Exercise session 1

Gibbs measures and phase transitions

1. Show that the grand canonical hard sphere partition function on any bounded measurable S satisfies $Z_S(\lambda) \le e^{\lambda \cdot \mathrm{vol}(S)}$.

2. Compute the hard-core partition function $Z_{S_k}(\lambda)$ for the graph S_k, the star on k leaves. Compute the probability that the center of the star is in the random independent set $\mathbf{I}$.

3. Compute the hard-core partition function $Z_{K_{d,d}}(\lambda)$ where $K_{d,d}$ is the complete d-regular bipartite graph on $2d$ vertices.

4. Compute the hard-core partition function $Z_G(\lambda)$ of the cycles C_3, C_4, C_5.

5. Compute the hard-core partition function of the cycle C_n.

6. Prove that the 1-dimensional hard-core model on $\mathbb{Z}$ does not exhibit a phase transition, by computing the limiting free energy and showing that it is a real analytic function of λ.

7. Suppose G is the disjoint union of two graphs H_1 and H_2. Show that $Z_G(\lambda) = Z_{H_1}(\lambda) \cdot Z_{H_2}(\lambda)$.

8. Suppose v is a vertex in some graph with no edges in its neighborhood. Call a vertex u uncovered with respect to an independent set I if $N(u) \cap I = \emptyset$. Consider the hard-core model on G at fugacity λ and calculate the probability that v is occupied given that v has exactly j uncovered neighbors.

9. Describe the line graph of the infinite graph $\mathbb{Z}^d$.

10. Construct an infinite graph that is claw-free but not the line graph of any graph.

11. Prove that the following is an alternative description of the hard-core model on G. Let $\mathbf{I}$ be a random subset of $V(G)$, each vertex included independently with probability $\frac{\lambda}{1+\lambda}$, conditioned on the event that the vertices form an independent set.

Exercise session 2

12. Let H be the graph consisting of two vertices each with self loops but no edge joining the two. Show that $\hom(K_{d+1}, H)^{1/(d+1)} > \hom(K_{d,d}, H)^{1/2d}$.

13. Let G be any graph. Let $G \times K_2$ be the *bipartite double cover* of G. That is, $G \times K_2$ has vertex set $V(G) \times \{0,1\}$ with an edge between (u,i) and (v,j) if $uv \in E(G)$ and $i = 0, j = 1$ or $i = 1, j = 0$. Prove that

$$i(G)^2 \leq i(G \times K_2).$$

Deduce that Kahn's theorem on independent sets in d-regular bipartite graphs can be extended to all d-regular graphs. This is Zhao's proof.

14. Prove (in an elementary way) that for all d-regular G on n vertices,

$$i_{n/2}(G) \leq i_d(K_{d,d})^{n/2d}.$$

15. What is the probability (in the hard-core model) that a vertex v is uncovered, given that the subgraph induced by its uncovered neighbors is isomorphic to a graph H?

16. Let G, H be two graphs on n vertices each. Prove that if $Z_G^M(\lambda) \geq Z_H^M(\lambda)$ for all $\lambda > 0$, then $m_{\text{perf}}(G) \geq m_{\text{perf}}(H)$.

Exercise session 3

17. Prove that for any graph G on at most d vertices,

$$\mathbb{E}_{G,\lambda}\left[|\mathbf{I}|\,\middle|\,|\mathbf{I}|\geq 1\right] \leq \frac{\lambda(1+\lambda)^{d-1}}{(1+\lambda)^d-1}\,.$$

18. Suppose G is a d-regular graph and suppose $v \in V(G)$ does not belong to a $K_{d,d}$ component. Give a lower bound (in terms of d, λ) on the probability that the uncovered neighborhood of v is not the empty graph or the graph of d isolated vertices. (This proves uniqueness in the independent set theorem in a quantitative way).

19. Suppose a vertex v of a graph G has d neighbors (but we make no assumption on the presence or absence of edges in its neighborhood). Do the hard-core model on G at fugacity λ, and let $p_k = \Pr[|\mathbf{I} \cap N(v)| = k]$. Give a lower bound on p_{k-1} in terms of p_k, d, and λ. Is the bound tight in any graph?

20. Use the tight bound from the previous question, for $k = 2, \ldots d$, to prove that $\alpha_G(\lambda) \leq \alpha_{K_{d,d}}(\lambda)$ for any d-regular G.

21. Prove that $\alpha_G(\lambda) \geq \alpha_{K_{d+1}}(\lambda)$ for any d-regular G.