# Exercise session 1

*Stirling's formula and Laplace's method*

**1.** The *Narayana numbers* $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ count (among other things) plane trees with $n+1$ vertices and $k$ leaves. Use Stirling's formula to prove that they satisfy a central limit theorem (i.e., they can be approximated by a normal distribution): for $k = \frac{n}{2} + a\sqrt{n}$, we have

$$\frac{1}{n}\binom{n}{k}\binom{n}{k-1} \sim \frac{2 \cdot 4^n}{\pi n^2} e^{-4a^2}.$$

**2.** Determine one more term in Stirling's formula by refining the Taylor-approximation in the central region of the integral $\int_0^\infty x^\alpha e^{-x}\,dx$:

$$\Gamma(\alpha+1) = \sqrt{2\pi\alpha} \cdot \frac{\alpha^\alpha}{e^\alpha}\left(1 + \frac{1}{12\alpha} + o\left(\frac{1}{\alpha}\right)\right).$$

**3.** Use the result of the previous problem to determine a constant $K$ (that depends on $\alpha$) such that

$$[x^n](1-x)^\alpha = \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\left(1 + \frac{K}{n} + o\left(\frac{1}{n}\right)\right),$$

provided that $\alpha \notin \{0, 1, 2, 3, \ldots\}$.

**4.** Prove: if the function $f(x)$ has only nonnegative values on $[a, b]$ and a unique maximum at $x_0 \in (a, b)$, and if the Taylor expansion at $x_0$ is given by

$$f(x) = f(x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + O(|x - x_0|^3),$$

then we have

$$\int_a^b f(x)^n\,dx \sim \sqrt{-\frac{2\pi}{f''(x_0)n}} \cdot f(x_0)^{n+1/2}$$

as $n \to \infty$.

**5.** Determine (using integration by parts and induction) an explicit formula for the integral

$$\int_{-\pi/2}^{\pi/2} \cos^n x\,dx.$$

Now determine an asymptotic formula for the integral in two ways: by means of Stirling's formula, and using the result of the previous problem.

## Exercise session 2

*Singularity Analysis: Meromorphic Functions*

**6.** Use singularity analysis to determine an asymptotic approximation for the secant and tangent numbers, which are given by

$$\sec x = \sum_{n=0}^{\infty} \frac{s_n x^{2n}}{(2n)!}$$

and

$$\tan x = \sum_{n=0}^{\infty} \frac{t_n x^{2n+1}}{(2n+1)!}.$$

**7.** The *Hadamard product* of two power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ is

$$A(x) \odot B(x) = \sum_{n=0}^{\infty} a_n b_n x^n.$$

Prove: if $A(x)$ and $B(x)$ are meromorphic functions on $\mathbb{C}$, then $A(x) \odot B(x)$ is also meromorphic on all of $\mathbb{C}$ [Hint: use information on the asymptotic behaviour of the coefficients.]

**8.** The generating function for the number of possible representations of a positive integers as an ordered sum of primes is given by

$$R(x) = \sum_{n=0}^{\infty} r_n x^n = \frac{1}{1 - \sum_{p \text{ prime}} x^p}$$

(why?). For example, $r_8 = 6$, since there are six representations: $5+3, 3+5, 3+3+2, 3+2+3, 2+3+3, 2+2+2+2$.

Prove that there exist constants $a, b$ with $1 < b < 2$ such that $r_n \sim a \cdot b^n$.

**9.** The exponential generating function for the number of ordered set partitions (number of ways to split a set of $n$ elements into groups, where the groups are ordered) is

$$S(x) = \sum_{n=0}^{\infty} \frac{s_n}{n!} x^n = \frac{1}{2 - e^x}.$$

Determine the asymptotic behaviour of $s_n$.

# Exercise session 3

*Singularity Analysis and Symbolic Method*

**10.** Use singularity analysis to find an asymptotic approximation for the coefficients of the following functions:

(a) $f(x) = 1 - \sqrt{1-2x}$,

(b) $g(x) = \dfrac{1}{\sqrt{1-3x-4x^2}}$,

(c) $h(x) = (\sqrt{1-x} - \sqrt{1-2x})^3$.

**11.** Show: the generating function for the number of 0, 1-strings without a substring of $k$ consecutive 1s is

$$\frac{1-x^k}{1-2x+x^{k+1}}.$$

Now show that for every $k$, there exist constants $a_k, b_k$ with $0 < b_k < 1$ such that the probability for a 0, 1-string not to contain $k$ consecutive 1s is asymptotically equal to $a_k \cdot b_k^n$.

**12.** Unary-binary trees are rooted trees with three types of nodes: leaves, nodes with a single child (unary), and nodes with two children (binary). For example, the evaluation of an arithmetic expression such as $\sqrt{2\cdot 8}+5$ can be expressed in terms of such a tree, where unary nodes stand for unary operations (such as $\sqrt{\cdot}$), and binary nodes for binary operations (such as $+$).

Prove: the generating function for the number of unary-binary trees with $n$ nodes is

$$\frac{1-x-\sqrt{1-2x-3x^2}}{2x}$$

and determine an asymptotic formula for the number of such trees.

**13.** Find an example of a function $A(x) = \sum_{n\geq 0} a_n x^n$ with positive coefficients $a_n$ that has a singularity at 1, even though $\lim_{x\to 1^-} A^{(k)}(x) < \infty$ for all $k \geq 0$.