t-T^2}.$$

$$C(z,z) = z + \frac{z^2}{1-(z+T^2)} = z + \frac{z^2}{1-T} = z + zT.$$

$$T = \frac{z}{1-T}$$

$$T - T^2 = z \iff z + T^2 = T$$

$$\frac{z(1-T^2)}{1-z-T^2} = z\cdot\left(\frac{1-T^2}{1-T}\right) = z(1+T) = z + zT$$

$$= \sum_{n\geq 1} z^n \cdot C_{n-2} \,.$$

$$(C_{-1} = 1)$$

Goal: D_n ... "Age of a (unif. random) CS-tree of size n"

\# of growth steps req. to grow the tree from o.

register function.