On Reductions of Plane Trees

Benjamin Hackl

joint work with
Clemens Heuberger, *Sara Kropf*, *Helmut Prodinger*

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April 7, 2017

mso DK modeling simulation optimization

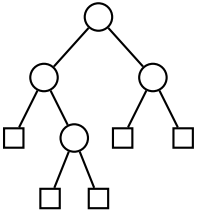
KARL POPPER KOLLEG

FWF Der Wissenschaftsfonds.

Trimming binary trees

Binary trees can be "trimmed" by the following strategy:

- Remove all leaves
- Merge nodes with only one descendant



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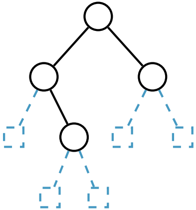

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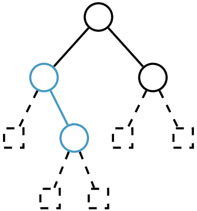

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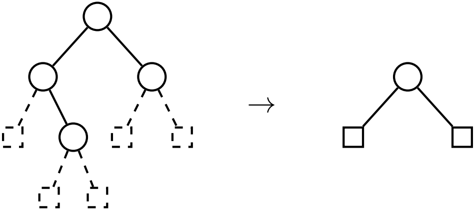

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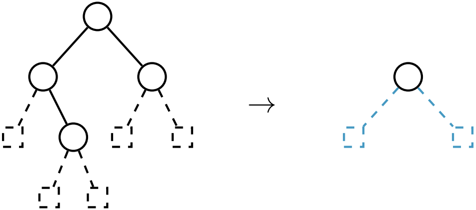

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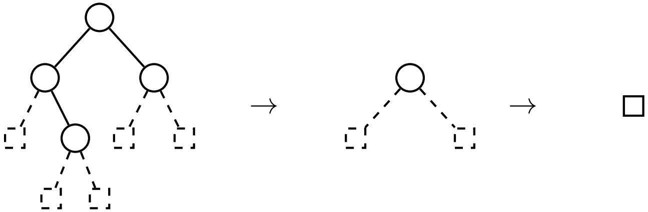

"Surviving" nodes

Label all nodes in the tree by the following rules:

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- val(left child) = val(right child) $\rightarrow$ increase by 1
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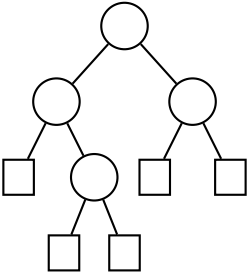

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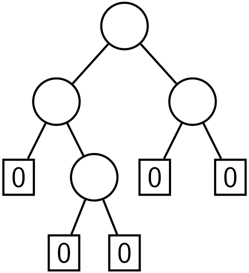

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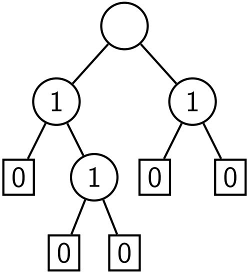

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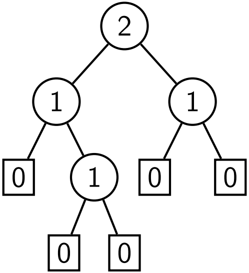

The register function

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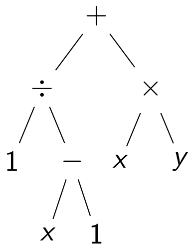

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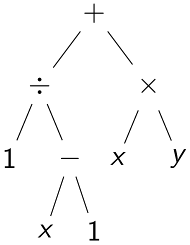

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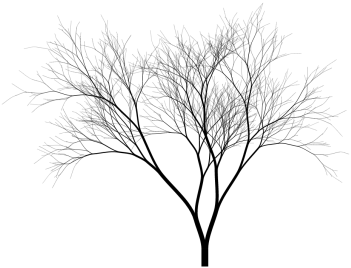

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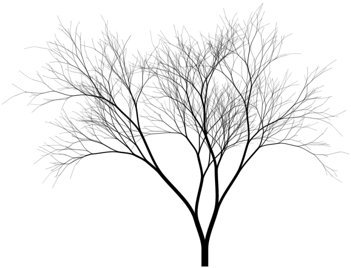

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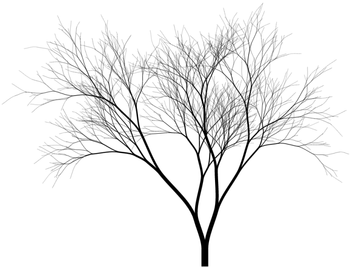

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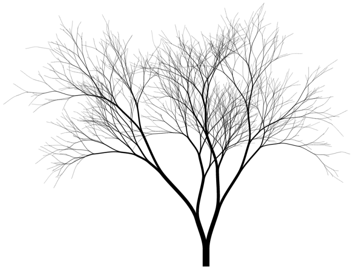

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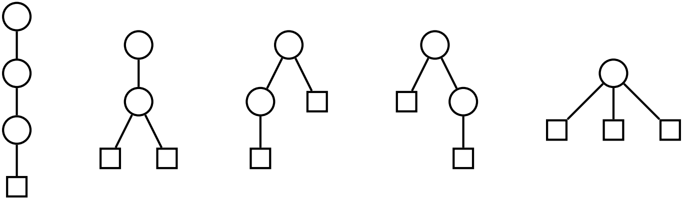

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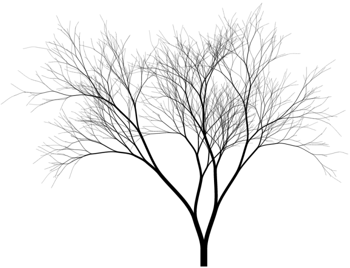

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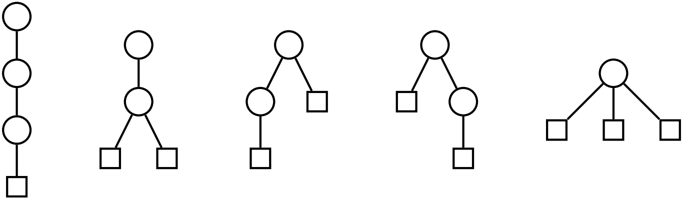

- $C_n = \frac{1}{n+1}\binom{2n}{n}$ plane trees of size n

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Growing plane trees

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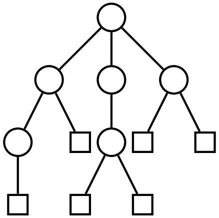

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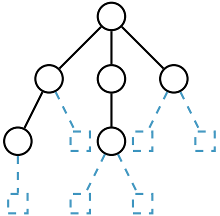

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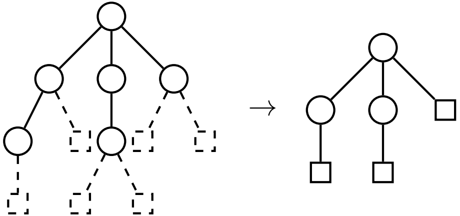

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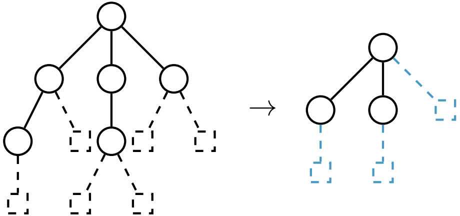

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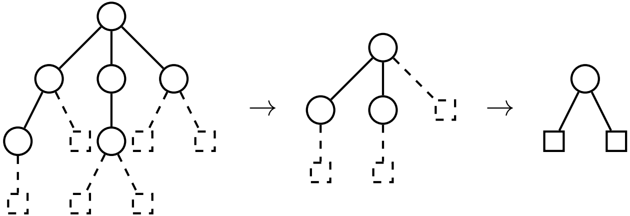

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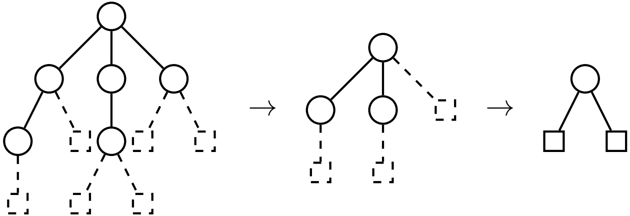

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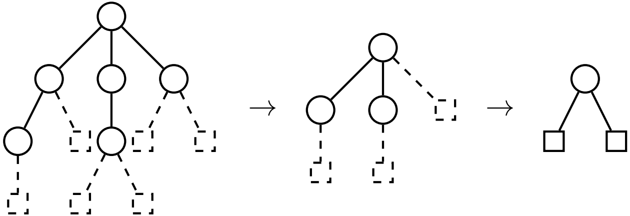

- Growing trees:
  - grow new leaves out of current leaves and inner nodes

“What?” and “How?”

- **Aim:** analysis of tree structure under iterated reduction

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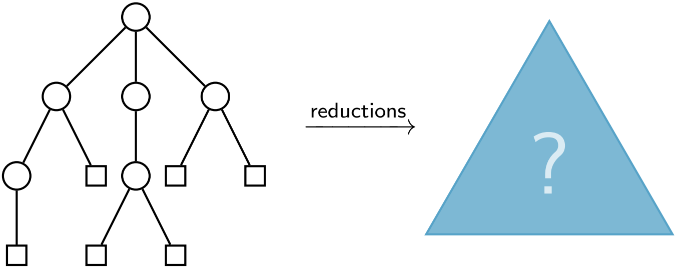

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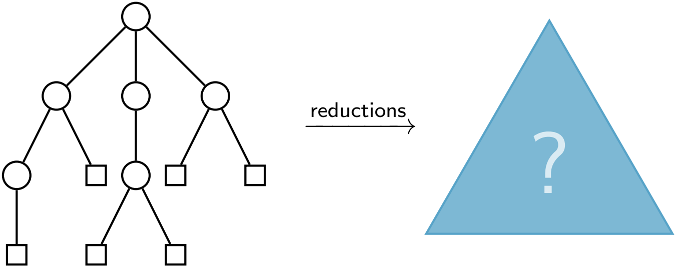

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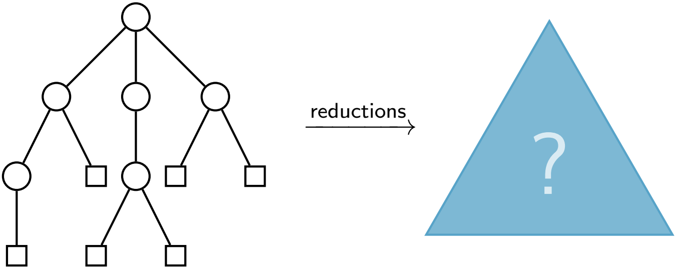

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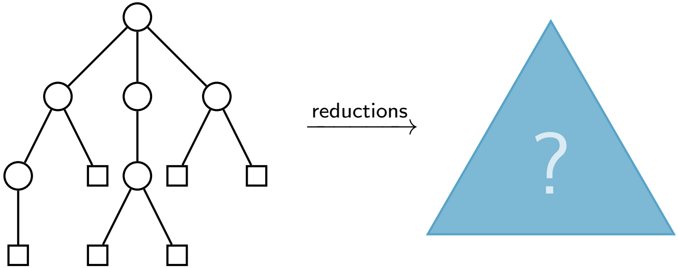

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- Investigation of "tree expansion" $\rightsquigarrow$ GF
- Coefficient extraction; Parameter distribution

BGF for plane trees

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Proof. Symbolic equation

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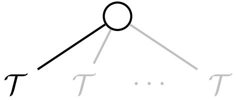

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which can be solved explicitly.

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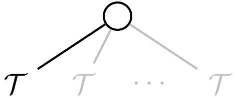

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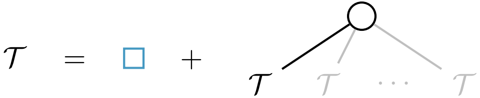

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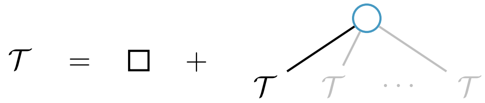

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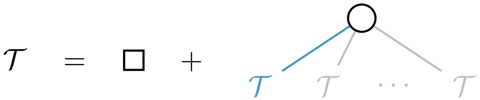

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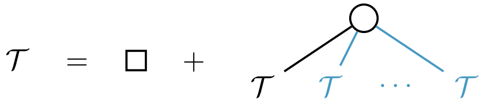

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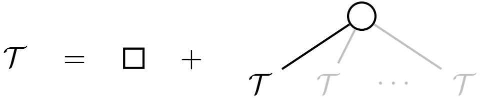

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- $N_{n-1}(t)\ldots$ Narayana polynomial, counts trees of size n (i.e. $n-1$ edges) w.r.t. number of leaves

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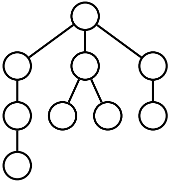
Interpretation: half of all nodes among all trees of size n are leaves.

Proof – Rotation correspondence

Construction of "Left-Child Right-Sibling"-tree:



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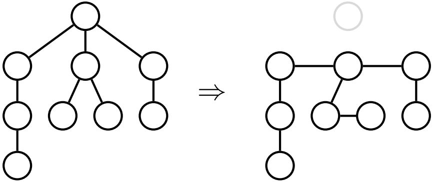

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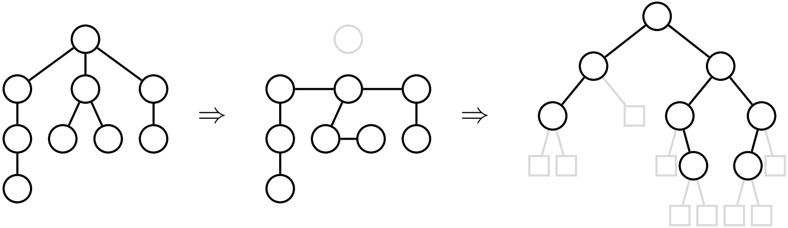

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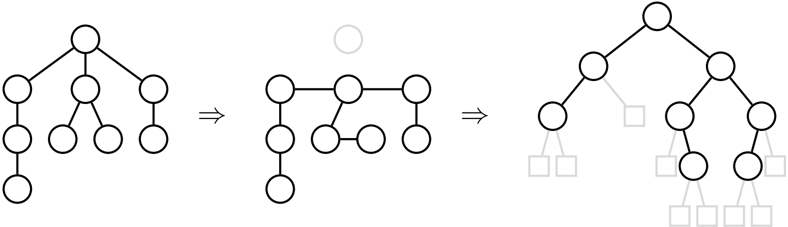

Observations:

- left leaves (binary tree) $\leftrightsquigarrow$ leaves (plane tree)

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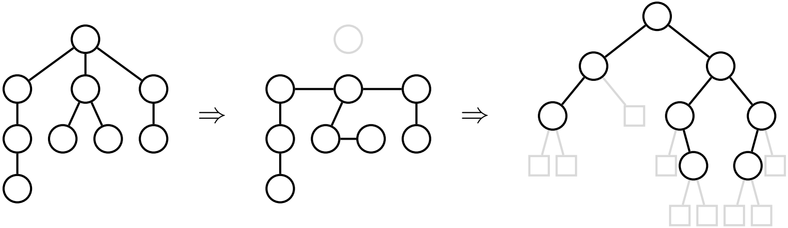

Observations:

- left leaves (binary tree) $\leftrightsquigarrow$ leaves (plane tree)
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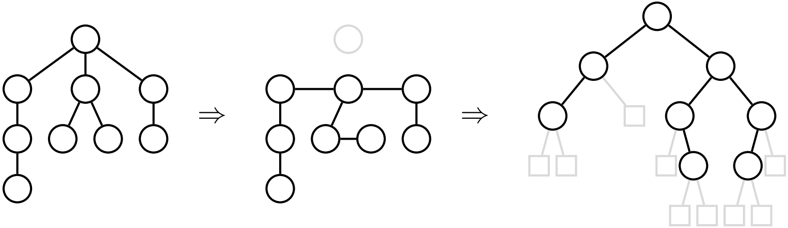

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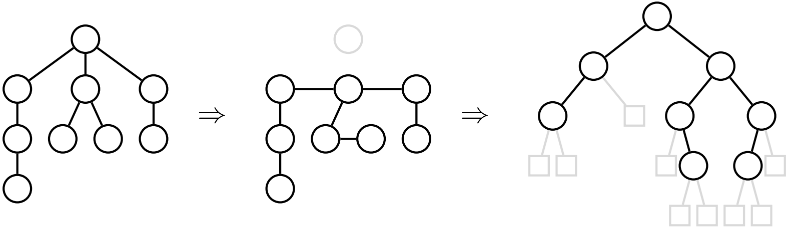

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- Proof 2: symmetry: equally many left as right leaves

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How do we cut our trees?

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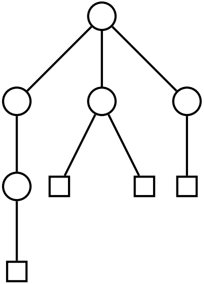

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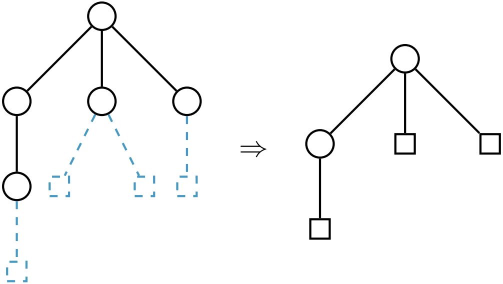

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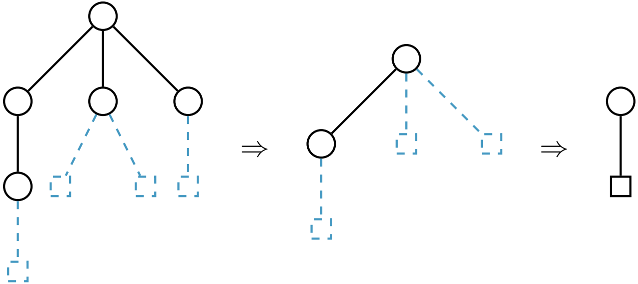

Expansion operators

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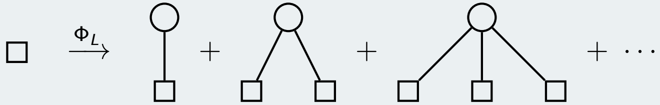
- inverse operation to leaf reduction
  - attach leaves to all current leaves (necessary)
  - attach leaves to inner nodes (optional)



$$\Rightarrow \quad \Phi_L(t) = zt + zt^2 + zt^3 + \cdots$$

Leaf expansion operator Φ_L

Proposition

$$\Phi_L(f(z,t)) = (1-t)f\left(\frac{z}{(1-t)^2}, \frac{zt}{(1-t)^2}\right)$$

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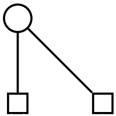

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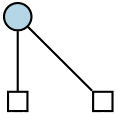

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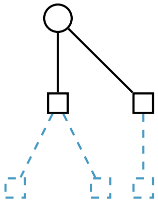

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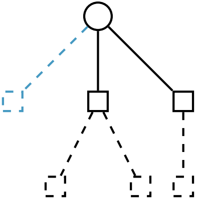

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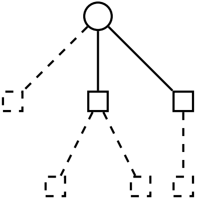

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- As Φ_L is linear, this proves the proposition.

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- This allows extracting the coefficient of $z^n q^d$

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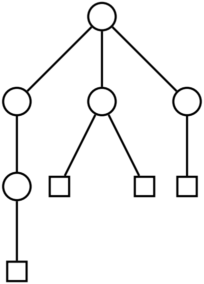
How do we cut our trees? (2)

- Remove all paths that end in a leaf!



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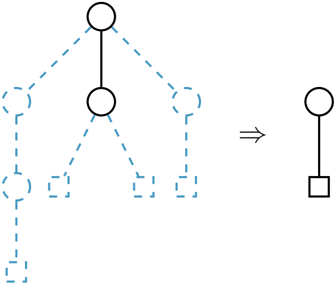

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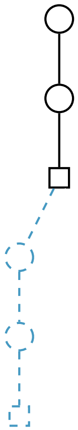

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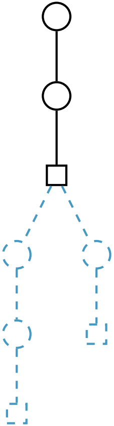

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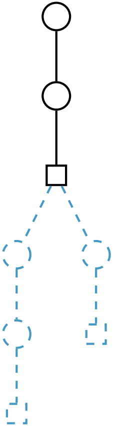

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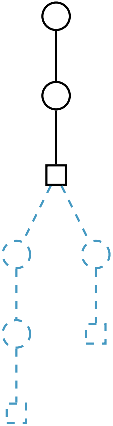

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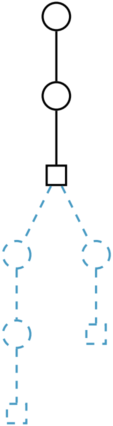

Proposition

The linear operator given by

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is the path expansion operator.

Generating function for path reductions

Proposition

BGF for size comparison ($z \rightsquigarrow$ original size, $v \rightsquigarrow$ r-fold path reduced size) is

$$\frac{1 - u^{2^{r+1}}}{(1 - u^{2^{r+1}-1})(1+u)} T\Big(\frac{u(1 - u^{2^{r+1}-1})^2}{(1 - u^{2^{r+1}})^2} v, \frac{u^{2^{r+1}-1}(1-u)^2}{(1 - u^{2^{r+1}})^2} v\Big),$$

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Observation. This is the BGF for leaf reductions

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with $r \mapsto 2^{r+1}-2$.

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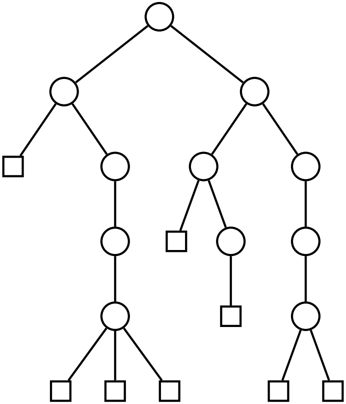
Furthermore, $X_{n,r}$ is asymptotically normally distributed.

- Factorial moments are known as well
- Proof: subsequence of RV's from cutting leaves

Counting total number of paths

- Trees can be partitioned into paths ($\rightsquigarrow$ branches)!



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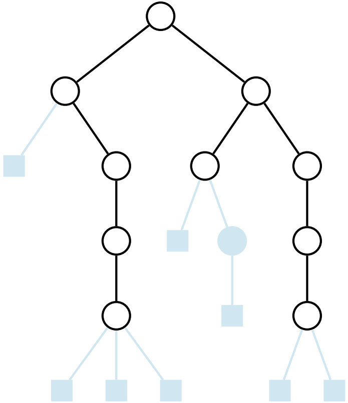

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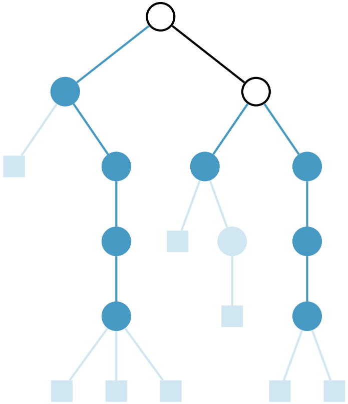

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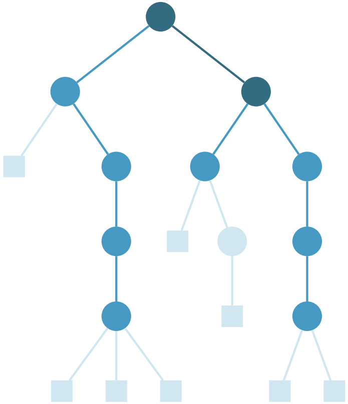

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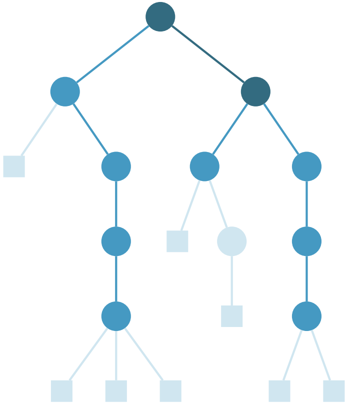

- Average number of paths?

Cutting paths – total number of paths

Theorem (H.–Heuberger–Kropf–Prodinger, 2017)

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- **Proof:** Sum of leaves in all reductions, Mellin-transform, singularity analysis.

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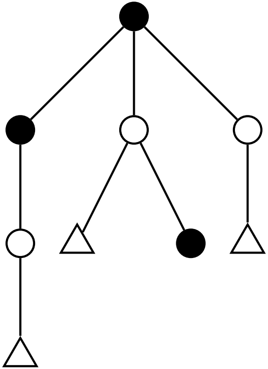

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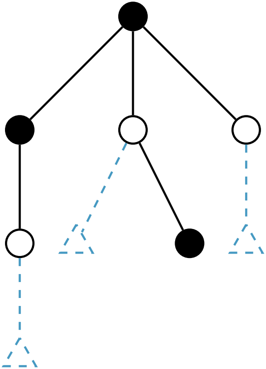

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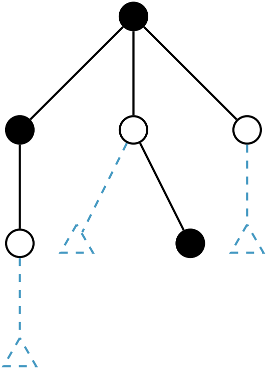

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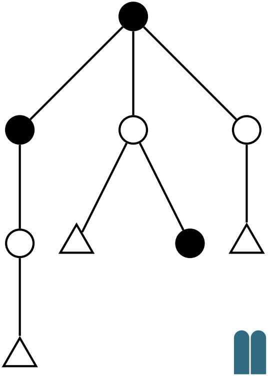

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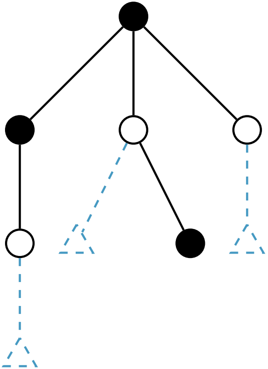

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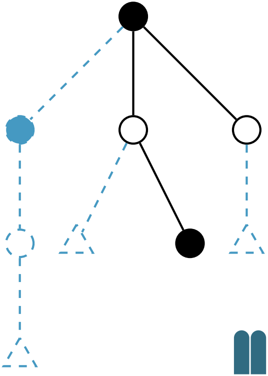

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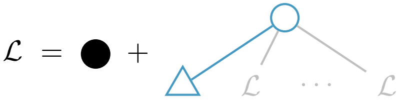

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translation; series expansion of the root.

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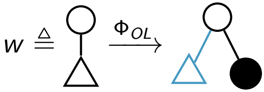

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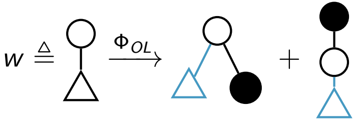

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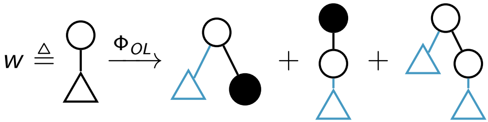

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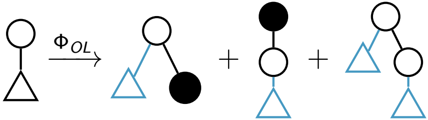

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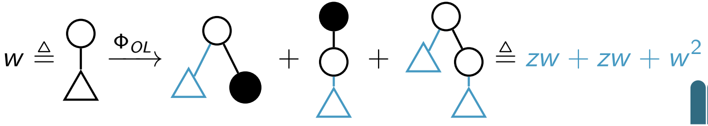

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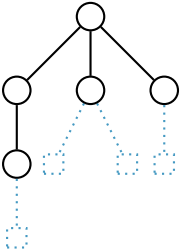

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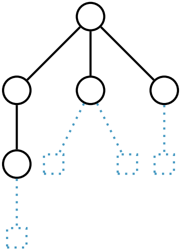

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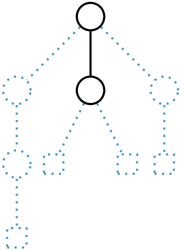

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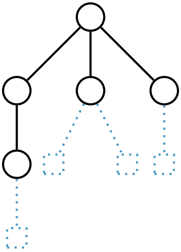

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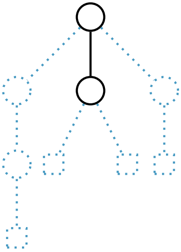

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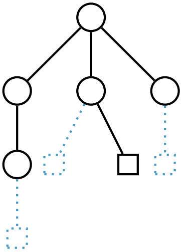

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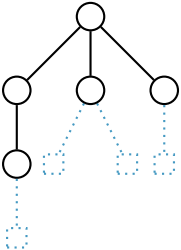

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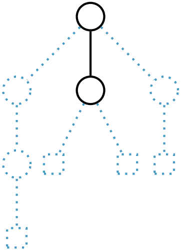

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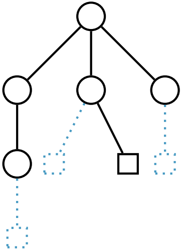

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limit law: ???