On a Reduction of Lattice Paths

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joint work in progress with
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mso DK modeling simulation optimization

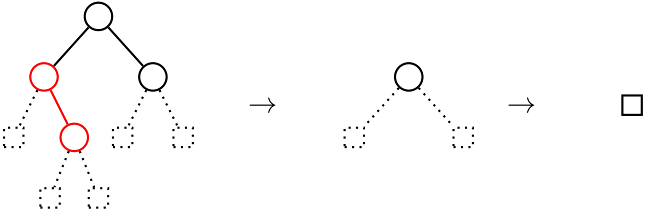
KARL POPPER KOLLEG

FWF Der Wissenschaftsfonds.

Trimming binary trees

Binary trees can be “trimmed” by the following strategy:

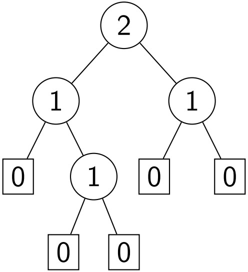
- Remove all leaves
- Merge nodes with only one descendant



"Surviving" nodes

Label all nodes in the tree by the following rules:

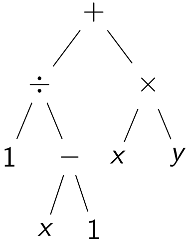
- Leaves → 0 (they do not survive a single reduction)
- val(left child) = val(right child) → increase by 1
- Otherwise: take the maximum



The register function

Number in the root of the tree: *Register function*, a.k.a. *Horton-Strahler* number.

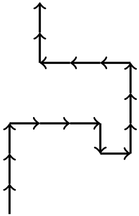
- Register function = maximal number of tree trimmings
- Applications:
  - Required stack size for evaluating an expression
  - Branching complexity of river networks (e.g. Danube: 9)

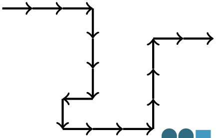


Reduction of lattice paths

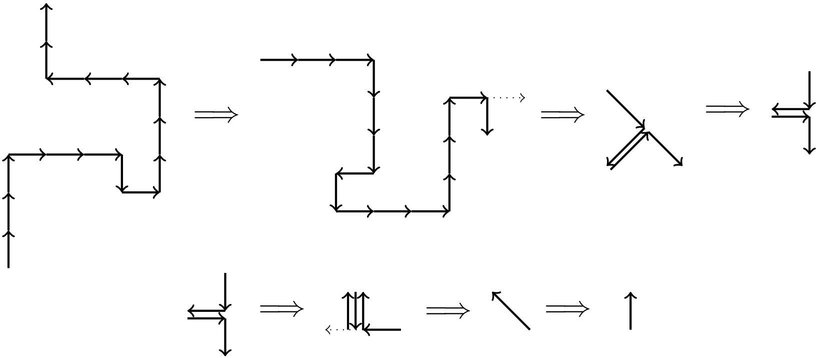
Reduction of a simple, two-dimensional lattice path (i.e. a sequence of $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$):

- If the path starts with $\uparrow$ or $\downarrow$: rotate it
- If the path ends with $\rightarrow$ or $\leftarrow$: rotate the last step
- Consider the pairs of horizontal-vertical segments:
    - Replace $\rightarrow \ldots \uparrow \ldots$ by $\nearrow$,
    - $\rightarrow \ldots \downarrow \ldots$ by $\searrow$,
    - $\leftarrow \ldots \downarrow \ldots$ by $\swarrow$,
    - $\leftarrow \ldots \uparrow \ldots$ by $\nwarrow$.
- Rotate the entire path again





Reduction – Example



Compactification degree and functional equation

- Compactification degree: number of reductions until a path is compactified to an atomic step $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$

Proposition

The generating function of simple two-dimensional lattice paths of length ≥ 1, $L(z) = \frac{4z}{1-4z}$, fulfills the functional equation

$$L(z) = 4z + 4L\Big(\frac{z^2}{(1-2z)^2}\Big).$$

Can be checked directly—or proven combinatorially!

Functional equation (combinatorial proof)

Read the reduction *backwards*:

- Replace $\rightarrow$ by $\rightarrow \dots \uparrow \dots$ and so on...
- Optionally rotate the entire path and/or the last step

Regular expression for $\rightarrow \dots \uparrow \dots$:

$$\rightarrow (\rightarrow \text{ or } \leftarrow)^* \uparrow (\uparrow \text{ or } \downarrow)^*$$

$\Rightarrow$ Replacement corresponds to $z \mapsto \frac{z^2}{(1-2z)^2}$.
Optional rotations: factor 4.

$$4L\Big(\frac{z^2}{(1-2z)^2}\Big)$$

counts all reducible paths.
Adding $4z$ (for $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$) then counts all paths. $\square$

Compactification degree – Recursion

- $L_r^{=}(z)$ ... OGF for paths with compactification degree r
- Only $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$ have comp. deg. 0 $\Rightarrow L_0^{=}(z) = 4z$
- Recursion:

$$L_r^{=}(z) = 4L_{r-1}^{=}\Big(\frac{z^2}{(1-2z)^2}\Big), \quad r \geq 1$$

- "Magic substitution" $z = \frac{u}{(1+u)^2}$: $z \mapsto \frac{z^2}{(1-2z)^2}$ becomes $u \mapsto u^2$
- Overall:

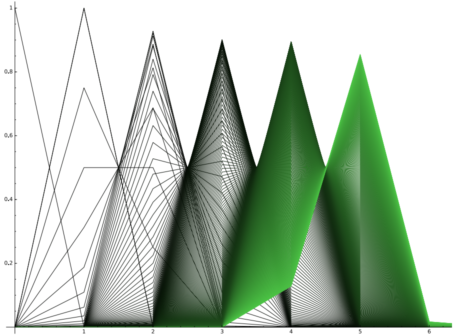
$$L_r^{=}(z) = 4^{r+1} \frac{u}{(1+u)^2}\Big|_{u \mapsto u^{2^r}} = 4^{r+1} \frac{u^{2^r}}{(1+u^{2^r})^2}$$

Compactification degree – Random variables

- X_n ... compactification degree of a (uniformly) random lattice path of length n

$$\Rightarrow \mathbb{P}(X_n = r) = \frac{[z^n]L_r^{=}(z)}{4^n}$$

- Probability densities of X_1 up to X_{512}:



Analysis of $\mathbb{E}X_n$ (1)

- As we have $\mathbb{E}X_n = 4^{-n}[z^n]\sum_{r\geq 0} rL_r^{=}(z)$, we analyze

$$G(z) = \sum_{r\geq 0} rL_r^{=}(z)$$

- With $z = \frac{u}{(1+u)^2}$ and $u = e^{-t}$, we have

$$G(z) = \sum_{r,\lambda\geq 0} r4^{r+1}(-1)^{\lambda-1}\lambda e^{-t\lambda 2^r}$$

$\rightsquigarrow$ Local expansion for $t \to 0$ ($z \to \frac{1}{4}$)?

Mellin transformation

- Mellin transformation of $(0, \infty)$-integrable $f(x)$:

$$\mathcal{M}(f)(s) = f^*(s) := \int_0^\infty x^{s-1} f(x)\, dx$$

- Important properties:
  - *Harmonic sums:*

$$\mathcal{M}\Big(\sum_{k\geq 0} \lambda_k f(x\mu_k)\Big)(s) = \Big(\sum_{k\geq 0} \lambda_k \mu_k^{-s}\Big) f^*(s)$$

  - *Asymptotic translation:*

    Asymptotic expansion of $f(x)$ $\leftrightsquigarrow$ Poles of $f^*(s)$
  - *Inversion formula:*

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s}\, ds$$

Analysis of $\mathbb{E}X_n$ (2)

- By basic properties of the Mellin transform we find

$$G^*(s) = \Gamma(s)\zeta(s-1)\frac{2^{2-s}}{1-2^{2-s}}$$

- Double pole at $s=2$, simple poles at $s = 2 + \frac{2\pi i}{\log 2}k = 2 + \chi_k$ for $k \in \mathbb{Z} \setminus \{0\}$
- Mellin inversion:

$$G(z) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} \Gamma(s)\zeta(s-1)\frac{2^{2-s}}{1-2^{2-s}} t^{-s}\, ds$$

- Basic idea: shift line of integration to the left, collect residues!

Analysis of $\mathbb{E}X_n$ (3)

- Residue at $s = 2$:

$$-\frac{4}{\log 2} t^{-2} \log t + \left(\frac{4}{\log 2} - 2\right) t^{-2}$$

- Substituting z back for t and expanding locally for $z \to \frac{1}{4}$ yields

$$-\frac{\log(1-4z)}{\log 2\,(1-4z)} + \frac{2 - 3\log 2}{\log 2\,(1-4z)} + \frac{\log 2 - 1}{\log 2} + \frac{\log(1-4z)}{3\log 2} + O(1-4z)$$

Singularity Analysis

“Singularity Analysis” by Flajolet and Odlyzko: extract expansion for coefficients from the singularities.

In particular:

$$[z^n](1-rz)^{-\alpha}\left(\frac{1}{rz}\log\left(\frac{1}{1-rz}\right)\right)^{\beta} \sim r^n \frac{n^{\alpha-1}}{\Gamma(\alpha)}\log(n)^{\beta},$$

more terms (inclusive error terms) are available.

Assumption: analyticity in a “Pacman region” (!)

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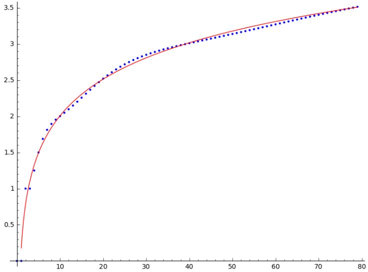
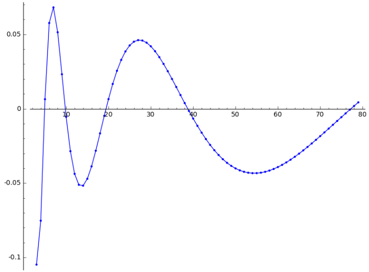
Analysis of $\mathbb{E}X_n$ (4)

- After division by 4^n, the local expansion translates into

$$\log_4 n + \frac{\gamma + 2 - 3\log 2}{2\log 2} + O(n^{-2}).$$

- Plot against exact values (left: comparison, right: difference):





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Analysis of $\mathbb{E}X_n$ (5)

Collecting the contributions at $s = 2 + \chi_k$ yields:

Theorem (H.–Heuberger–Prodinger, 2016)

The expected compactification degree among all simple 2D lattice paths of length n admits the asymptotic expansion

$$\mathbb{E}X_n = \log_4 n + \frac{\gamma + 2 - 3\log 2}{2\log 2} + \delta_1(\log_4 n) + O(n^{-1}),$$

where

$$\delta_1(x) = \frac{1}{\log 2} \sum_{k \neq 0} \frac{\Gamma(2+\chi_k)\zeta(1+\chi_k)}{\Gamma(1+\chi_k/2)} e^{2k\pi i x}$$

is a small 1-periodic fluctuation.

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Analysis of $\mathbb{V}X_n$

Similarly: variance $\mathbb{V}X_n$ can be determined.

Theorem (H.–Heuberger–Prodinger, 2016)

The corresponding variance is given by

$$\mathbb{V}X_n = \frac{\pi^2 - 24\log^2\pi - 48\zeta''(0) - 24}{24\log^2 2} - \frac{2\log\pi}{\log 2} - \frac{11}{12} + \delta_2(\log_4 n) + \frac{\gamma + 2 - 3\log 2}{\log 2}\delta_1(\log_4 n) + \delta_1^2(\log_4 n) + O\Big(\frac{1}{\log n}\Big),$$

where $\delta_1(x)$ is defined as above and $\delta_2(x)$ is a small 1-periodic fluctuation as well.

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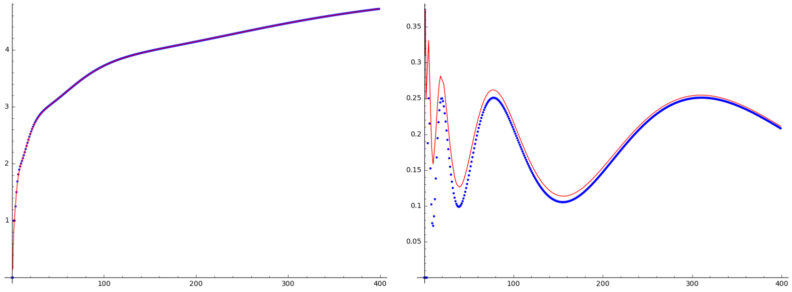
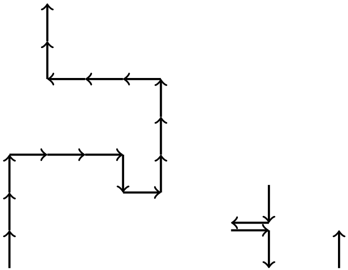
Expectation and Variance: exact vs. asymptotic



Fringes

- Size of rth fringe. . . length of rth lattice path reduction



- How large is the rth fringe and the entire fringe on average?

Bivariate generating function

- $H_r(z, v) \ldots$ BGF counting path length (with z) and rth fringe size (with v)
- Recursion:

$$H_0(z, v) = \frac{4zv}{1 - 4zv}, \quad H_r(z, v) = 4H_{r-1}\left(\left(\frac{z}{1 - 2z}\right)^2, v\right)$$

- Explicit solution with $z = \frac{u}{(1+u)^2}$:

$$H_r(z, v) = \frac{4^{r+1} u^{2^r} v}{(1 + u^{2^r})^2 - 4u^{2^r} v}$$

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Size of rth fringe

Theorem (H.–Heuberger–Prodinger, 2016)

The expectation $E^L_{n;r}$ and variance $V^L_{n;r}$ of the rth fringe size of a random path of length n have the asymptotic expansions

$$E^L_{n;r} = \frac{n}{4^r} + \frac{1-4^{-r}}{3} + O(n^3\theta_r^{-n}),$$

$$V^L_{n;r} = \frac{4^r-1}{3\cdot 16^r}n + \frac{-2\cdot 16^r - 5\cdot 4^r + 7}{45\cdot 16^r} + O(n^5\theta_r^{-n}),$$

where $\theta_r^{-n} = \frac{4}{2+2\cos(2\pi/2^r)} > 1$.

For $r > 0$, the random variables modeling the rth fringe size of lattice paths of length n are asymptotically normally distributed.

Overall fringe size

Strategy: sum over $H_r(z, v)$, expansion via Mellin transform, singularity analysis.

Theorem (H.–Heuberger–Prodinger, 2016)

The expected fringe size E_n^L for a random path of length n admits the asymptotic expansion

$$E_n^L = \frac{4}{3}n + \frac{1}{3}\log_4 n + \frac{5 + 3\gamma - 11\log 2}{18\log 2} + \delta(\log_4 n) + O(n^{-1}\log n),$$

where $\delta(x)$ is a 1-periodic fluctuation of mean zero with

$$\delta(x) = \frac{2}{3\sqrt{\pi}\log 2}\sum_{k\neq 0}\Gamma\Big(\frac{3+\chi_k}{2}\Big)\big(2\zeta(\chi_k - 1) + \zeta(\chi_k + 1)\big)e^{2k\pi i x}.$$