Growing and Cutting down Rooted Plane Trees

Benjamin Hackl

joint work in progress with
Sara Kropf and *Helmut Prodinger*

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT | WIEN GRAZ

DK Seminar of the Karl-Popper-Kolleg

January 11, 2017

mso DK modeling simulation optimization

KARL POPPER KOLLEG

FWF Der Wissenschaftsfonds.

Procedural Tree Generation

- Computer Graphics: "How to generate trees that look like trees efficiently?"

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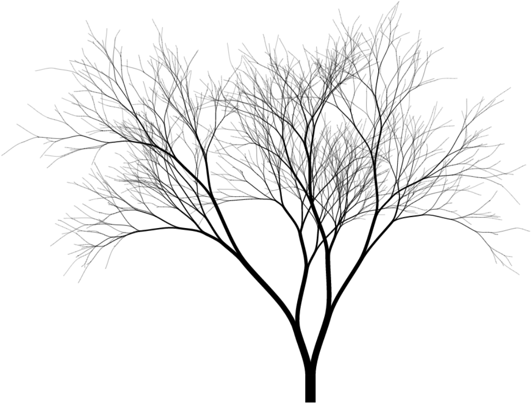

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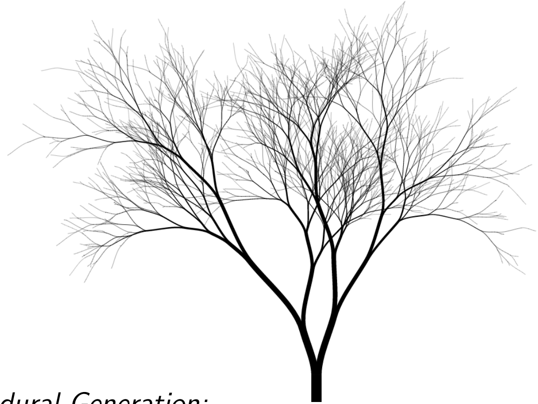

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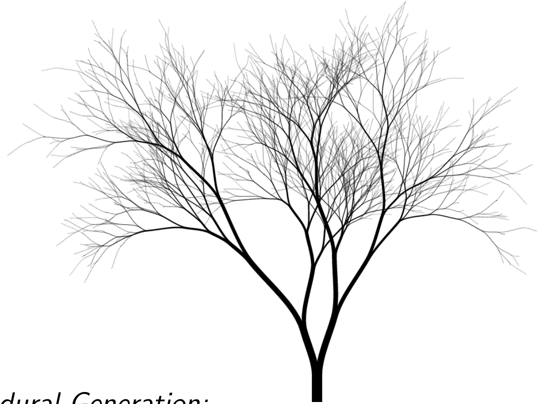

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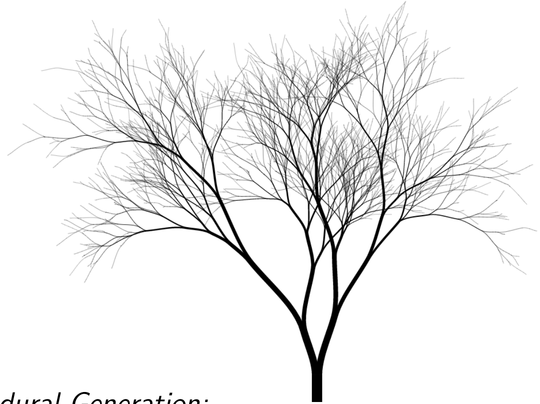

- *Procedural Generation:*
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Growing binary trees

- How can we grow binary trees?

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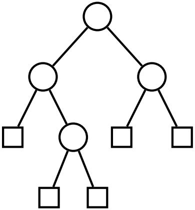

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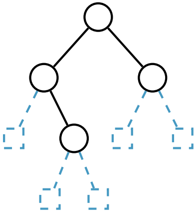

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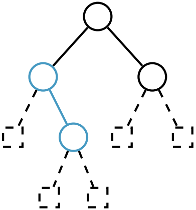

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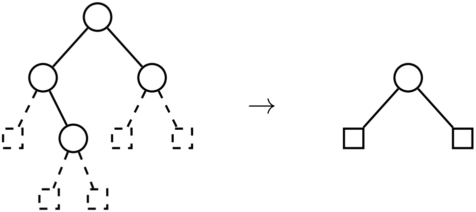

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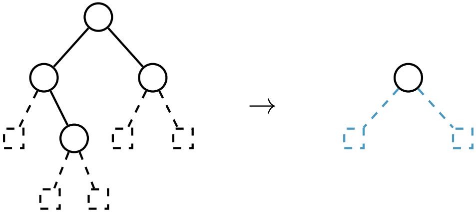

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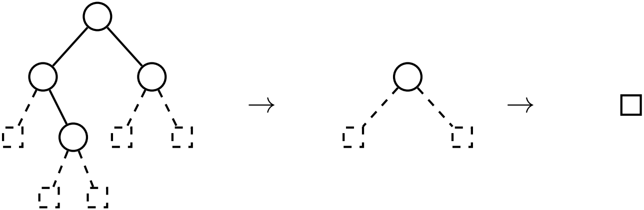

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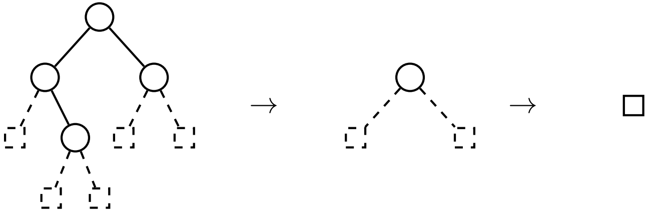

- Growing trees: attach paths to all leaves.

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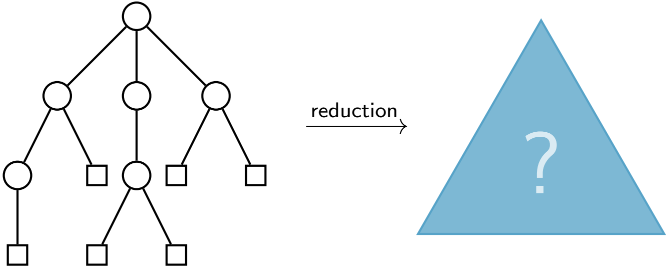

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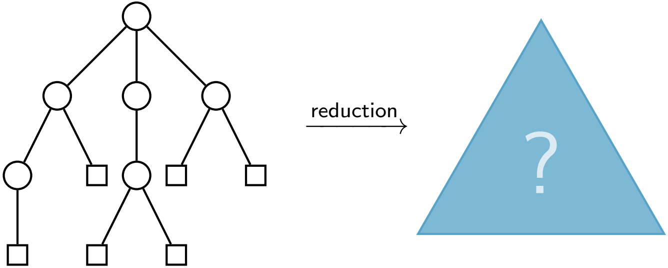

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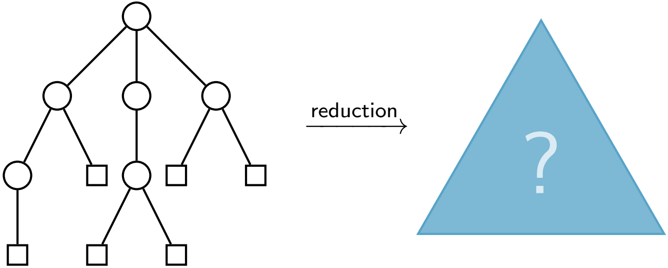

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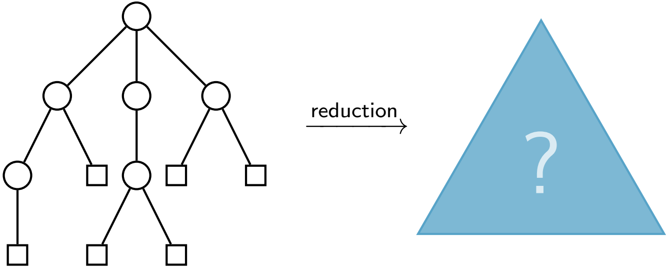

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- Coefficient extraction; Parameter distribution

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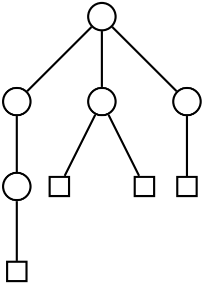

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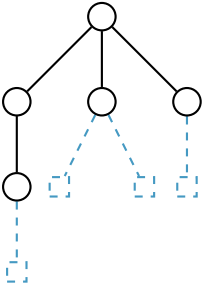

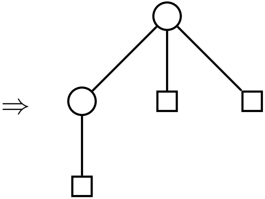

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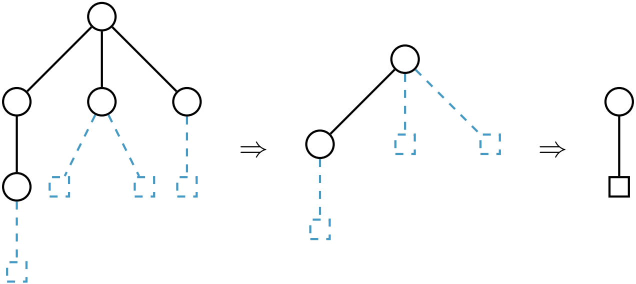

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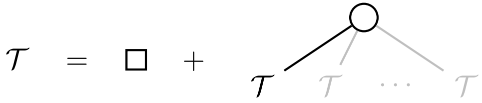
Proof. Symbolic equation



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$$T(z,t) = t + z \cdot \frac{T(z,t)}{1-T(z,t)}$$

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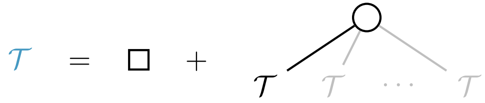

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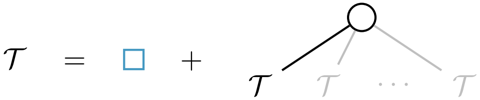

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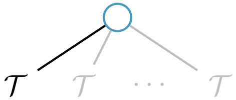

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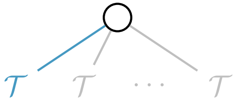
$\mathcal{T} \quad = \quad \square \quad +$ 

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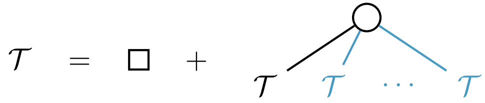

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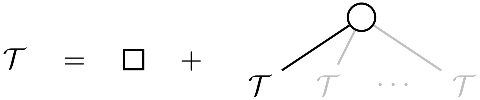

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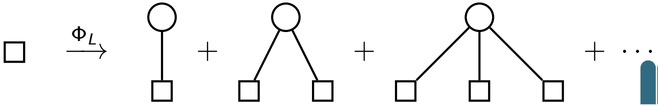

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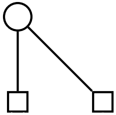
- Tree with n inner nodes and k leaves $\rightsquigarrow z^n t^k$



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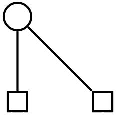

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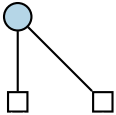

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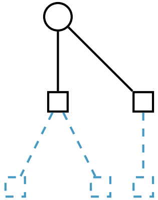

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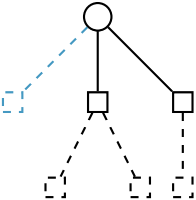

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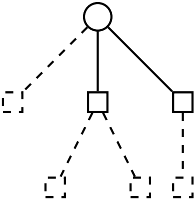

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- Linear extension of Φ_L proves the proposition.

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Then the expected size of the reduced tree and the corresponding variance are

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Cutting leaves

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- Asymptotic normality: $X_{n,r}$ is a tree parameter with small toll function, limit law by Wagner (2015)

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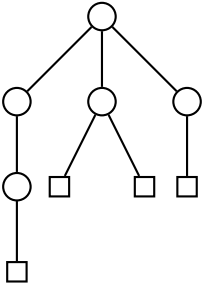
How do we cut our trees? (2)

- Remove all paths that end in a leaf!



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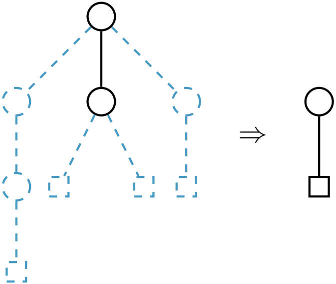

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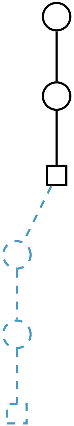

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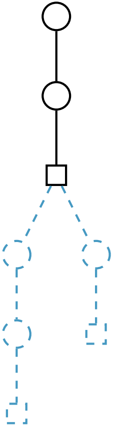

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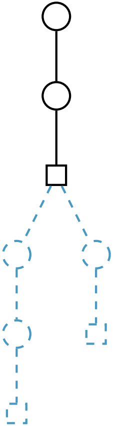

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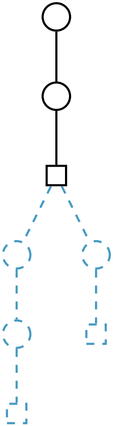

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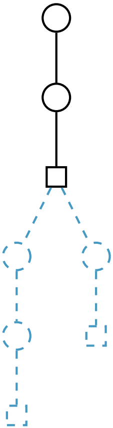

Proposition

The linear operator given by

$$\Phi_P(f(z,t)) = (1-p) f\left(\frac{z}{(1-p)^2}, \frac{zp^2}{(1-p)^2}\right)$$

is the path expansion operator.

Generating function for path reductions

Proposition

BGF for size comparison ($z \rightsquigarrow$ original size, $v \rightsquigarrow$ r-fold path reduced size) is

$$\frac{1-u^{2^{r+1}}}{(1-u^{2^{r+1}-1})(1+u)}\, T\Big(\frac{u(1-u^{2^{r+1}-1})^2}{(1-u^{2^{r+1}})^2}v, \frac{u^{2^{r+1}-1}(1-u)^2}{(1-u^{2^{r+1}})^2}v\Big),$$

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Observation. This is the BGF for leaf reductions

$$\frac{1 - u^{r+2}}{(1 - u^{r+1})(1+u)} T\Big(\frac{u(1 - u^{r+1})^2}{(1 - u^{r+2})^2} v, \frac{u^{r+1}(1-u)^2}{(1 - u^{r+2})^2} v\Big)$$

with $r \mapsto 2^{r+1} - 2$.

Cutting paths – Pruning

Theorem (H.–Kropf–Prodinger, 2016)

- *$r \dots$ number of reductions, fixed*

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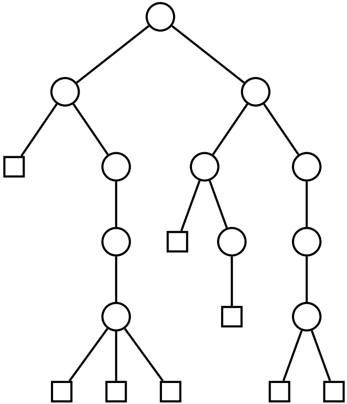
- Proof: subsequence of RV's from cutting leaves

Counting total number of paths

- Trees can be partitioned into paths!



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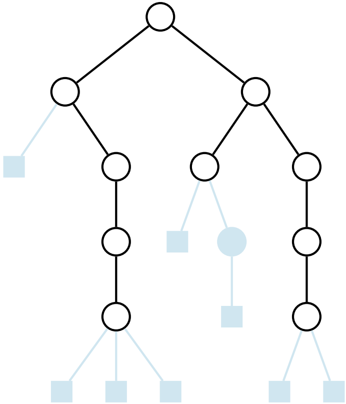

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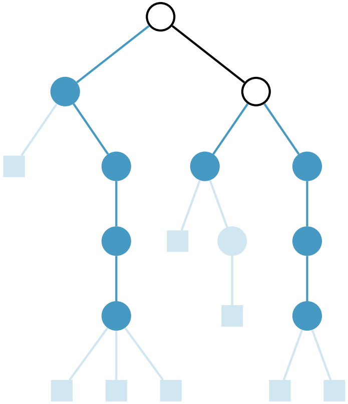

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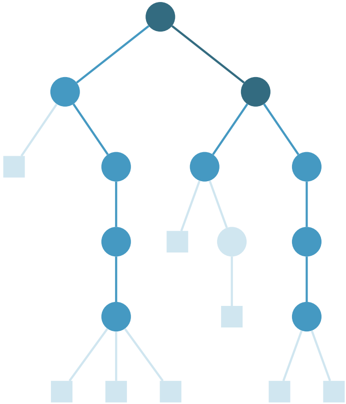

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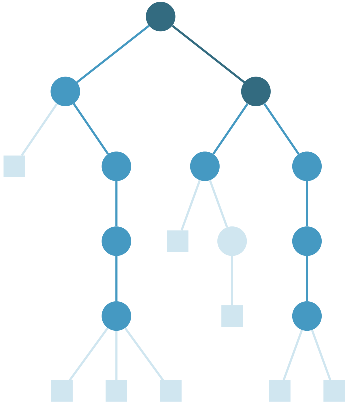

- Average number of paths?

Cutting paths – total number of paths

Theorem (H.–Kropf–Prodinger, 2017)

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- **Proof:** Sum of leaves in all reductions, Mellin-transform, singularity analysis.

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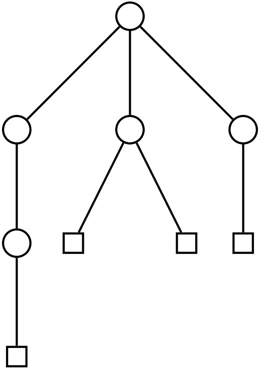

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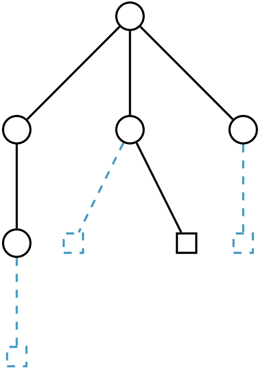

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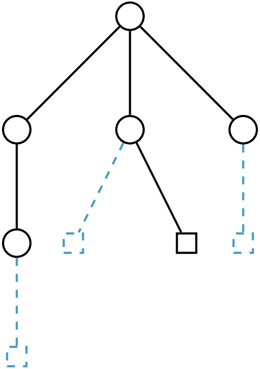

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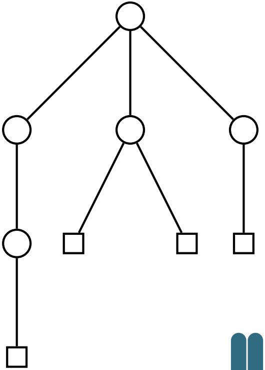

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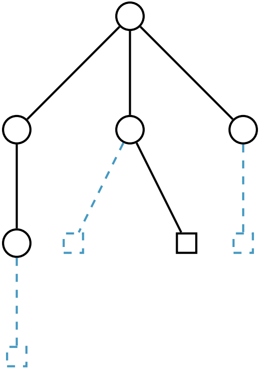

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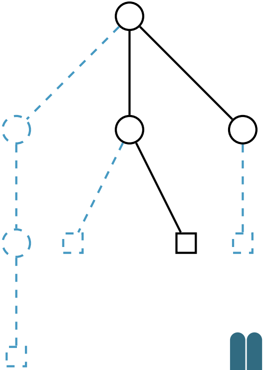

Preliminaries

Proposition

- $\mathcal{L}$... *rooted plane trees*

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and there are $C_{k-1}\binom{n-2}{n-2k}2^{n-2k}$ *trees of size* n *with* k *old leaves.*

Proof. Symbolic equation

$$\mathcal{L} = \square$$

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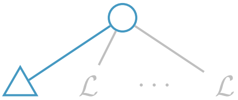

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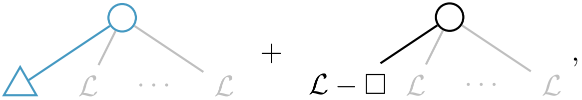

translation; Lagrange inversion.

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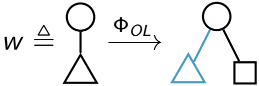

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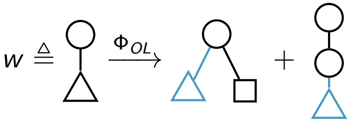

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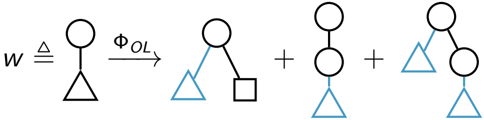

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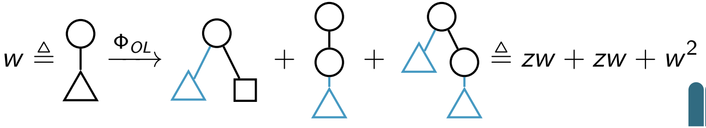

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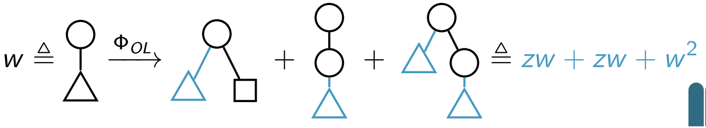
$$\Phi_{OP}(f(z,w)) = f\left(z + \frac{w}{1-z}, \left(z+\frac{w}{1-z}\right)\frac{w}{1-z}\right),$$

respectively.

Proof for old leaves.



Cutting old leaves

Theorem (H.–Kropf–Prodinger, 2016)

- *$r\ldots$ number of reductions, fixed*

Cutting old leaves

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- $r \ldots$ *number of reductions, fixed*
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Then the expected reduced tree size after r "old leaf"-reductions and the corresponding variance are given by

$$\mathbb{E}X_{n,r} = (2 - B_{r-1}(1/4))n - \frac{B'_{r-1}(1/4)}{8} + O(n^{-1}),$$

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Note. Via Flajolet, Odlyzko (1982):

$$B_r(1/4) = 2 - \frac{4}{r} - \frac{4 \log r}{r^2} + O(r^{-3}), \quad r \to \infty$$

Cutting old paths

Theorem (H.–Kropf–Prodinger, 2016)

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Cutting old paths

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Summary

Leaves

$\mathbb{E} \sim \frac{n}{r+1}$

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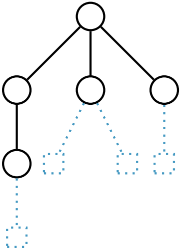
limit law: ✓



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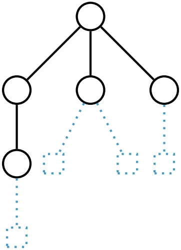

Paths

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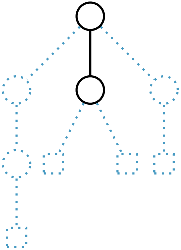

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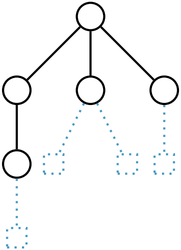

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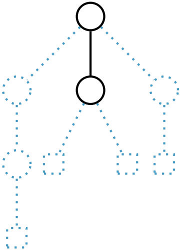
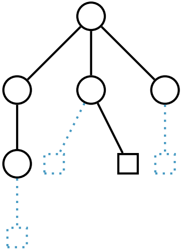


Old leaves

$\mathbb{E} \sim (2 - B_{r-1}(1/4))n$

$\mathbb{V} = \Theta(n)$

limit law: ???



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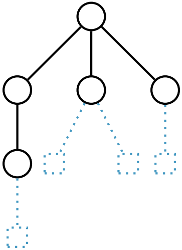

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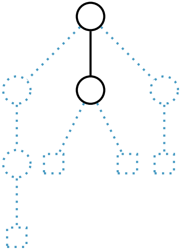

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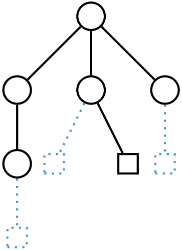

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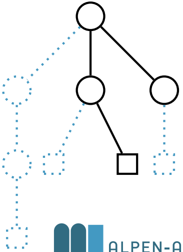

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