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Asymptotic Analysis of Shape Parameters of Trees and Lattice Paths

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Abstract

This thesis belongs to the mathematical field of *Analytic Combinatorics*, which is concerned with the asymptotic analysis of parameters of discrete structures (here, primarily *trees* and *lattice paths*) using analytic methods. After modeling the parameter of interest as a random variable, the properties brought to light by means of such a rigorous investigation include high-precision asymptotic (sometimes even explicit) formulas for the expected value, the corresponding variance—and, if possible, higher moments and the characterization of a limiting distribution as well.

Predominantly, the shape parameters under investigation are associated to deterministic reduction procedures defined on families of plane trees and lattice paths, respectively. To be more precise, a suitable deterministic reduction naturally induces some sort of *age* on the objects (in the sense that "older" objects require more reductions until they are "irreducible"). A prominent example for a parameter that can be modeled in this way is the well-known register function for binary trees. Both the age itself as well as the object size after a fixed number of reductions are studied in different context within this thesis.

Another interesting shape parameter is defined for so-called *Łukasiewicz paths*, i.e., two-dimensional simple lattice paths with a unique down step. These paths have a very nice structure, as they are strongly related to plane trees whose node degrees are contained in a predefined set. In this fairly general setting we are interested in *ascents*—maximal sequences of non-negative steps.

Something all of our investigations have in common is that they all contain some comprehensive computational aspects. To this end, we make heavy use of the free open-source computer mathematics software system SageMath and its included module for computations with asymptotic expansions developed by Clemens Heuberger, Daniel Krenn, and the author. For all results obtained with the help of this module, there is a corresponding worksheet containing the computations available for download.

Zusammenfassung

Die vorliegende Dissertation ist dem mathematischen Teilgebiet der *analytischen Kombinatorik* zuzuordnen, welches sich mit der präzisen asymptotischen Analyse von Parametern diskreter Strukturen (in dieser Arbeit konkret *Bäumen* und *Gitterpfaden*) mittels analytischer Methoden beschäftigt. Die Ergebnisse der Untersuchungen dieser als Zufallsvariablen modellierten Parameter umfassen asymptotische Ausdrücke hoher Präzision (sowie gegebenenfalls sogar explizite Ausdrücke) für den Erwartungswert, die Varianz — und, sofern möglich, auch für höhere Momente (was gegebenenfalls auch Rückschlüsse auf eine Grenzverteilung erlaubt).

Die in erster Linie untersuchten Parameter hängen mit deterministischen Reduktionsprozeduren zusammen, die auf Familien von geordneten Wurzelbäumen, beziehungsweise auf Familien von Gitterpfaden, definiert werden. Genauer gesagt induziert eine geeignete deterministische Reduktion auf natürliche Art und Weise ein *Alter* auf den jeweiligen Objekten (in dem Sinne, dass „ältere“ Objekte öfter reduziert werden müssen, bis sie „irreduzibel“ sind). Ein prominentes Beispiel für einen Parameter, der als ein solches Alter gesehen werden kann, ist die wohlbekannte Registerfunktion binärer Bäume. Sowohl das Alter als auch die Objektgröße nach einer festen Anzahl von Reduktionen werden im Rahmen dieser Arbeit in verschiedenen Kontexten untersucht.

Ein weiterer Parameter von Interesse ist für sogenannte Łukasiewicz-Pfade definiert. Das sind einfache zweidimensionale Gitterpfade, bei denen die Menge der erlaubten Schritte nur einen einzigen Schritt nach unten enthält. Die Struktur von Łukasiewicz-Pfaden ist besonders reichhaltig, weil sie stark mit jenen geordneten Wurzelbäumen, deren Knotengrade in einer vorgegebenen Menge enthalten sind, zusammenhängen. In diesem relativ allgemeinen Rahmen werden *Aufstiege* — das sind maximale Folgen nicht-negativer Schritte — untersucht.

Ein Aspekt, der allen Analysen in dieser Dissertation gleicherweise innewohnt, ist, dass immer wieder rechentechnisch sehr aufwändige Berechnungen vorkommen. Aus diesem Grund wird starker Gebrauch vom freien, quelloffenen Computermathematiksystem SageMath und insbesondere dem darin enthaltenen Modul für asymptotische Entwicklungen, welches von Clemens Heuberger, Daniel Krenn und dem Autor entwickelt wurde, gemacht. Für alle Resultate, die mit Hilfe dieses Moduls erhalten wurden, steht ein SageMath-Worksheet mit den zugehörigen Berechnungen zum Download zur Verfügung.

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1 Introduction

Maybe in our world there lives a happy little tree over there...

Bob Ross, The Joy of Painting

I may not have gone where I intended to go,
but I think I have ended up where I needed to be.

Douglas Adams, The Long Dark Tea-Time of the Soul

Trees and lattice paths are two fundamental, inherently connected, combinatorial structures with a plethora of applications both within Mathematics as well as within other fields. As an example, the concept of trees is particularly interesting in Computer Science, where it has incarnations in the form of file systems, general data structures, and of course also algorithms (see, e.g., [40, 58] for examples). Another (more or less) obvious application of trees is within Biology and Life Sciences for the modeling of branching processes [38] such as the development of some population. Similarly, applications of lattice paths can, amongst others, be found in Biology, Chemistry, as well as Physics (see, e.g., [57, Chapter 5], [7]), where, for example, they are used to model the motion of particles.

Given that trees and lattice paths are so fundamentally involved in a variety of applications, it is (not only for the sake of mathematical satisfaction) of great interest to rigorously study different aspects of their structure—which is, in a nutshell, the topic of this thesis.

To be more precise, this thesis is devoted to the analysis of certain *shape parameters* within special families of trees and lattice paths, respectively.

In the scope of this thesis, we are particularly interested in parameters related to reduction procedures. However, while cutting down trees according to some given probabilistic model until some condition is satisfied (e.g., until the root is isolated) has been a popular research theme during the last decades (see, for example, [32, 44, 49]), this thesis features a different approach. Here, the reductions are fully deterministic.

Think of the following example: Let $\mathcal{T}$ be the combinatorial class of plane trees (i.e., trees with a special root node where the children of each node are ordered). Now, consider a reduction operator acting on any non-trivial tree from $\mathcal{T}$ (i.e., any tree besides the one consisting just of the root) by removing all of its leaves. Iterated application of this operator to some given tree is illustrated in Figure 1.1.

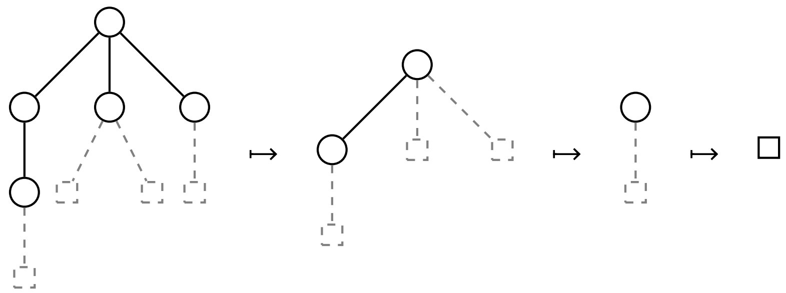


Figure 1.1: Iterated application of a simple deterministic tree reduction operator.

For such a given deterministic reduction, we declare two different parameters of interest. On the one hand, we are interested in studying the number of iterated reductions required to reduce a given object to a object that cannot be reduced any further—in the case of the "removing leaves"-example, this would just be the trivial tree consisting of just the root. On the other hand, we are interested in determining how "fast" a given procedure reduces an object, which can be studied by investigating the object size after some fixed number of reductions.

By inversion, a given deterministic reduction procedure induces a growth process on the combinatorial family under investigation, where a given object can grow into one of many (if not even infinitely many, depending on the reduction) different "successors". Figure 1.2 illustrates the first few generations in the context of the "growing leaves" growth process (which is induced by the "cutting leaves" operator discussed above).

In this alternate growth process-based view, the first parameter can be seen as the "age" of a given object. The second parameter corresponds to the size of the ancestor from some given number of generations ago.

It is not too difficult to see that in the context of the simple reduction introduced above, the age of a tree corresponds to its height—an important parameter that has been studied, e.g., in [8, 50]. A rigorous analysis of the tree size after iteratively reducing the tree a fixed number of times, however, will be carried out within this thesis (cf. Chapter 3 for the analysis

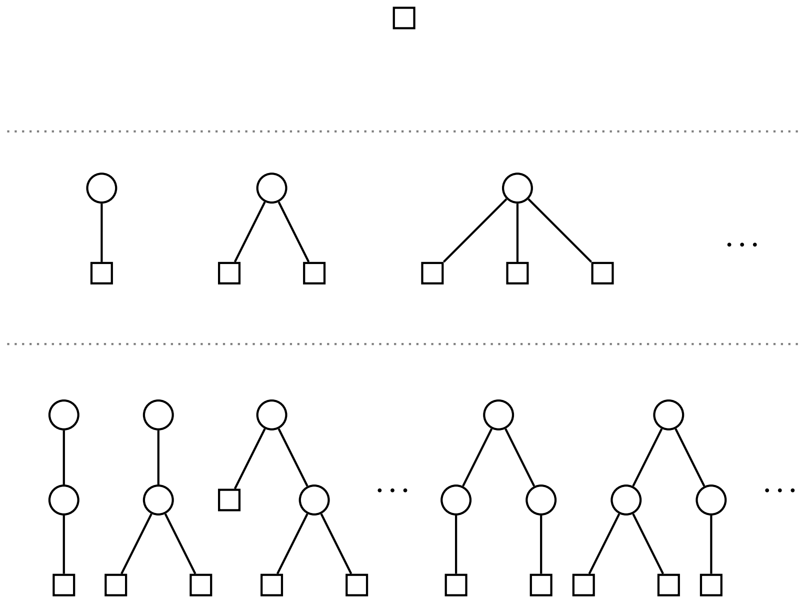


Figure 1.2: Descendents of the trivial tree in the “growing leaves” procedure. Different generations are separated by dotted lines.

of the “cutting leaves”-reduction). For these parameters, results of the following type can be expected (see Theorems 3.2.13 and 3.2.14):

Theorem.
Let $r \in \mathbb{N}_0$ be fixed and let $X_{n,r}$ be the random variable modeling the number of nodes left after reducing a random tree with n nodes r times by cutting away all leaves. Then the expected size and the corresponding variance behave for $n \to \infty$ according to the asymptotic expansions

$$\mathbb{E}X_{n,r} = \mu n - \frac{r(r-1)}{6(r+1)} + O(n^{-1}) \qquad \text{and} \qquad \mathbb{V}X_{n,r} = \sigma^2 n + O(1),$$

where

$$\mu = \frac{1}{r+1} \qquad \text{and} \qquad \sigma^2 = \frac{r(r+2)}{6(r+1)^2}.$$

In addition, $X_{n,r}$ is, after standardization, asymptotically normally distributed for $n \to \infty$. To be more precise, for $x \in \mathbb{R}$ we have

$$\mathbb{P}\Big(\frac{X_{n,r} - n\mu}{\sqrt{\sigma^2 n}} \le x\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt + O(n^{-1/2}).$$

All O-constants depend implicitly on r.

Throughout Chapters 2, 3, and 4, the analysis of the shape parameters can be seen in a common framework. The dual interpretation of the setting (either based on a growth- or a reduction procedure) plays an integral part. This is because first of all, we switch to the growth-based view and translate the procedure yielding a new generation from some given family into the language of generating functions, resulting in a (for our setting) linear operator Φ acting on the generating function.

The next step consists of finding an explicit formula for the iterated application of the expansion operator Φ to some generating function. This allows us to determine explicit representations of both the generating function enumerating objects of up to some certain age as well as the bivariate generating function enumerating objects with respect to their own size and the size of their rth predecessor.

These generating functions are the basis for the actual analysis of the shape parameters, which can now be conducted by utilizing and/or combining different techniques from analytic combinatorics (like, for example, singularity analysis [18] and the Mellin transform [16]).

In Chapter 2 we study two different reduction procedures and the corresponding shape parameters: the first one, defined on the family of binary trees, cuts away all leaves and repairs the resulting structure, such that once again a binary tree is obtained. It is noteworthy that for this reduction the age parameter corresponds with the well-known *register function*.

The second reduction is defined on the family of two-dimensional lattice paths with step set $\mathcal{S} = \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$; both the age as well as the ancestor size are analyzed.

Chapter 3 is devoted to the rigorous analysis of the ancestor size for four different reduction procedures defined for plane trees. Apart from the "cutting leaves" example discussed above, the reduction removing all leaves from a given plane tree, we consider the following variants:

- "Pruning" trees, which means that the reduction removes all linear graphs ending in leaves,
- "Cutting old leaves", where the reduction removes all leaves that are simultaneously the leftmost children,
- "Cutting old paths", where all linear graphs ending in leaves that also start as leftmost children are removed.

In Chapter 4 we demonstrate that by restricting ourselves to a rather special subclass of plane trees and defining a reduction procedure that is allowed to cut away very large substructures, the behavior of the shape parameters is fundamentally different than in the previous chapters. For both age and ancestor size, the limiting distribution degenerates to a discrete limiting distribution.

Then, in Chapter 5 we shift the focus of our interest away from the deterministic reduction-based parameters investigated in the previous chapters and concentrate on a new setting. The combinatorial objects that we are interested in this chapter are so-called non-negative Łukasiewicz paths, i.e., a family of one-dimensional lattice paths with a finite step set $\mathcal{S} \subseteq \mathbb{Z}$ such that $-1 \in \mathcal{S}$ is the unique negative allowed step.

Łukasiewicz paths are particularly nice lattice paths, because there is a bijection inherently linking the family of Łukasiewicz excursions with respect to the step set $\mathcal{S}$ (i.e., non-negative paths ending on the starting altitude with steps in $\mathcal{S}$) to the family of plane trees where the nodes of the tree have outdegrees in $1+\mathcal{S}$. An example of this correspondence is the well-known bijective relation between Dyck paths (which are Łukasiewicz excursions with respect to the step set $\mathcal{S} = \{-1, 1\}$) and (plane) binary trees, where all nodes are either leaves (outdegree 0) or internal nodes (outdegree 2). As a consequence of this bijection, the recursive tree structure is, in some sense, carried over to the structure of Łukasiewicz excursions.

The shape parameter we are interested in for these objects is the number of ascents of given length. An ascent is an inclusion-wise maximal sequence of steps different from the down step $\searrow$. If an ascent consists of precisely r non-negative steps, then it is called an r-ascent.

The special structure of Łukasiewicz paths enables us to carry out a precise analysis for the number of r-ascents in three special subfamilies: meanders (i.e., all non-negative paths), excursions, and dispersed excursions (i.e., excursions where horizontal steps are not allowed except on the horizontal axis). It is interesting to note that deriving the corresponding generating function allowing us to carry out this analysis is possible via two different approaches. In a purely analytic manner by means of the so-called kernel method (see [1] for a formal description and [52] for examples) combined with the "adding a new slice" approach (see [54, Section 2.5]), and by following a more combinatorial approach based on the recursive structure of the underlying plane trees.

Throughout this thesis, we often have to deal with extensive computations with asymptotic expansions. This is why the computer mathematics system SageMath [59] with its module for computing with asymptotic expansions implemented by Clemens Heuberger, Daniel Krenn, and the author [23] also plays an integral part in this thesis.

Besides "trivial" arithmetic operations with asymptotic expansions like addition, multiplication, and inversion, this module also allows us to easily carry out more complex operations (e.g. singularity analysis, or singular inversion [21, Chapter VI.7]) as well.

For all calculations in this thesis that were carried out with our asymptotic expansion module, there is a corresponding worksheet that can be downloaded, which allows to easily reproduce the results in this thesis. Details on where these worksheets can be found are given in the introductory section of the respective chapters.

2

Reductions of Binary Trees and Lattice Paths induced by the Register Function

The register function (or Horton-Strahler number) of a binary tree is a well-known combinatorial parameter. We study a reduction procedure for binary trees which offers a new interpretation for the register function as the maximal number of reductions that can be applied to a given tree. In particular, the precise asymptotic behavior of the number of certain substructures (“branches”) that occur when reducing a tree repeatedly is determined.

In the same manner we introduce a reduction for simple two-dimensional lattice paths from which a complexity measure similar to the register function can be derived. We analyze this quantity, as well as the (cumulative) size of an (iteratively) reduced lattice path asymptotically.

This chapter is an adapted version of [26], which is joint work with Clemens Heuberger and Helmut Prodinger.

2.1 Introduction

The aim of this chapter is to investigate local substructures that appear within discrete objects after reducing according to, in some sense, intrinsic rules. In particular, there are two reductions we focus on: a reduction for binary trees, as well as a reduction for simple two-dimensional lattice paths.

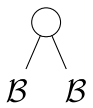
In order to give a summary of our results we will briefly sketch both reductions and explain the nature of the local structures emerging when applying the reduction repeatedly.

As announced in the introductory chapter, the SageMath [59] worksheets associated to this chapter as well as instructions on how to use them can be found at `https://arxiv.org/src/1612.07286v1/anc`.

2.1.1 Binary Trees

Binary trees are either a leaf or a root together with a left and a right subtree which are binary trees. This recursive definition can be written as a symbolic equation (□ and ○ mark leaves and inner nodes, respectively):

$$\mathcal{B} = \square +$$



By using the symbolic method (cf. [21, Part A]), this equation can be translated into a functional equation for the generating function counting binary trees with respect to their size (i.e. the number of inner nodes). The corresponding functional equation is given by

$$B(z) = 1 + zB(z)^2,$$

which leads to the well-known expansion

$$B(z) = \frac{1 - \sqrt{1-4z}}{2z} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} z^n.$$

This means that the number of binary trees with n inner nodes is given by the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

By simple algebraic manipulations, it is easy to verify that the generating function $B(z)$ satisfies the identity

$$B(z) = 1 + \frac{z}{1-2z} B\left(\frac{z^2}{(1-2z)^2}\right).$$

However, as we will see in Section 2.2, we can justify this identity from a combinatorial point of view as well, and the most important part of this combinatorial interpretation is a reduction procedure for binary trees.

Essentially, this procedure first removes all leaves from the tree and then "repairs" the resulting object by collapsing chains of nodes with only one child into one node. More details on this reduction are provided in Section 2.2.

With the help of this reduction we can assign labels to all nodes in a given tree by tracking how many iterated tree reductions it takes until the node is deleted. Note that collapsing some nodes into one node does not count as deleting the node. In Section 2.2 we prove that these labels are intimately linked with a very well-known and well-studied branching complexity measure of binary trees: the register function.

The local structures we are interested in also become visible after labeling a tree as described above: the so-called r-branches of a binary tree are the connected subgraphs of nodes with label r. The number of these r-branches in a random tree of size n is modeled by the random variable $X_{n;r} : \mathcal{B}_n \to \mathbb{N}_0$, where $\mathcal{B}_n$ is the set of all binary trees of size n. A proper definition as well as results on r-branches can be found in Section 2.2.2.

In the context of binary tree reductions we are interested in precise analyses of the random variables $X_{n;r}$ as well as $X_n := \sum_{r\geq 0} X_{n;r}$, which models the total number of branches in a random tree of size n. This quantity is investigated closely in Section 2.2.3.

Table 2.1 gives an overview of the results of our investigation. The results for the register function are well-known, which is why we refer to external literature instead. Additionally, Theorem 2.2.8 proves asymptotic normality for the number of r-branches $X_{n;r}$.

	Register function	r-branches ($X_{n;r}$)	branches total (X_n)
range	[20, Sec. 1.1, Sec 2]	Proposition 2.2.5	Proposition 2.2.9
explicit formula	[20, Theorem 1]	Proposition 2.2.7	Proposition 2.2.10
asymptotic formula	[20, Theorem 3]	Theorem 2.2.6	Theorem 2.2.11

Table 2.1: Results: binary trees.

2.1.2 Lattice Paths

Let $\mathcal{L}$ be the combinatorial class of simple two-dimensional lattice paths, i.e., the set of all nonempty sequences over $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$. It is easy to see that the corresponding generating function is

$$L(z) = \frac{4z}{1-4z}.$$

Similarly to before, it is easy to check by algebraic manipulation that $L(z)$ satisfies the functional equation

$$L(z) = 4L\left(\frac{z^2}{(1-2z)^2}\right) + 4z.$$

However, as in the case of binary trees, we will see in Section 2.3 that the combinatorial interpretation of this equation is much more fruitful and gives rise to a reduction procedure for lattice paths.

In this case it takes a bit more to fully describe the reduction. The core idea is to reduce a given path by collapsing an entire horizontal-vertical segment (i.e. a path segment that consists of a sequence of horizontal movements followed by a sequence of vertical movements) into a single step.

The first parameter of interest in this context is the reduction degree of a random path of length n, which is the number of repeated reductions that it takes until the entire path is reduced to a single step. We will model this parameter with the random variable $D_n\colon \mathcal{L}_n \to \mathbb{N}_0$, where $\mathcal{L}_n$ consists of all simple two-dimensional lattice paths of length n.

As an analogue to the number of r-branches in a given binary tree we consider the length of the rth fringe, i.e., the rth reduction of a given lattice path. This quantity is modeled by the random variable $X_{n;r}^L\colon \mathcal{L} \to \mathbb{N}_0$.

By summation of the length of the rth fringe for $r \geq 0$ we obtain the total fringe size $X_n^L := \sum_{r\geq 0} X_{n;r}^L$. In some sense, the total fringe size measures the complexity of horizontal-vertical direction changes of a given lattice path. Both, the rth fringe size as well as the total fringe size are analyzed in Section 2.3.2.

Table 2.2 gives an overview of the results of our investigation.

	Reduction degree (D_n)	r-fringes ($X_{n;r}^L$)	total fringe size (X_n^L)
range	Proposition 2.3.6	Proposition 2.3.9	Proposition 2.3.12
explicit formula	Corollary 2.3.5	Proposition 2.3.11	Corollary 2.3.13
asymptotic formula	Theorem 2.3.8	Theorem 2.3.10	Theorem 2.3.14

Table 2.2: Results: lattice paths.

2.2 Tree Reductions and the Register Function

2.2.1 Motivation and Preliminaries

As mentioned in the introduction, we want to find a combinatorial proof of the following proposition.

Proposition 2.2.1.
The generating function counting binary trees by the number of inner nodes, $B(z) = \frac{1-\sqrt{1-4z}}{2z}$, satisfies the identity

$$B(z) = 1 + \frac{z}{1-2z} B\Big(\frac{z^2}{(1-2z)^2}\Big). \tag{2.1}$$

Proof. We consider the following reduction of a binary tree t, which we write as $\rho(t)$:

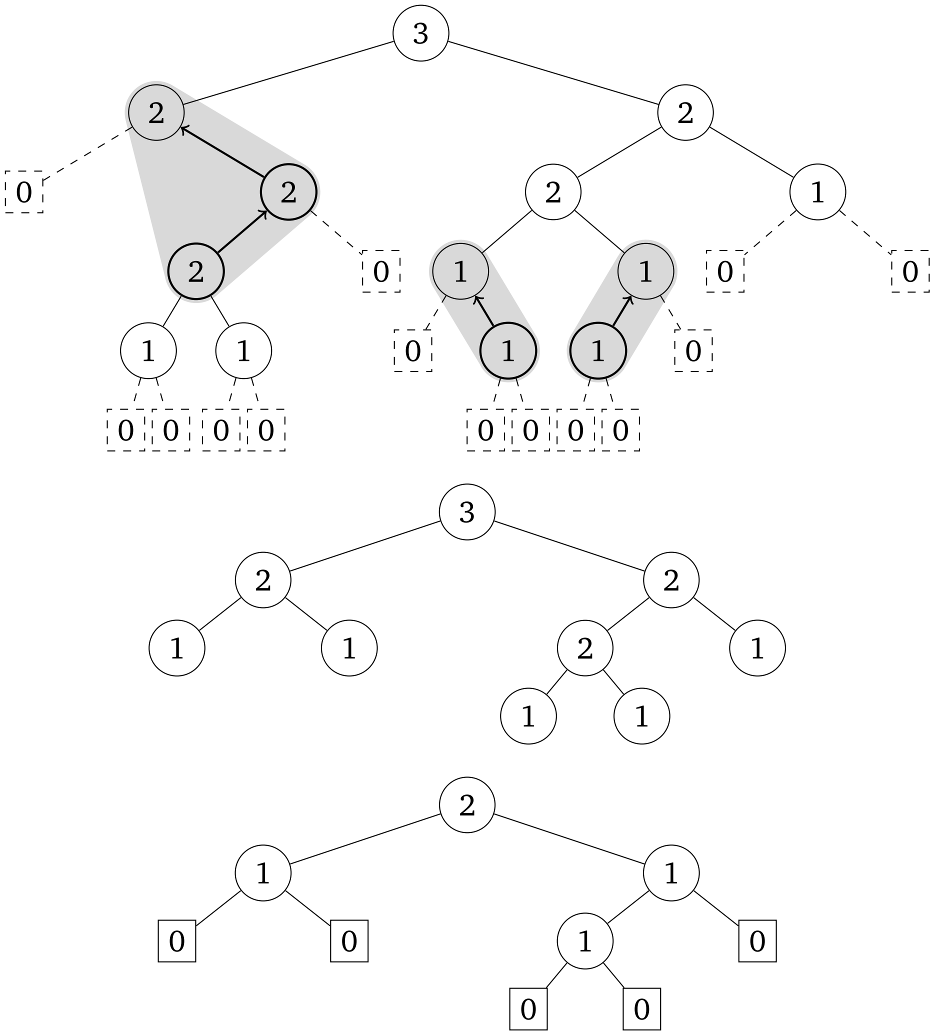


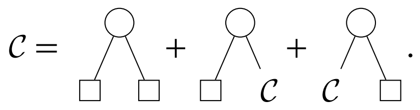
Figure 2.1: Illustration of the tree reduction ρ: in the first tree, the leaves are deleted (dashed nodes) and nodes with exactly one child are merged (gray overlay). The second tree shows the result of these operations. Finally, in the last tree all nodes without children are marked as leaves.

First, all leaves of t are erased. Then, if a node has only one child, these two nodes are merged; this operation will be repeated as long as there are such nodes. The leaves of the reduced tree are precisely the nodes without children.

This operation was introduced in [71]. The various steps of the reduction are depicted in Figure 2.1. The numbers attached to the nodes will be explained later.

Note that $\rho(\square)$ is undefined, so this is a partial function. Of course, many different trees are mapped to the same binary tree. However, they can all be obtained from a given reduced tree by the following operations:

All leaves and all internal nodes in the tree are replaced by chains of internal nodes. In such a chain, there has to be at least one leaf attached to every internal node; the symbolic equation for chains is



Obviously, these substitutions do not only restore the (previously deleted) leaves, but can also "unmerge" previously merged nodes. Thus, all trees that reduce to some tree t' can be reconstructed from t'.

From the symbolic equation of chains above, we find that the generating function $C(z)$ counting chains with respect to their size (i.e. number of internal nodes) satisfies the equation $C(z) = z + 2zC(z)$ and thus, we obtain

$$C(z) = \frac{z}{1-2z}.$$

Finally, if $F(z)$ is a generating function counting some family of binary trees, then the bivariate generating function $vF(zv)$ counts the same family with respect to size (variable z) and number of leaves (variable v). This is a direct consequence of the fact that binary trees with n inner nodes have $n+1$ leaves.

Therefore, replacing all nodes of a binary tree with chains corresponds to the substitutions $v \mapsto \frac{z}{1-2z}$ and $z \mapsto \frac{z}{1-2z}$ in the language of generating functions. Therefore, all binary trees that can be reconstructed from a reduced version of itself are counted by

$$\frac{z}{1-2z} B\left(\frac{z^2}{(1-2z)^2}\right).$$

By all these considerations, (2.1) can be interpreted combinatorially as the following statement: a binary tree is either just $\square$, or it can be reconstructed from another binary tree where all nodes are replaced by chains. $\square$

Remark.
Note that (2.1) can be used to find a very simple proof for a well-known identity for Catalan numbers:

Comparing the coefficients of z^{n+1}, (2.1) leads to

$$\begin{aligned} C_{n+1} &= [z^{n+1}] \sum_{k \geq 0} C_k \frac{z^{2k+1}}{(1-2z)^{2k+1}} = \sum_{k \geq 0} C_k [z^{n-2k}] \sum_{j \geq 0} 2^j \binom{2k+j}{j} z^j \\ &= \sum_{0 \leq k \leq n/2} C_k 2^{n-2k} \binom{n}{2k}, \end{aligned}$$

which is known as Touchard's identity [61, 65].

With this interpretation in mind, (2.1) can also be seen as a recursive process to generate binary trees by repeated substitution of chains. This process can be modeled by the generating functions

$$B_0(z) = 1, \quad B_r(z) = 1 + \frac{z}{1-2z} B_{r-1}\Big(\frac{z^2}{(1-2z)^2}\Big), \quad r \geq 1. \tag{2.2}$$

By construction, $B_r(z)$ is the generating function of all binary trees that can be constructed from $\square$ with up to r expansions—or, equivalently—all binary trees that can be reduced to $\square$ by applying ρ up to r times.

Expanding the first few functions gives

$$\begin{aligned}
B_1(z) &= 1 + z + 2z^2 + 4z^3 + 8z^4 + 16z^5 + 32z^6 + 64z^7 + 128z^8 + 256z^9 + 512z^{10} + \cdots, \\
B_2(z) &= 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 428z^7 + 1416z^8 + 4744z^9 + \cdots, \\
B_3(z) &= 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7 + 1430z^8 + 4862z^9 + \cdots.
\end{aligned}$$

As it turns out, these generating functions are inherently linked with the *register function* (also known as the Horton-Strahler number) of binary trees. In order to understand this connection, we introduce the register function and prove a simple property regarding the tree reduction ρ.

The register function is recursively defined: for the binary tree consisting of only a leaf we have $\operatorname{Reg}(\square) = 0$, and if a binary tree t has subtrees t_1 and t_2, then the register function is defined to be

$$\operatorname{Reg}(t) = \begin{cases} \max\{\operatorname{Reg}(t_1), \operatorname{Reg}(t_2)\} & \text{for } \operatorname{Reg}(t_1) \neq \operatorname{Reg}(t_2), \\ \operatorname{Reg}(t_1) + 1 & \text{otherwise.} \end{cases}$$

In particular, the numbers attached to the nodes in Figure 2.1 represent the values of the register function of the subtree rooted at the respective node.

Historically, the idea of the register function originated (as the Horton-Strahler numbers) in [30, 64] in the study of the complexity of river networks. However, the very same concept also occurs within a computer science context: arithmetic expressions with binary operators can be expressed as a binary tree with data in the leaves and operators in the internal nodes. Then, the register function of this binary expression tree corresponds to the minimal number of registers needed to evaluate the expression.

There are several publications in which the register function and related concepts are investigated in great detail, for example Flajolet, Raoult, and Vuillemin [20], Kemp [36], Flajolet and Prodinger [19], Nebel [46], Drmota and Prodinger [13], and Viennot [66]. For a detailed survey on the register function and related topics see [56]. In contrast to those papers, we do not study the register function in this chapter, but we focus on the enumeration of branches of the binary tree. Details are given in Sections 2.2.2 and 2.2.3. Note that our methods could also be used to re-prove the known results on the register function.

We continue by observing that the tree reduction ρ is a very natural operation regarding the register function:

Proposition 2.2.2.
Let t be a binary tree with $\operatorname{Reg}(t) = r \geq 1$. Then $\rho(t)$ is well-defined and the register function of the reduced tree is $\operatorname{Reg}(\rho(t)) = r - 1$.

Proof. First, observe that all trees with at least one internal node have a node with two leaves attached. Therefore, this node has register function 1—and thus, only $\square$ has register function 0. Consequently, if we have $\operatorname{Reg}(t) \geq 1$, t cannot be $\square$, meaning that $\rho(t)$ is well-defined.

Now take an arbitrary binary tree t with at least one internal node and assume that we have $\operatorname{Reg}(\rho(t)) = r$. As described above, the tree t can be reconstructed from $\rho(t)$ by replacing all nodes (i.e. leaves and internal nodes) by chains of internal nodes.

When replacing internal nodes with chains of internal nodes, nothing changes for the register function: the value is just propagated up along the chain. However, if all leaves are replaced by chains, the register function of all subtrees that are rooted at a internal node increases by 1, resulting in $\operatorname{Reg}(t) = r + 1$. This proves the proposition. $\square$

As an immediate consequence of Proposition 2.2.2 we find that ρ can be applied r times repeatedly to some binary tree t if and only if $\operatorname{Reg}(t) \geq r$ holds. In particular, we obtain

$$\rho^r(t) = \square \quad \Longleftrightarrow \quad \operatorname{Reg}(t) = r. \tag{2.3}$$

With (2.3), the link between the generating functions $B_r(z)$ from above and the register function becomes clear: $B_r(z)$ is exactly the generating function of binary trees with register function $\leq r$.

In order to analyze these recursively defined generating functions an explicit representation is convenient. As it turns out, the substitution $z = \frac{u}{(1+u)^2} =: Z(u)$ is a helpful tool in this context.

Proposition 2.2.3.
Consider the complex functions

$$Z(u) = \frac{u}{(1+u)^2} \qquad \text{for } u \in \mathbb{C} \setminus \{-1\},$$
$$U(z) = \frac{1 - \sqrt{1-4z}}{2z} - 1 \qquad \text{for } z \in \mathbb{C},$$

where the principal branch of the square root function is chosen as usual, i.e., as a holomorphic function on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ such that $\sqrt{1} = 1$. Then the following properties hold:

(a) Let $\mathcal{Z} = \mathbb{C} \setminus [1/4, \infty)$ and $\mathcal{U} = \{u \in \mathbb{C} \mid |u| < 1\}$. Then $U\colon \mathcal{Z} \to \mathcal{U}$ and $Z\colon \mathcal{U} \to \mathcal{Z}$ are bijective holomorphic functions which are inverses of each other.

(b) Let $\overline{\mathcal{U}} = \mathcal{U} \cup \{\exp(-t\pi i) \mid 0 \le t < 1\}$. Then $U\colon \mathbb{C} \to \overline{\mathcal{U}}$ is bijective with inverse Z.

(c) The relations

$$Z'(u) = \frac{1-u}{(1+u)^3} \quad \text{and} \quad \frac{Z(u)}{1-2Z(u)} = \frac{u}{1+u^2}$$

hold for $u \in \mathbb{C} \setminus \{-1\}$.

(d) For the function $\sigma\colon \mathbb{C} \setminus \{\frac{1}{2}\} \to \mathbb{C}$ with $\sigma(z) = \frac{z^2}{(1-2z)^2}$, the diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "\sigma"] & \mathcal{Z} \\
\mathcal{U} \arrow[u, "Z"] \arrow[r, "u \mapsto u^2"'] & \mathcal{U} \arrow[u, "Z"']
\end{tikzcd}$$

commutes, i.e., we have $\sigma \circ Z = Z \circ (u \mapsto u^2)$.

(e) Let $\alpha \in \mathbb{C} \setminus \{0, -1\}$, $u \in \mathbb{C} \setminus \{\alpha, 1/\alpha\}$ and $z = Z(u)$. Then

$$\frac{u}{(u-\alpha)(u-\frac{1}{\alpha})} = -\frac{zZ(\alpha)}{z - Z(\alpha)}.$$

For $\alpha = -1$ we find $\frac{u}{(1+u)^2} = Z(u) = z$.

Proof.

(a) We first note that Z is well-defined and holomorphic on $\mathcal{U}$ with $Z'(u) \neq 0$ for all $u \in \mathcal{U}$. If $|u| = 1$, then

$$Z(u) = \frac{1}{u + \frac{1}{u} + 2} = \frac{1}{2 + 2\operatorname{Re} u}.$$

Thus, the image of the unit circle without $u = -1$ is the interval $[1/4, \infty)$.

For every $z \in \mathbb{C} \setminus \{0\}$, $z = Z(u)$ is equivalent to

$$u^2 + u\left(2 - \frac{1}{z}\right) + 1 = 0 \tag{2.4}$$

which has two not necessarily distinct solutions u_1, $u_2 \in \mathbb{C}$ with $u_1 u_2 = 1$. W.l.o.g., $|u_1| \le |u_2|$. Thus either $u_1 \in \mathcal{U}$ and $|u_2| > 1$ or $|u_1| = |u_2| = 1$. In the latter case, we have $z \in [1/4, \infty)$. For $z = 0$, $z = Z(u)$ is equivalent to $u = 0$. This implies that $Z\colon \mathcal{U} \to \mathcal{Z}$ is bijective.

Furthermore, $Z\colon \mathcal{U} \to \mathcal{Z}$ has a holomorphic inverse Z^{-1} defined on the simply connected region $\mathcal{Z}$. Solving (2.4) explicitly yields

$$u = \frac{1 \pm \sqrt{1-4z}}{2z} - 1.$$

In a neighborhood of zero, we must have $Z^{-1}(z) = U(z)$, because

$$\frac{1+\sqrt{1-4z}}{2z} - 1$$

has a pole at $z = 0$. Altogether this proves that U is the inverse of Z.

(b) [28]. For $z \in [1/4, \infty)$, we know that $U(z)$ is on the unit circle. It is easily checked that $\operatorname{Im} U(z) = -\sqrt{|1-4z|}/(2z)$ for these z, thus $\operatorname{Im} U(z) \in \overline{\mathcal{U}}$.

(c) The two relations follow directly from the definition of Z.

(d) [28]. This can be shown by straightforward computation: we obtain

$$\sigma(Z(u)) = \left(\frac{Z(u)}{1-2Z(u)}\right)^2 = \left(\frac{u}{1+u^2}\right)^2 = Z(u^2),$$

where (c) is used.

(e) [28]. By writing (2.4) as

$$u + \frac{1}{u} = \frac{1}{z} - 2,$$

we have

$$\begin{aligned}\frac{u}{(u-\alpha)(u-\frac{1}{\alpha})} &= -\frac{\alpha}{(u-\alpha)(\frac{1}{u}-\alpha)} = -\frac{\alpha}{1-\alpha(u+\frac{1}{u})+\alpha^2} \\ &= -\frac{\alpha}{(\alpha+1)^2-\frac{\alpha}{z}} = -\frac{zZ(\alpha)}{z-Z(\alpha)}.\end{aligned}$$

□

In a nutshell, the fact that $\sigma \circ Z = Z \circ (u \mapsto u^2)$ means that applying σ in the "z-world" corresponds to squaring in the "u-world". As we will see in a moment, this is very useful for expressing recursively defined generating functions like the one encountered above explicitly.

Proposition 2.2.4.
Let F_0, D, and E be complex functions that are analytic in a neighborhood of 0. Then the recursively defined functions

$$F_r(z) = D(z) + E(z)F_{r-1}(\sigma(z)), \quad r \geq 1, \tag{2.5}$$

can be written explicitly by means of the substitution $z = \frac{u}{(1+u)^2}$ as

$$F_r(z) = \sum_{j=0}^{r-1} D\left(\frac{u^{2^j}}{(1+u^{2^j})^2}\right)\prod_{k=0}^{j-1} E\left(\frac{u^{2^k}}{(1+u^{2^k})^2}\right) + F_0\left(\frac{u^{2^r}}{(1+u^{2^r})^2}\right)\prod_{k=0}^{r-1} E\left(\frac{u^{2^k}}{(1+u^{2^k})^2}\right). \tag{2.6}$$

Proof. Let $j \in \mathbb{N}$. Observe that by repeated application of Property (d) of Proposition 2.2.3 we can write

$$\frac{u^{2^j}}{(1+u^{2^j})^2} = Z(u^{2^j}) = \sigma(Z(u^{2^{j-1}})) = \cdots = \sigma^j(Z(u)) = \sigma^j(z),$$

where $\sigma^j(z)$ denotes the j-fold application of σ to z. This lets us write (2.6) as

$$F_r(z) = \sum_{j=0}^{r-1} D(\sigma^j(z)) \prod_{k=0}^{j-1} E(\sigma^k(z)) + F_0(\sigma^r(z)) \prod_{k=0}^{r-1} E(\sigma^k(z)).$$

This expression follows from (2.5) by induction over r. $\square$

With Proposition 2.2.4 we have an appropriate tool for analyzing $B_r(z)$, the generating function enumerating binary trees with register function $\leq r$. With $D(z) = 1$, $E(z) = \frac{z}{1-2z}$, and Property (c) of Proposition 2.2.3 the recurrence in (2.2) yields

$$B_r(z) = \frac{1-u^2}{u} \sum_{j=0}^{r} \frac{u^{2^j}}{1-u^{2^{j+1}}}. \tag{2.7}$$

Note that at this point, we can determine the generating function $B_r^=(z)$ counting binary trees with register function equal to r with respect to their size as

$$B_r^=(z) = B_r(z) - B_{r-1}(z) = \frac{1-u^2}{u} \frac{u^{2^r}}{1-u^{2^{r+1}}}. \tag{2.8}$$

This explicit representation of $B_r^=(z)$ could be used to determine the asymptotic behavior of the register function. However, as these properties are well-known (cf. [20]), we will continue in a different direction by studying the number of so-called r-branches—where we will also encounter the generating function $B_r^=(z)$ again.

2.2.2 r-branches

The register function associates a value to each node (internal nodes as well as leaves), and the value at the root is the value of the register function of the tree. An r-branch is a maximal chain of nodes labeled r. This must be a chain, since the merging of two such chains would already result in the higher value $r+1$. The nodes of the tree are partitioned into such chains, from $r = 0, 1, \ldots$. Figure 2.2 illustrates this situation for a tree of size 13.

The goal of this section is the study of the parameter "number of r-branches", in particular, the average number of them, assuming that all binary trees of size n are equally likely.

Formally, we investigate this parameter via the family of random variables $(X_{n;r})_{\substack{n\geq 0\\ r\geq 0}}$ where $X_{n;r} : \mathcal{B}_n \to \mathbb{N}_0$ counts the number of r-branches in binary trees of size n.

This parameter was the main object of the paper [71], and some partial results were given that we are now going to extend. In contrast to this paper, our approach relies heavily on generating functions which, besides allowing us to verify the results in a relatively straightforward way, also enables us to extract explicit formulas for the expectation (and, in principle, also for higher moments).

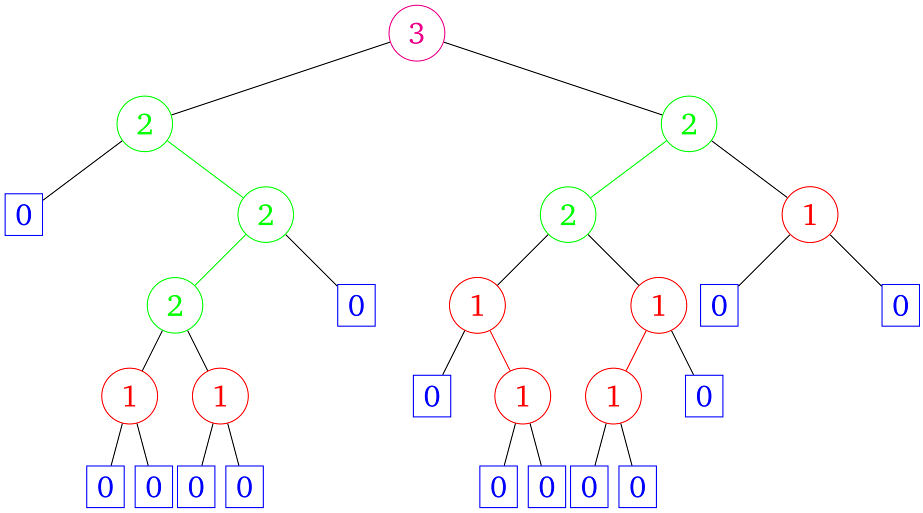


Figure 2.2: Binary tree with colored r-branches.

A parameter that was not investigated in [71] is the total number of r-branches, for any r, i.e., the sum over $r \geq 0$. Here, asymptotics are trickier, and the basic approach from [71] cannot be applied. However, in this chapter we use the Mellin transform, combined with singularity analysis of generating functions, a multi-layer approach that also allowed one of us several years ago to solve a problem by Yekutieli and Mandelbrot, cf. [51]. The origins of singularity analysis can be found in [18], and for a detailed survey see [21].

In particular, note that the value of the register function in [71] differs by one to the value we consider here, and that n generally refers there to the number of leaves, not nodes as here.

According to our previous considerations, after r iterations of ρ, the r-branches become leaves (or, equivalently, 0-branches).

We begin our detailed analysis of the random variables enumerating r-branches by studying sharp bounds for this parameter.

Notation.
We use the Iversonian notation

$$[\![\text{expr}]\!] = \begin{cases} 1 & \text{if expr is true,} \\ 0 & \text{otherwise,} \end{cases}$$

popularized in [22, Chapter 2].

Proposition 2.2.5.
Let $n, r \in \mathbb{N}_0$. If $r = 0$, then $X_{n;0}$ is a deterministic quantity with $X_{n;0} = n + 1$. For $r > 0$, the bound

$$[\![n > 0 \text{ and } r = 1]\!] \leq X_{n;r} \leq \left\lfloor \frac{n+1}{2^r} \right\rfloor$$

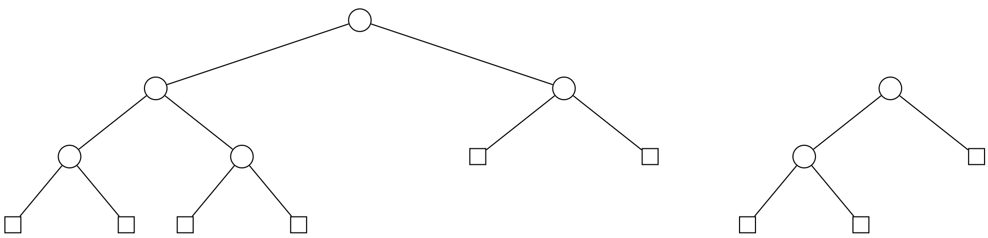
holds and is sharp.

Proof. First, recall that r-branches are nothing but leaves in the r-fold reduced tree. Thus, $X_{n;0}$ counts the number of leaves in a binary tree with n inner nodes—and it is a well-known fact that binary trees with n inner nodes always have $n+1$ leaves.

For the lower bound we observe that in every tree with at least one inner node, there is a node to which two leaves are attached. This node is part of a (possibly larger) 1-branch. Therefore, 1 is a lower bound for $X_{n;1}$ where $n > 0$. Chains are an example for arbitrarily large binary trees where the lower bounds 1 and 0 are attained for $r = 1$ and $r > 1$, respectively.

As there are finitely many binary trees of size n, there is a tree t for which $X_{n;r}$ attains its maximum value $M \in \mathbb{N}_0$, meaning that the r-fold reduced tree $\rho^r(t)$ has M leaves. In order to obtain an estimate between M and n we expand the reduced tree r times by successively replacing leaves by cherries, which are chains of size one. By doing so, the number of leaves doubles after every iteration, which means that our new tree has $M \cdot 2^r$ leaves—or, equivalently, $M \cdot 2^r - 1$ inner nodes. Because t cannot be smaller than the tree we have just constructed, the inequality $M \cdot 2^r - 1 \leq n$ has to hold. This proves the upper bound in the statement above.

In order to show that the upper bound is sharp as well, we consider the family of binary trees $(B_m)_{m\geq 1}$, where B_m denotes the unique almost complete binary tree with m leaves, which is constructed by adding the nodes layer-to-layer from left to right. For these trees, we can



(a) B_6. (b) B_3.

Figure 2.3: Almost complete binary trees.

prove that $\rho(B_m) = B_{\lfloor m/2 \rfloor}$: in case m is even, reducing the tree is equivalent to replacing all cherries on the lower levels by leaves, effectively halving the number of leaves. If $m = 2k+1$ is odd, there is a node whose left and right child are an inner node and a leaf, respectively. In particular, the subtree in question looks like B_3 illustrated in Figure 2.3b. When reducing this tree, the left child has to be merged with its parent. This shows that in total, $\rho(B_{2k+1})$ has k leaves.

By applying $\rho(B_m) = B_{\lfloor m/2 \rfloor}$ iteratively, and by

$$\left\lfloor \frac{m}{2^r} \right\rfloor = \left\lfloor \frac{1}{2} \left\lfloor \frac{m}{2^{r-1}} \right\rfloor \right\rfloor$$

we see that B_{n+1}, which is a binary tree of size n, attains the upper bound for the number of r-branches. □

Next we analyze the asymptotic behavior of the expectation and variance of $X_{n;r}$.

Theorem 2.2.6.
Let $r \in \mathbb{N}_0$ be fixed. The expected number $\mathbb{E}X_{n;r}$ of r-branches in binary trees of size n and the corresponding variance $\mathbb{V}X_{n;r}$ have the asymptotic expansions

$$\mathbb{E}X_{n;r} = \frac{n}{4^r} + \frac{1}{6}\left(1 + \frac{5}{4^r}\right) + \frac{1}{20n}\left(4^r - \frac{1}{4^r}\right) + \frac{1}{12n^2}\left(\frac{5 \cdot 16^r}{21} - \frac{7 \cdot 4^r}{10} + \frac{97}{210 \cdot 4^r}\right) + O(n^{-3}), \tag{2.9}$$

$$\mathbb{V}X_{n;r} = \frac{4^r - 1}{3 \cdot 16^r} n - \frac{2 \cdot 16^r - 25 \cdot 4^r + 23}{90 \cdot 16^r} - \frac{13 \cdot 64^r - 14 \cdot 16^r + 7 \cdot 4^r - 6}{420 \cdot 16^r n} + O(n^{-2}). \tag{2.10}$$

Remark.
The main terms (without error terms) of the asymptotic expansions for the expectation and the variance of the number of r-branches have already been determined in [45].

Proof. We begin our asymptotic analysis by constructing the generating function of the total number of leaves in all trees of size n. First, observe that the bivariate generating function allowing us to count the leaves of the binary trees is $vB(zv)$. Hence, the generating function counting the total number of leaves among all trees of size n is given by

$$\frac{\partial}{\partial v} vB(zv)\Big|_{v=1} = \frac{1}{\sqrt{1-4z}} = \frac{1+u}{1-u}.$$

Following the same recursive procedure as described in the proof of Proposition 2.2.1 and replacing all nodes of a given tree by chains, the leaves become 1-branches. Generally speaking, expanding a tree lets the r-branches become $(r+1)$-branches. In particular, this means that after r iterations of the tree expansion, the leaves have become r-branches.

With this in mind, we want to construct the generating function $F_r^{(1)}(z)$ that enumerates the sum of the number of r-branches over all trees with the same size, which is marked by z. As 0-branches are leaves, the expression determined above is precisely $F_0^{(1)}(z)$. Applying the tree expansion operator r times to $F_0^{(1)}(z)$ yields $F_r^{(1)}(z)$. This is justified by the argument

$$\begin{aligned} F_r^{(1)}(z) = \sum_{t \in \mathcal{B}} \#(r\text{-branches of } t) z^{|t|} &= \sum_{t' \in \mathcal{B}} \sum_{\substack{t \in \mathcal{B} \\ \rho(t) = t'}} \#(r\text{-branches of } t) z^{|t|} \\ &= \sum_{t' \in \mathcal{B}} \#((r-1)\text{-branches of } t') \sum_{\substack{t \in \mathcal{B} \\ \rho(t) = t'}} z^{|t|} \\ &= \frac{z}{1-2z} F_{r-1}^{(1)}\left(\frac{z^2}{(1-2z)^2}\right), \end{aligned}$$

where $|t|$ denotes the size of a tree $t \in \mathcal{B}$.

Altogether, we obtain the recursion

$$F_0^{(1)}(z) = \frac{1}{\sqrt{1-4z}}, \quad F_r^{(1)}(z) = \frac{z}{1-2z} F_{r-1}^{(1)}\left(\frac{z^2}{(1-2z)^2}\right), \quad r \geq 1.$$

By construction, dividing the nth coefficient of $F_r^{(1)}(z)$ by C_n yields

$$\mathbb{E}X_{n;r} = \frac{1}{C_n}[z^n]F_r^{(1)}(z),$$

which is the expected number of r-branches in a random tree of size n.

In order to analyze $F_r^{(1)}(z)$ we rewrite it using Proposition 2.2.4 and the fact that $D(z) = 0$ and $E(z) = \frac{z}{1-2z}$. Thus, we obtain

$$F_r^{(1)}(z) = \frac{1-u^2}{u} \frac{u^{2^r}}{(1-u^{2^r})^2}. \tag{2.11}$$

The generating function $F_r^{(1)}(z)$ has a singularity at $z = 1/4$, so we have to locally expand the function in terms of $\sqrt{1-4z}$ such that the methods of singularity analysis can be applied.

Expansion yields

$$F_r^{(1)}(z) = \frac{1}{4^r\sqrt{1-4z}} + \frac{1}{3}(4^{-r}-1)\sqrt{1-4z} + \frac{1}{15}(4^{1-r}-5+4^r)(1-4z)^{3/2} + O((1-4z)^{5/2}).$$

Singularity analysis [21, Chapter VI] guarantees that one can read off coefficients in this expansion:

$$[z^n]F_r^{(1)}(z) = \frac{4^n}{\sqrt{\pi}}\left(\frac{1}{4^r\sqrt{n}} + \frac{1}{6n^{3/2}}\left(1 - \frac{7}{4^{r+1}}\right) + \frac{1}{n^{5/2}}\left(\frac{4^r}{20} - \frac{3}{16} + \frac{93}{640\cdot 4^r}\right) + O(n^{-7/2})\right).$$

The asymptotics of C_n are straightforward, especially for a computer. By performing singularity analysis on the generating function $B(z)$ we obtain

$$C_n = \frac{4^n}{\sqrt{\pi}}\left(\frac{1}{n^{3/2}} - \frac{9}{8n^{5/2}} + \frac{145}{128n^{7/2}} + O(n^{-9/2})\right).$$

Division of the two expansions yields (2.9). In principle, any number of terms would be available.

We also determine the variance by virtually the same approach. In this case we determine the variance using the second factorial moment. Let $F_r^{(2)}(z)$ be the generating function of the unnormalized second factorial moment of the number of r-branches, i.e.,

$$F_r^{(2)}(z) = \sum_{n\geq 0} C_n \mathbb{E}\big(X_{n;r}(X_{n;r}-1)\big)z^n.$$

By analogous argumentation as before we know that $F_0^{(2)}(z)$ can be obtained by differentiating the bivariate generating function $vB(vz)$ two times with respect to v and setting $v = 1$. This gives

$$\frac{\partial^2}{\partial v^2} vB(zv)\Big|_{v=1} = \frac{2z}{(1-4z)^{3/2}} = \frac{2u(1+u)}{(1-u)^3}.$$

Furthermore, we know that the recurrence

$$F_0^{(2)}(z) = \frac{2z}{(1-4z)^{3/2}}, \quad F_r^{(2)}(z) = \frac{z}{1-2z} F_{r-1}^{(2)}\left(\frac{z^2}{(1-2z)^2}\right)$$

has to hold. Again, with the help of Proposition 2.2.4, we find

$$F_r^{(2)}(z) = 2\frac{1-u^2}{u} \frac{u^{2^{r+1}}}{(1-u^{2^r})^4},$$

which can be locally expanded to

$$F_r^{(2)}(z) = \frac{1}{2 \cdot 16^r (1-4z)^{3/2}} - \frac{1+2\cdot 4^r}{6 \cdot 16^r \sqrt{1-4z}} - \frac{1+10\cdot 4^r - 11\cdot 16^r}{90 \cdot 16^r} \sqrt{1-4z} + O((1-4z)^{3/2}).$$

After determining the asymptotic contribution of these coefficients by means of singularity analysis and dividing the result by the asymptotic expansion of the Catalan numbers, we arrive at an expansion for the second factorial moment:

$$\mathbb{E}X_{n;r}(X_{n;r}-1) = \frac{1}{C_n}[z^n]F_r^{(2)}(z) = \frac{n^2}{16^r} + \frac{4-4^r}{3\cdot 16^r} n + \frac{61 - 50\cdot 4^r - 11 \cdot 16^r}{180 \cdot 16^r} + O(n^{-1}).$$

Computing the variance by means of the well-known formula $\mathbb{V}X_{n;r} = \mathbb{E}X_{n;r}(X_{n;r}-1) + \mathbb{E}X_{n;r} - (\mathbb{E}X_{n;r})^2$ yields (2.10) and thus, concludes the proof. □

Of course, the expected number of r-branches can also be computed explicitly by using Cauchy's integral formula. This yields the following result:

Proposition 2.2.7.
The expected number $\mathbb{E}X_{n;r}$ of r-branches in binary trees of size n is given by the explicit formula

$$\mathbb{E}X_{n;r} = \frac{n+1}{\binom{2n}{n}} \sum_{\lambda \geq 1} \lambda \left[\binom{2n}{n+1-\lambda 2^r} - 2\binom{2n}{n-\lambda 2^r} + \binom{2n}{n-1-\lambda 2^r}\right]. \tag{2.12}$$

Proof. Let γ be a small circle around the origin. Then, $\tilde{\gamma}$, the image of γ under the substitution

$z = \frac{u}{(1+u)^2}$, is a closed curve in the interior of the unit circle that winds around the origin once. Now, applying Cauchy's integral formula yields

$$\begin{aligned}
[z^n]F_r^{(1)}(z) &= \frac{1}{2\pi i}\oint_\gamma \frac{1-u^2}{u}\frac{u^{2^r}}{(1-u^{2^r})^2}\frac{dz}{z^{n+1}} \\
&= \frac{1}{2\pi i}\oint_{\tilde{\gamma}} \frac{1-u^2}{u}\frac{u^{2^r}}{(1-u^{2^r})^2}\frac{(1-u)(1+u)^{2n+2}}{(1+u)^3}\frac{du}{u^{n+1}} \\
&= \frac{1}{2\pi i}\oint_{\tilde{\gamma}} \frac{(1-u)^2(1+u)^{2n}}{u^{n+2}}\Big(\sum_{\lambda\ge 0}\lambda u^{\lambda 2^r}\Big)du,
\end{aligned}$$

where we used

$$\frac{x}{(1-x)^2} = \sum_{\lambda\ge 0}\lambda x^\lambda. \tag{2.13}$$

Then, interchanging summation and integration and applying Cauchy's integral formula once again yields

$$[z^n]F_r^{(1)}(z) = \sum_{\lambda\ge 1}\lambda[u^{n+1-\lambda 2^r}](1-u)^2(1+u)^{2n},$$

which, after extracting the coefficient and dividing by $C_n = \frac{1}{n+1}\binom{2n}{n}$, proves the statement. $\square$

We are also interested in the limiting distribution of $X_{n;r}$ for fixed $r \in \mathbb{N}_0$ and $n \to \infty$. Note that as $X_{n;0} = n+1$ is a deterministic quantity, we focus on the case that $r \ge 1$.

Theorem 2.2.8.
Let $r \in \mathbb{N}$ be fixed. Then $X_{n;r}$, the random variable modeling the number of r-branches in a binary tree of size n, is asymptotically normally distributed for $n \to \infty$. In particular, for $x \in \mathbb{R}$ we have

$$\mathbb{P}\Big(\frac{X_{n;r} - \mathbb{E}X_{n;r}}{\sqrt{\mathbb{V}X_{n;r}}} \le x\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt + O(n^{-1/2}).$$

Remark.
For the special case of 1-branches, i.e., $r = 1$, a central limit theorem has been proved in [68]. Additionally, numerical evidence for the validity of a general central limit theorem like the one we obtained above has been provided in [70].

Proof of Theorem 2.2.8. The central idea behind this proof is that $X_{n;r}$ can be interpreted as an *additive tree parameter*, meaning that the parameter can be evaluated as the sum of the parameters corresponding to the subtrees rooted at the children of the root of the original tree and an additional so-called *toll function*.

In our case, it is straightforward to see that the number of r-branches in a binary tree of size n can be computed as the sum of the number of r-branches in the left and right subtree. Only in the case where both subtrees have register function $r-1$, the root itself is an r-branch that is not accounted for in the subtrees.

Hence, the random variable $X_{n;r}$ satisfies the distributional recurrence relation

$$X_{n;r} = X_{I_n;r} + X^*_{n-1-I_n;r} + T_{n;r},$$

where $X^*_{n;r}$ is an independent copy of $X_{n;r}$, $I_{n;r}$ is a random variable modeling the size of the left subtree with

$$\mathbb{P}(I_n = j) = \frac{C_j C_{n-1-j}}{C_n} \qquad \text{where } j \in \{0, 1, \ldots, n-1\},$$

and where $T_{n;r}$ is a toll function depending on $X_{n;r}$ satisfying

$$T_{n;r} = \begin{cases} 1 & \text{if the register function of both rooted subtrees is } r-1, \\ 0 & \text{otherwise.} \end{cases}$$

Asymptotic normality of $X_{n;r}$ can now be obtained by showing that the expectation of the toll function decays exponentially, according to [67].

In order to show that this condition is satisfied we consider $B^=_r(z)$, the generating function for binary trees with register function equal to r. By (2.8) we have

$$B^=_r(z) = \frac{1-u^2}{u} \frac{u^{2^r}}{1-u^{2^{r+1}}}.$$

By means of Property (e) in Proposition 2.2.3 we can write

$$B^=_r(z) = \frac{(u-1)(u+1)u^{2^r-1}}{\prod_{0 \le k < 2^{r+1}} (u - \omega_k)} = -\frac{z^{2^r-1} \prod_{0<k<2^r} Z(\omega_k)}{\prod_{0<k<2^r} (z - Z(\omega_k))},$$

where $\omega_k := \exp(2\pi i k/2^{r+1})$. In particular, this proves that $B^=_r(z)$ is a rational function. As we have $Z(\omega_k) = \frac{1}{2+2\cos(\pi k/2^r)}$, the dominant singularity of $B^=_r(z)$ can uniquely be identified as $Z(\omega_1) = \frac{1}{2+2\cos(\pi/2^r)} > 1/4$, which proves that the ratio of trees with register function r among all binary trees of size n decays exponentially.

The exponential decay of the expected value of the toll function $T_{n;r}$ now follows from the fact that $\mathbb{E}T_{n;r}$ equals the ratio of the trees whose children both have register function $r-1$ among all trees of size n. These trees form a subset of those counted by $B^=_r(z)$, which means that their ratio has to decay exponentially as well.

The asymptotic normality of $X_{n;r}$ now follows from [67, Theorem 2.1]. All that remains to show is that the speed of convergence is $O(n^{-1/2})$. In order to do so, we observe that the proof for asymptotic normality in Wagner's theorem basically relies on [11, Theorem 2.23], which uses a formulation of Hwang's Quasi-Power Theorem without quantification of the speed of convergence (cf. [11, Theorem 2.22]). By replacing this argument with a quantified version (cf. [31] or [29] for a generalization to higher dimensions) of the Quasi-Power Theorem, we find that the speed of convergence in Wagner's result—and therefore, in our result as well—is $O(n^{-1/2})$. □

2.2.3 Total Number of Branches

So far, we were dealing with fixed r, and the number of r-branches in trees of size n, for large n. Now we consider the total number of such branches, i.e., the sum over $r \geq 0$, which was not considered in [71]. Formally, this corresponds to the analysis of the random variable $X_n \colon \mathcal{B}_n \to \mathbb{N}_0$ where

$$X_n := \sum_{r\geq 0} X_{n;r}.$$

By definition, X_n enumerates the total number of branches in binary trees of size n.

With the help of the bounds for the number of r-branches obtained in Proposition 2.2.5 we can characterize the range of X_n as well.

Proposition 2.2.9.
Let $n \in \mathbb{N}_0$ and let $w_2(n)$ denote the binary weight, i.e. the number of non-zero digits in the binary expansion of n. Then the bound

$$n + 1 + [\![n > 0]\!] \leq X_n \leq 2n + 2 - w_2(n+1) \leq 2n + 1$$

holds for the random variable X_n and is sharp.

Remark.
The sharp upper bound $2n + 2 - w_2(n+1)$ is enumerated by sequence A005187, shifted by one, in [47].

Proof. We begin by observing that for fixed $n \in \mathbb{N}$, the random variable $X_{n;r}$ vanishes for sufficiently large r. As the bounds from Proposition 2.2.5 are sharp, we are allowed to sum up the inequalities in order to obtain

$$n + 1 + [\![n > 0]\!] \leq \sum_{r\geq 0} X_{n;r} \leq \sum_{r\geq 0} \left\lfloor \frac{n+1}{2^r} \right\rfloor.$$

This immediately proves the lower bound from the statement.

In order to prove the upper bound we investigate the sum $\sum_{r\geq 0} \lfloor m/2^r \rfloor$ for $m \in \mathbb{N}$. Consider the binary digit expansion of m, denoted by $(x_k \dots x_1 x_0)_2$. In this context, the sum can be written as

$$\sum_{r=0}^{k}(x_k \dots x_{r+1}x_r)_2 = \sum_{r=0}^{k} x_r(1+2+4+\dots+2^r) = \sum_{r=0}^{k} x_r(2^{r+1}-1)$$
$$= 2\cdot(x_k \dots x_1 x_0)_2 - \sum_{r=0}^{k} x_r = 2m - w_2(m).$$

By setting $m = n+1$ we see that the upper bound holds as well. The fact that $2n+2-w_2(n+1) \le 2n+1$ is a direct consequence of $w_2(n+1) \ge 1$ for all $n \in \mathbb{N}_0$.

It is easy to see that the bounds are sharp for $n = 0$. For $n > 0$, the lower bound is attained by any chain of size n: they consist of $n+1$ leaves (which are 0-branches) and exactly one additional 1-branch which connects all the leaves. The upper bound is attained by the family of almost complete binary trees constructed in the proof of Proposition 2.2.5, which follows from the fact that $X_{n;r}$ attains its maximum in the tree B_{n+1}, which does not depend on the value of r. □

First, to get an explicit formula, the results from Proposition 2.2.7 can be summed.

Corollary 2.2.10.
The expected number of branches in binary trees of size n is given by the explicit formula

$$\mathbb{E}X_n = \frac{n+1}{\binom{2n}{n}} \sum_{k=1}^{n+1} (2-2^{-\nu_2(k)})k\left[\binom{2n}{n+1-k} - 2\binom{2n}{n-k} + \binom{2n}{n-1-k}\right],$$

where $\nu_2(k)$ is the dyadic valuation of k, i.e., the highest exponent ν such that 2^ν divides k.

Proof. To simplify the double summation, we consider

$$\psi(k) := \sum_{\substack{\lambda\ge 0,\, r\ge 0:\\ \lambda 2^r = k}} \lambda.$$

This sum can be simplified to some degree. We write $k = 2^{\nu_2(k)}(2j+1)$, such that we have

$$\psi(k) = \sum_{r=0}^{\nu_2(k)} 2^{\nu_2(k)-r}(2j+1) = (2^{\nu_2(k)+1}-1)(2j+1) = (2-2^{-\nu_2(k)})k,$$

which proves the result. □

While it is absolutely possible to work out the asymptotic growth from this explicit formula, at it was done in earlier papers [20, 36], we choose a faster method, like in [19]. It works on the level of generating functions and uses the Mellin transform together with singularity analysis of generating functions [21, 54].

The following theorem describes the asymptotic behavior for the expected number of branches in a binary tree.

Theorem 2.2.11.
The expected value of the total number of branches in a random binary tree of size n admits the asymptotic expansion

$$\mathbb{E}X_n = \frac{4n}{3} + \frac{1}{6}\log_4 n - \frac{2\zeta'(-1)}{\log 2} - \frac{\gamma}{12\log 2} - \frac{1}{6\log 2} + \frac{43}{36} + \delta(\log_4 n) + O\Big(\frac{\log n}{n}\Big),$$

where

$$\delta(x) := \frac{1}{\log 2}\sum_{k\neq 0}\Gamma\Big(\frac{\chi_k}{2}\Big)\zeta(\chi_k - 1)(\chi_k - 1)e^{2\pi i k x}$$

is a 1-periodic function of mean zero, given by its Fourier series expansion with $\chi_k = \frac{2\pi i k}{\log 2}$.

Remark.
Note that the value of the derivative of the zeta function is given by $\zeta'(-1) = -\frac{1}{12} - \log A \approx -0.1654211437$, where A is the Glaisher-Kinkelin constant (cf. [14, Section 2.15]).

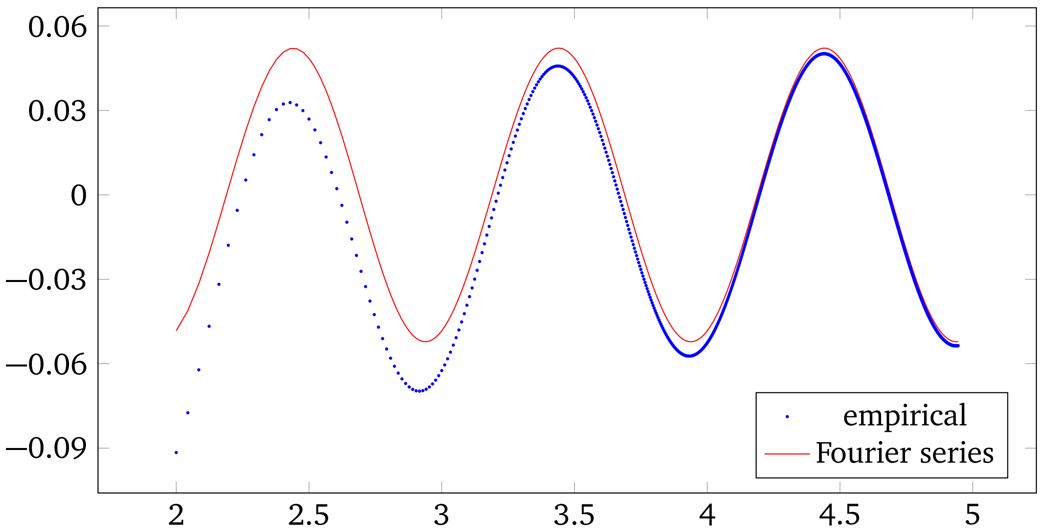


Figure 2.4: Partial Fourier series (20 summands) compared with the empirical values of the function δ from Theorem 2.2.11.

Remark.
The occurrence of the periodic fluctuation δ where the argument is logarithmic in n is actually not surprising: while this phenomenon is already very common in the context of the register function, fluctuations appear very often in the asymptotic analysis of sums.

Proof. By using (2.11) and (2.13), the generating function of interest can be written as

$$F(z) = \sum_{r\geq 0} F_r^{(1)}(z) = \sum_{r\geq 0}\frac{1-u^2}{u}\frac{u^{2^r}}{(1-u^{2^r})^2} = \frac{1-u^2}{u}\sum_{r,\lambda\geq 0}\lambda u^{\lambda 2^r}.$$

To find the asymptotic behavior of the sum, we set $u = e^{-t}$, consider the function

$$f(t) := \sum_{r,\lambda\geq 0} \lambda e^{-t\lambda 2^r},$$

and compute its Mellin transform

$$f^*(s) = \sum_{r,\lambda\geq 0} \lambda^{1-s} 2^{-rs}\Gamma(s) = \Gamma(s)\zeta(s-1)\frac{1}{1-2^{-s}}.$$

The fundamental strip of $f^*(s)$ is $\langle 2,\infty\rangle$. Then, by the inversion formula for the Mellin transform we obtain

$$f(t) = \frac{1}{2\pi i}\int_{5-i\infty}^{5+i\infty} \Gamma(s)\zeta(s-1)\frac{1}{1-2^{-s}} t^{-s}\, ds, \tag{2.14}$$

which is valid for real, positive $t \to 0$, which gives an expansion for $u \to 1^-$, or, equivalently $z \to (1/4)^-$. In order to use this representation of $f(t)$ for our purposes (i.e. in order to apply singularity analysis), we need to have analyticity in a larger region (cf. [18]), e.g. in a complex punctured neighborhood of $1/4$ with[1] $|\arg(z-1/4)| > 2\pi/5$. In particular, the expansion

$$t = -\log(U(z)) = 2\sqrt{1-4z} + \frac{2}{3}(1-4z)^{3/2} + O((1-4z)^{5/2}) \tag{2.15}$$

implies

$$|\arg t| = \frac{1}{2}|\arg(1-4z)| + o(1),$$

such that we have the bound $|\arg t| < 2\pi/5$ for $t \to 0$, given that the restriction on the argument in the z-world is satisfied.

Then, given that $\mathrm{Re}(s) = 5$ or $\mathrm{Re}(s) = -3$ holds we find that we have the estimate

$$|f^*(s)t^{-s}| = O\Big(|\mathrm{Im}(s)|^5 |t|^{-\mathrm{Re}(s)} \exp\Big(-\frac{\pi}{10}|\mathrm{Im}(s)|\Big)\Big) \tag{2.16}$$

for the integrand in (2.14). The very same estimate also holds for $-3 \leq \mathrm{Re}(s) \leq 5$ where $\mathrm{Im}(s) = \frac{2\pi i}{\log 2}\big(k+\frac{1}{2}\big)$ and $k \in \mathbb{Z}$ tends towards ∞ or $-\infty$. This is a consequence of the behavior of $\Gamma(s)$ as given in [10, 5.11.3], estimates for $\zeta(s)$ as given in [69, 13.51], and the fact that $\frac{1}{1-2^{-s}}$ is bounded for s in the given ranges.

Together with the identity theorem for analytic functions (cf. [19] for a similar argumentation) this means that the inverse Mellin transform remains valid for complex z in a neighborhood of $1/4$ with $|\arg(1-4z)| > 2\pi/5$, which justifies the following approach.

We can evaluate (2.14) by shifting the line of integration from $\mathrm{Re}(s) = 5$ to $\mathrm{Re}(s) = -3$ and collecting the residues of the poles we cross. This yields

$$f(t) = \sum_{p\in P} \mathrm{Res}_{s=p}(f^*(s)t^{-s}) + \frac{1}{2\pi i}\int_{-3-i\infty}^{-3+i\infty} f^*(s)t^{-s}\, ds,$$

[1]Note that the bound $2\pi/5$ is somewhat arbitrary: the argument just needs to be less than $\pi/2$.

where $P = \{-2, 0, 2\} \cup \{\chi_k \mid k \in \mathbb{Z} \setminus \{0\}\}$ and $\chi_k := \frac{2\pi i k}{\log 2}$. Multiplying this representation of $f(t)$ with $\frac{1-u^2}{u}$ and expanding everything locally for $z \to 1/4$ yields a singular expansion from which coefficient growth can be extracted by means of singularity analysis.

For the error term we use the estimate above and find

$$\frac{1}{2\pi i} \int_{-3-i\infty}^{-3+i\infty} f^*(s) t^{-s}\, ds = O(|t|^3).$$

However, for the sake of simplicity we want to use the contribution of the residue collected from the pole at $s = -2$ as the error term. We immediately find

$$\operatorname{Res}_{s=-2}(f^*(s)t^{-s}) = O(|t|^2).$$

We compute the remaining residues explicitly with the help of SageMath [59] and obtain

$$\begin{aligned} f(t) &= \sum_{p \in P \setminus \{-2\}} \operatorname{Res}_{s=p}(f^*(s)t^{-s}) + O(|t|^2) \\ &= \Big(\frac{4}{3t^2} + \frac{\log t}{12 \log 2} + \frac{\zeta'(-1)}{\log 2} + \frac{\gamma}{12 \log 2} - \frac{1}{24}\Big) + \sum_{k \neq 0} \frac{\Gamma(\chi_k)\zeta(\chi_k - 1)}{\log 2} t^{-\chi_k} + O(|t|^2), \end{aligned}$$

where we used the Laurent expansion for the Gamma function at $s = 0$ (cf. [62, 43:6:1]) and the fact that $\zeta(-1) = -1/12$ (cf. [10, 25.6.3]). When translating this expansion in terms of $t \to 0$ to an expansion in terms of $z \to 1/4$, we have to be particularly careful with respect to the sum of the residues at $s = \chi_k$ as we have to check that the sum of the errors is still controllable.

We do so by considering the expansion

$$t^{-\chi_k} = (1 - 4z)^{-\chi_k/2}\big(1 + O(1 - 4z)\big)^{-\chi_k/2}.$$

With the well-known inequality

$$|\exp(z) - 1| \le |z| \exp|z|$$

we find

$$\begin{aligned} |(1 + O(1 - 4z))^{-\chi_k/2} - 1| &= \Big|\exp\Big(-\frac{\chi_k}{2} \log(1 + O(1 - 4z))\Big) - 1\Big| \\ &\le \Big|\frac{\chi_k}{2}\Big| |\log(1 + O(1 - 4z))| \exp\Big(\frac{2\pi}{\log 2} |k| |\log(1 + O(1 - 4z))|\Big) \\ &= \Big|\frac{\chi_k}{2}\Big| O(1 - 4z) \exp\Big(\frac{2\pi}{\log 2} |k| O(1 - 4z)\Big). \end{aligned} \tag{2.17}$$

This proves that the errors we sum up are of order $O(|k|(1 - 4z)\exp(|k|O(1 - 4z)))$. Thus, if z is chosen sufficiently close to $1/4$, this exponential growth is slow enough to vanish within the exponential decay established in (2.16).

Finally, by considering

$$\frac{1-u^2}{u} = 4\sqrt{1-4z} + 4(1-4z)^{3/2} + O((1-4z)^{5/2})$$

we find that

$$F(z) = \frac{4}{3\sqrt{1-4z}} + \left(\frac{\gamma}{3\log 2} + \frac{4\zeta'(-1)}{\log 2} + \frac{11}{18} + \frac{\log(1-4z)}{6\log 2}\right)\sqrt{1-4z} \\ + \frac{4}{\log 2}\sum_{k\neq 0}\Gamma(\chi_k)\zeta(\chi_k - 1)(1-4z)^{1/2-\chi_k/2} + O((1-4z)^{3/2}\log(1-4z)).$$

Applying singularity analysis, normalizing the result by C_n and rewriting the coefficients of the contributions from the poles at χ_k via the duplication formula for the Gamma function (cf. [10, 5.5.5]) then proves the asymptotic expansion for $\mathbb{E}X_n$. $\square$

While this multi-layer approach enabled us to analyze the expected value of the number of branches in binary trees of size n, the same strategy fails for computing the variance. This is because the random variables modeling the number of r-branches are correlated for different values of r—and thus, the sum of the variances (which we compute by our approach) differs from the variance of the sum.

This concludes our study of the number of branches per binary tree. In the next section, we analyze a quantity that has similar properties as the register function, but is defined on simple two-dimensional lattice paths.

2.3 Reduction of Lattice Paths

2.3.1 Iterative Reductions and an Analogue to the Register Function

Recall that the register function describes the number of reductions of a binary tree required in order to reduce the tree to a leaf. By defining a similar process for simple two-dimensional lattice paths, a function that plays a similar role as the register function is obtained.

Simple two-dimensional lattice paths are sequences of the symbols $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$. It is easy to see that the generating function counting these paths (without the path of length 0) is

$$L(z) = \frac{4z}{1-4z} = 4z + 16z^2 + 64z^3 + 256z^4 + 1024z^5 + \cdots.$$

Proposition 2.3.1.
The generating function $L(z) = \frac{4z}{1-4z}$ satisfies the functional equation

$$L(z) = 4L\left(\frac{z^2}{(1-2z)^2}\right) + 4z. \tag{2.18}$$

Remark.
It is easy to verify this result by means of substitution and expansion. However, we want to give a combinatorial proof—this approach also motivates the definition of a recursive reduction process for lattice paths, similar to the process for binary trees from above.

The principle is easy: a sequence of consecutive horizontal steps followed by a sequence of consecutive vertical steps is replaced by a diagonal step which is determined by the first horizontal step and the first vertical step. The details are slightly more technical because we have to ensure that any given path can be reduced.

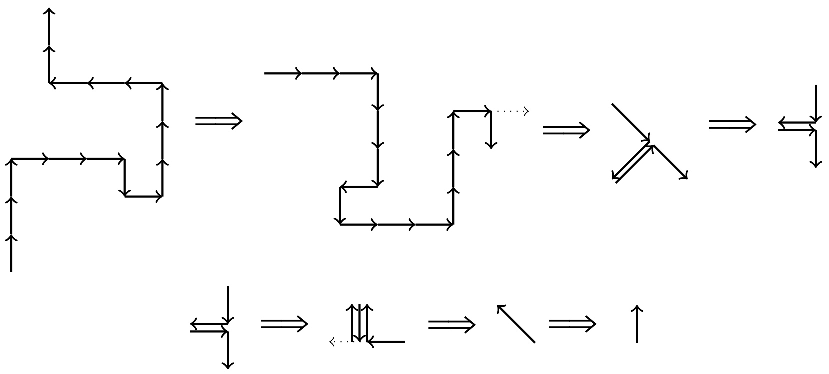


Figure 2.5: Repeated application of the reduction ρ_L on a path with reduction degree 2.

Proof. We show that the right-hand side of (2.18) counts simple two-dimensional lattice paths (excluding the path of length 0) as well. In order to do so, we introduce a reduction of lattice paths denoted by ρ_L, that works on a given path ℓ with length ≥ 2 as follows:

First, if ℓ starts vertically (i.e. with $\uparrow$ or $\downarrow$), rotate the entire path clockwise such that it starts horizontally.

Second, if the (possibly rotated) path ends horizontally, rotate the very last step (which has to be $\rightarrow$ or $\leftarrow$) once again clockwise.

Now, the path can be reduced by collapsing each pair of horizontal-vertical path segments into a path of length 1 as follows:

- If a segment starts with $\rightarrow$ and the first vertical step is $\uparrow$, replace it by $\nearrow$,
- if a segment starts with $\rightarrow$ and the first vertical step is $\downarrow$, replace it by $\searrow$,
- if a segment starts with $\leftarrow$ and the first vertical step is $\downarrow$, replace it by $\swarrow$,
- and if a segment starts with $\leftarrow$ and the first vertical step is $\uparrow$, replace it by $\nwarrow$.

Finally, rotate the obtained path with the diagonal steps by 45° clockwise such that $\nearrow$

becomes $\rightarrow$ and so on. The resulting path is the reduction $\rho_L(\ell)$. This process is visualized in Figure 2.5.

As it is the case with the reduction ρ of binary trees, ρ_L is a partial function as well: $\rho_L(s)$ is undefined for $s \in \{\uparrow, \rightarrow, \downarrow, \leftarrow\}$. Furthermore, a reduced path can be expanded (although not uniquely) to its original path again by rotating a given path to the left such that it is given in diagonal steps, reading the replacements from above from right to left, and then optionally rotating the very last step and/or the entire path to the left again.

We find that the generating function for lattice paths consisting of sequences of horizontal-vertical segments is given by $L\big(\frac{z^2}{(1-2z)^2}\big)$. In order to see this, consider the expansion of a path of length 1, for example

$$\rightarrow \quad \Longrightarrow \quad \nearrow \quad \Longrightarrow \quad \rightarrow(\rightarrow + \leftarrow)^* \uparrow(\uparrow + \downarrow)^*,$$

where the regular expression describes a sequence of horizontal steps starting with $\rightarrow$, followed by a sequence of vertical steps starting with $\uparrow$. As path length is marked by z, the expansion above translates to the substitution

$$z \mapsto \frac{z^2}{(1-2z)^2}.$$

As all four expansion variants lead to the same variable substitution, $L\big(\frac{z^2}{(1-2z)^2}\big)$ precisely enumerates all lattice paths consisting of sequences of horizontal-vertical segments.

The factor 4 in (2.18) is explained by the four path variants obtained by either rotating just the last step and/or the entire path.

Putting all of this together, (2.18) can be interpreted combinatorially as the following statement: a simple two-dimensional lattice path is either a simple step, or can be obtained by expanding another simple two-dimensional lattice path. This proves the proposition. $\square$

The process described in the proof of Proposition 2.3.1 allows us to assign a unique number to each lattice path:

Definition 2.3.2.
Let ℓ be a simple two-dimensional lattice path consisting of at least one step. We define the *reduction degree* of ℓ, denoted as $\operatorname{rdeg}(\ell)$ as

$$\operatorname{rdeg}(\ell) = n \quad \Longleftrightarrow \quad \rho_L^n(\ell) \in \{\uparrow, \rightarrow, \downarrow, \leftarrow\}.$$

Remark.
The parallels between the reduction degree and the register function are obvious: both count the number of times some given mathematical object can be reduced according to some rules until an atomic form of the respective object is obtained. Therefore, both functions describe, in some sense, the complexity of a given structure.

In the remainder of this section we want to derive some asymptotic results for the reduction degree, namely the expected degree of a lattice path of given length as well as the corresponding variance.

Analogously to our strategy for (2.1), we want to interpret (2.18) as a recursive generation process as well and therefore set

$$L_0(z) = 4z, \quad L_r(z) = 4L_{r-1}\left(\frac{z^2}{(1-2z)^2}\right) + 4z, \quad r \geq 1.$$

This yields the functions

$$\begin{aligned}
L_1(z) &= 4z + 16z^2 + 64z^3 + 192z^4 + 512z^5 + 1280z^6 + 3072z^7 + 7168z^8 + \cdots, \\
L_2(z) &= 4z + 16z^2 + 64z^3 + 256z^4 + 1024z^5 + 4096z^6 + 16384z^7 + 65280z^8 + \cdots, \\
L_3(z) &= 4z + 16z^2 + 64z^3 + 256z^4 + 1024z^5 + 4096z^6 + 16384z^7 + 65536z^8 + \cdots.
\end{aligned}$$

Due to the construction, the function $L_r(z)$ is the generating functions of those lattice paths with reduction degree $\leq r$.

By using Proposition 2.2.4 with $D(z) = 4z$ and $E(z) = 4$, the generating functions $L_r(z)$ can be written explicitly in terms of $u = U(z)$—with $U(z)$ as given in Proposition 2.2.3—as

$$L_r(z) = \sum_{j=0}^{r} 4^{j+1} \frac{u^{2^j}}{(1+u^{2^j})^2}.$$

The generating function $L_r^{=}(z)$ of lattice paths with reduction degree equal to r can then be found by considering the difference $L_r(z) - L_{r-1}(z)$, or, alternatively, by dropping the summand $4z$ in the recursion above. Both approaches lead to

$$L_r^{=}(z) = 4^{r+1} \frac{u^{2^r}}{(1+u^{2^r})^2}. \tag{2.19}$$

The coefficients of this function can be extracted explicitly by applying Cauchy's integral formula.

Proposition 2.3.3.
The number of two-dimensional simple lattice paths of length n that have reduction degree r is given by

$$[z^n]L_r^{=}(z) = 4^{r+1} \sum_{\lambda \geq 0} \lambda(-1)^{\lambda-1} \left[\binom{2n-1}{n-\lambda 2^r} - \binom{2n-1}{n-\lambda 2^r - 1} \right].$$

Proof. The proof is straightforward and uses the same approach as the proof of Proposition 2.2.7. □

In fact, by studying the substitution $z = Z(u)$ closely, the asymptotic behavior of the coefficients of $L_r^{=}(z)$ can be extracted as well.

Proposition 2.3.4 ([28]).
Let $r \geq 1$ be fixed. Then $L_r^{=}(z)$ is a rational function in z with poles at

$$z_k = \frac{1}{4\cos^2(\pi k 2^{-r-1})}$$

with singular expansions

$$L_r^{=}(z) = \frac{4\tan^2\left(\frac{k\pi}{2^{r+1}}\right)}{\left(1-\frac{z}{z_k}\right)^2} - \frac{4\sin^2\left(\frac{k\pi}{2^{r+1}}\right)+2}{\cos^2\left(\frac{k\pi}{2^{r+1}}\right)} \frac{1}{1-\frac{z}{z_k}} + O(1), \qquad z \to z_k,$$

for $1 \leq k < 2^r$, k odd.

Proof. First note that all of the following estimates are not uniform w.r.t. r, meaning that the constant in the O-term depends heavily on r.

By definition, $L_r^{=}(z)$ is a rational function. From (2.19) and Proposition 2.2.3(e), we obtain that the poles of $L_r^{=}(z)$ are located at $Z(\omega)$ where ω runs through the 2^rth roots of -1. By symmetry, we restrict ourselves to ω with $\operatorname{Im}\omega \leq 0$.

We now fix such an $\omega = \exp(-k\pi i 2^{-r})$ for some $1 \leq k < 2^r$, k odd. By expansion around ω, we get

$$\frac{4^{r+1}u^{2^r}}{(1+u^{2^r})^2} = -\frac{4\omega^2}{(u-\omega)^2} - \frac{4\omega}{u-\omega} + O(1) \quad \text{for } u \to \omega.$$

We know that $L_r^{=}(z)$ has a pole of order 2 at $z_k = Z(\omega)$, implying that expanding $L_r^{=}(z)$ for $z \to z_k$ yields an expansion of the form

$$L_r^{=}(z) = \frac{A}{\left(1-\frac{z}{z_k}\right)^2} + \frac{B}{1-\frac{z}{z_k}} + O(1) \quad \text{for } z \to z_k$$

where A and B are some constants depending on k and r. With the help of Cauchy's integral formula, the substitution $u = U(z)$, and the expansion from above we can determine the constants A and B and find

$$L_r^{=}(z) = \frac{-4(\omega-1)^2}{(\omega+1)^2} \frac{1}{\left(1-\frac{z}{z_k}\right)^2} + \frac{4(\omega^2-4\omega+1)}{(\omega+1)^2} \frac{1}{1-\frac{z}{z_k}} + O(1) \quad \text{for } z \to z_k.$$

Rewriting all complex exponentials in terms of trigonometric functions then yields the result. □

With the help of this characterization of the poles of $L_r^{=}$ the asymptotic behavior of the number of lattice paths with reduction degree equal to r can be obtained.

Corollary 2.3.5 ([28]).
Let $r \geq 1$ be fixed. The number of lattice paths with reduction degree equal to r admits the asymptotic expansion

$$[z^n]L_r^{=}(z) = (4\cos^2(\pi 2^{-r-1}))^n \Big(4\tan^2(\pi 2^{-r-1})n - \frac{2}{\cos^2(\pi 2^{-r-1})}\Big) + O\big((4\cos^2(3\pi 2^{-r-1}))^n n\big), \quad (2.20)$$

where the constant in the O-term depends on r.

Proof. We use the notation of Proposition 2.3.4. By means of singularity analysis and by considering that $L_r^{=}$ is a rational function, we find that the pole at z_k (for odd k) yields a contribution of (up to simplification)

$$z_k^{-n}\Big(4\tan^2(k\pi 2^{-r-1})(n+1) - \frac{4\sin^2(k\pi 2^{-r-1})+2}{\cos^2(k\pi 2^{-r-1})}\Big) = z_k^{-n}\Big(4\tan^2(k\pi 2^{-r-1})n - \frac{2}{\cos^2(k\pi 2^{-r-1})}\Big)$$

for sufficiently large n. ☐

We turn to the investigation of the expected reduction degree. Let $\mathcal{L}_n$ denote the set of simple two-dimensional lattice paths of size n. Consider the family of random variables $D_n \colon \mathcal{L}_n \to \mathbb{N}_0$ modeling the reduction degree of the lattice paths of length n under the assumption that all paths are equally likely.

Similar to the investigations we have conducted for the random variables in Sections 2.2.2 and 2.2.3, we want to characterize the range of the reduction degree for lattice paths of given length n as well.

Proposition 2.3.6.
Let $n \in \mathbb{N}$. Then the reduction degree for any simple two-dimensional lattice path of length n satisfies

$$[\![n > 1]\!] \leq D_n \leq \lfloor \log_2 n \rfloor,$$

and these bounds are sharp.

Proof. First, observe that for $n = 1$, we only have the atomic steps $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$, and all of them have reduction degree 0, which the lower and upper bound given above agree upon.

For $n > 1$ we find that, e.g., the path $(\rightarrow)^n$ has reduction degree 1. In combination with the fact that there are no paths of length greater than 1 with reduction degree 0, this establishes the lower bound and proves that it is sharp.

In order to prove the upper bound, we consider M to be the maximal reduction degree among all lattice paths of length n, i.e. the corresponding path can be obtained from one of the steps (of length 1) by expanding the path M times.

The shortest possible path after M expansions can be obtained by replacing every step of the path iteratively by a segment of length 2, meaning that the length doubles after every expansion. Thus, a minimally expanded path has length 2^M.

As the minimally expanded path has to be at most equally long as the original path, the inequality $2^M \leq n$ and therefore $M \leq \lfloor \log_2 n \rfloor$ holds, which proves the upper bound.

In order to construct a path of length n with reduction degree equal to $\lfloor \log_2 n \rfloor$, we consider the binary digit expansion $(x_k \ldots x_1 x_0)_2$ of n. Reading this expansion from left to right, starting at x_{k-1}, we construct the path as follows: we start with $\rightarrow$, if the current digit is 0 then we expand the path minimally, and otherwise we expand all but the last step of the path minimally; the last step is expanded by replacing it by a corresponding segment of length 3 (i.e. one additional step is added in contrast to minimal expansion). The digit $x_k = 1$ is not relevant for this construction, thus it is ignored.

It is easy to see that the length of the resulting path is n, as our construction corresponds to the "double-and-add"-strategy used to determine the value of the binary expansion. Furthermore, for each of the digits in $x_{k-1} \ldots x_1 x_0$ we have expanded our path once, which produces a path with reduction degree k. Finally, from the binary expansion it is easy to see that $k = \lfloor \log_2 n \rfloor$ holds, which proves that for all $n \in \mathbb{N}$, the upper bound above is attained for some lattice path of length n. □

The following results are immediate consequences of Proposition 2.3.3.

Corollary 2.3.7.
The probability that a lattice path of length n has reduction degree r is given by the explicit formula

$$\mathbb{P}(D_n = r) = \frac{[z^n] L_r^{=}(z)}{4^n} = 4^{r+1-n} \sum_{\lambda \geq 0} \lambda (-1)^{\lambda - 1} \left[\binom{2n-1}{n - \lambda 2^r} - \binom{2n-1}{n - \lambda 2^r - 1} \right],$$

and the expected reduction degree for paths of length n is given by

$$\mathbb{E} D_n = \sum_{k \geq 1} 8k (2^{v_2(k)} - 1) \left[\binom{2n-1}{n-k} - \binom{2n-1}{n-k-1} \right]. \tag{2.21}$$

Proof. Analogously to our approach in Section 2.2.3, the double sum

$$\mathbb{E} D_n = \sum_{r, \lambda \geq 0} 4^{r+1-n} r (-1)^{\lambda - 1} \lambda \left[\binom{2n-1}{n - \lambda 2^r} - \binom{2n-1}{n - \lambda 2^r - 1} \right]$$

can be simplified by considering

$$\psi(k) := 4 \sum_{\substack{\lambda,r\geq 0\\ \lambda 2^r = k}} 4^r r(-1)^{\lambda-1}\lambda.$$

We find

$$\psi(k) = 4k\left(2^{\nu_2(k)}\nu_2(k) - \sum_{r=0}^{\nu_2(k)-1} r2^r\right) = 8k(2^{\nu_2(k)}-1),$$

which proves (2.21). □

Remark.
The formula for $\mathbb{P}(D_n = r)$ is very similar to the results for the classical register function obtained by Flajolet (cf. [15]). It is likely that applying the techniques that were used in [42] could be used to determine expansions for arbitrary moments.

The following theorem characterizes the asymptotic behavior of the expected reduction degree and the corresponding variance.

Theorem 2.3.8.
The expected reduction degree of simple two-dimensional lattice paths of length n admits the asymptotic expansion

$$\mathbb{E}D_n = \log_4 n + \frac{\gamma + 2 - 3\log 2}{2\log 2} + \delta_1(\log_4 n) + O(n^{-1}), \tag{2.22}$$

and for the corresponding variance we have

$$\begin{aligned}\mathbb{V}D_n = \frac{\pi^2 - 24\log^2\pi - 48\zeta''(0) - 24}{24\log^2 2} - \frac{2\log\pi}{\log 2} - \frac{11}{12} + \delta_2(\log_4 n)\\ - \frac{\gamma + 2 - 3\log 2}{\log 2}\delta_1(\log_4 n) - \delta_1^2(\log_4 n) + O\left(\frac{\log n}{n}\right)\end{aligned} \tag{2.23}$$

where $\delta_1(x)$ and $\delta_2(x)$ are 1-periodic fluctuations of mean zero which are defined as

$$\delta_1(x) = \log 2 \sum_{k\neq 0} c_k e^{2k\pi i x} \tag{2.24}$$

and

$$\delta_2(x) = \sum_{k\neq 0}\left(d_k - c_k\psi\left(1 + \frac{\chi_k}{2}\right)\right)e^{2k\pi i x} \tag{2.25}$$

with $\chi_k = \frac{2\pi i k}{\log 2}$ and constants

$$c_k = \frac{2}{\sqrt{\pi}\log^2 2}\Gamma\left(\frac{3+\chi_k}{2}\right)\zeta(1+\chi_k)$$

and

$$d_k = \frac{4}{\sqrt{\pi}\log^2 2}\Gamma\left(\frac{3+\chi_k}{2}\right)\big(\psi(2+\chi_k)\zeta(1+\chi_k) + \zeta'(1+\chi_k)\big) - 3c_k\log 2.$$

Proof. In order to analyze the expected value $\mathbb{E}D_n$ asymptotically, we study the corresponding generating function $G^{(1)}(z) = \sum_{r\ge 0} rL_r^{=}(z)$, for which we have $\mathbb{E}D_n = \frac{1}{4^n}[z^n]G^{(1)}(z)$, with an approach that is similar to the one in Theorem 2.2.11.

With the substitution $u = e^{-t}$, we find

$$G^{(1)}(z) = \sum_{r\ge 1} r4^{r+1}\frac{u^{2^r}}{(1+u^{2^r})^2} = \sum_{r,\lambda\ge 1} r4^{r+1}(-1)^{\lambda-1}\lambda e^{-t\lambda 2^r},$$

where we used (2.13). Thus, the Mellin transform $g^{(1)}(s) = \mathcal{M}(G^{(1)})(s)$ of $G^{(1)}$ (which is a function in t) is given by

$$\begin{aligned} g^{(1)}(s) &= \sum_{r,\lambda\ge 1} r4^{r+1}(-1)^{\lambda-1}\lambda^{1-s}2^{-rs}\Gamma(s) = 4\Big(\sum_{r\ge 1} r2^{(2-s)r}\Big)\Big(\sum_{\lambda\ge 1}(-1)^{\lambda-1}\lambda^{1-s}\Big)\Gamma(s) \\ &= 4\frac{2^{2-s}}{(1-2^{2-s})^2}(1-2^{2-s})\zeta(s-1)\Gamma(s) = 4\Gamma(s)\zeta(s-1)\frac{2^{2-s}}{1-2^{2-s}}, \end{aligned}$$

which is analytic for $\operatorname{Re}(s) > 2$. Observe that $g^{(1)}(s)$ has a pole of order two at $s = 2$, simple poles at $s = 2+\chi_k$ for $k \in \mathbb{Z}\setminus\{0\}$ and further simple poles at $s \in -2\mathbb{N}_0$.

As the fundamental strip of $g^{(1)}(s)$ is given by $\langle 2, \infty\rangle$, the Mellin inversion formula yields

$$G^{(1)}(z) = \frac{1}{2\pi i}\int_{5-i\infty}^{5+i\infty} g^{(1)}(s)t^{-s}\, ds,$$

and we compute this integral by shifting the line of integration to $\operatorname{Re}(s) = -3$.

Note that analogously to the argumentation in the proof of Theorem 2.2.11, the Mellin inversion formula above is also valid for complex z in a punctured neighborhood of $1/4$ where $|\arg(4z-1)| > 2\pi/5$, which allows us to apply singularity analysis.

We compute the contributions of the singularities with the help of SageMath [59]. With an analogous estimation as in the proof of Theorem 2.2.11 we find that the integral (after the shift) contributes an error of $O(|t|^3)$. Again, for the sake of simplicity we take the contribution of the residue at -2 as the error term:

$$\operatorname{Res}_{s=-2}(g^{(1)}(s)t^{-s}) = O(t^2).$$

Thus, with $P = \{-2, 0, 2\} \cup \{\chi_k \mid k \in \mathbb{Z}\setminus\{0\}\}$ we find

$$\begin{aligned} \sum_{p\in P}\operatorname{Res}_{s=p}(g^{(1)}(s)t^{-s}) = -\frac{4}{\log 2}t^{-2}\log t + \Big(\frac{4}{\log 2} - 2\Big)t^{-2} \\ + \frac{4}{9} + \frac{4}{\log 2}\sum_{k\ne 0}\Gamma(2+\chi_k)\zeta(1+\chi_k)t^{-2-\chi_k} + O(t^2). \end{aligned}$$

Substituting back, controlling the error analogously to (2.17), applying singularity analysis, normalizing by 4^n, and rewriting the coefficients of the terms of growth $n^{\chi_k/2}$ with the duplication formula for the Gamma function (cf. [10, 5.5.5]) then proves (2.22) and (2.24).

For the analysis of the variance we turn our attention to the second moments, $\mathbb{E}D_n^2$. The related generating function is given by

$$G^{(2)}(z) = \sum_{r\geq 0} r^2 L_r^{=}(z) = \sum_{r,\lambda\geq 0} r^2 4^{r+1}(-1)^{\lambda-1}\lambda e^{-t\lambda 2^r}.$$

It is easy to check that the corresponding Mellin transform $g^{(2)}(s)$ is

$$g^{(2)}(s) = 4\Gamma(s)\zeta(s-1)\frac{(1+2^{2-s})2^{2-s}}{(1-2^{2-s})^2},$$

with a pole of order 3 at $s=2$, and poles of order two at $s=2+\chi_k$ for $k\in\mathbb{Z}\setminus\{0\}$, as well as simple poles at $s\in -2\mathbb{N}_0$. Analogously to above, the inversion formula is also valid for complex z in a punctured neighborhood around $1/4$ with $|\arg(1-4z)|>2\pi/5$. We shift the line of integration to $\operatorname{Re}(s)=-3$, which yields an error term of

$$\frac{1}{2\pi i}\int_{-3-i\infty}^{-3+i\infty} g^{(2)}(s)t^{-s}\,ds = O(|t|^3),$$

and collect residues.

We find that $\operatorname{Res}_{s=0}(g^{(2)}(s)t^{-s})$ does not yield a contribution in terms of z. The pole at $s=-2$ is the leftmost pole we shift the line of integration over. For the sake of simplicity, we use the contribution of the residue at this pole as the error term, which we find to be

$$\operatorname{Res}_{s=-2}(g^{(2)}(s)t^{-s}) = O(|t|^2).$$

Furthermore, the pole at $s=2$ yields a residue of

$$\begin{aligned}\operatorname{Res}_{s=2}(g^{(2)}(s)t^{-s}) = \frac{4}{\log^2 2}t^{-2}\log^2 t + \Big(\frac{4}{\log 2} - \frac{8}{\log^2 2}\Big)t^{-2}\log t \\ + \Big(\frac{\pi^2 - 12\log^2\pi - 24\zeta''(0)}{3\log^2 2} - \frac{8\log\pi + 4}{\log 2} - \frac{8}{3}\Big)t^{-2}\end{aligned}$$

which translates into a local expansion of

$$\begin{aligned}\frac{1}{4\log^2 2}\frac{\log^2(1-4z)}{1-4z} + \Big(\frac{3}{2\log 2} - \frac{1}{\log^2 2}\Big)\frac{\log(1-4z)}{1-4z} \\ + \Big(\frac{\pi^2 - 12\log^2\pi - 24\zeta''(0)}{12\log^2 2} - \frac{2\log\pi + 3}{\log 2} + \frac{4}{3}\Big)\frac{1}{1-4z} \\ + O(\log^2(1-4z)),\end{aligned}$$

and, after applying singularity analysis and dividing by 4^n, into an asymptotic contribution of

$$\log_4^2 n + \frac{\gamma + 2 - 3\log 2}{\log 2}\log_4 n + \frac{6\gamma^2 + \pi^2 - 24\log^2\pi + 24\gamma - 48\zeta''(0)}{24\log^2 2} - \frac{3\gamma + 4\log\pi + 6}{2\log 2} + \frac{4}{3} + O(n^{-1}\log n).$$

Observe that the logarithmic terms in this expansion cancel against the square of the expansion for $\mathbb{E}D_n$ as given in (2.22)—which is a common phenomenon.

Next we determine the contribution of the remaining poles. Locally expanding the sum of the corresponding residues in terms of $z \to 1/4$ and controlling the resulting error analogously to (2.17) yields

$$\sum_{k\neq 0}\operatorname{Res}_{s=2+\chi_k}(g^{(1)}(s)t^{-s}) = \sum_{k\neq 0}\Big(-\log(1-4z)(1-4z)^{-1-\chi_k/2}\Gamma(1+\chi_k/2)c_k + (1-4z)^{-1-\chi_k/2}\Gamma(1+\chi_k/2)d_k\Big) + O(\log(1-4z))$$

where the coefficients c_k and d_k are defined as in the theorem. Note that all estimates still work out as the product of the Gamma and the Digamma function decays exponentially and the derivative of the zeta function grows at most polynomially, which is easy to see by considering the derivative by means of Cauchy's integral formula.

Singularity analysis and dividing by 4^n then yields an asymptotic contribution of

$$2\delta_1(\log_4 n)\log_4 n + \delta_2(\log_4 n)$$

with δ_1 and δ_2 from (2.24) and (2.25), respectively.

Putting everything together we find

$$\mathbb{E}D_n^2 = \log_4^2 n + \frac{\gamma + 2 - 3\log 2}{\log 2}\log_4 n + 2\delta_1(\log_4 n)\log_4 n + \frac{6\gamma^2 + \pi^2 - 24\log^2\pi + 24\gamma - 48\zeta''(0)}{24\log^2 2} - \frac{3\gamma + 4\log\pi + 6}{2\log 2} + \frac{4}{3} + \delta_2(\log_4 n) + O\Big(\frac{\log n}{n}\Big).$$

With this result, we are able to find an expansion of the variance $\mathbb{V}D_n$ by considering the difference $\mathbb{E}D_n^2 - (\mathbb{E}D_n)^2$, which yields the expansion given in (2.23). □

2.3.2 Fringes

We define the rth *fringe* of a given lattice path ℓ of length ≥ 1 to be $\rho_L^r(\ell)$, i.e. the rth fringe is given by the rth reduction of the path. In particular, if ℓ can be reduced r times, we call the length of $\rho_L^r(\ell)$ the size of the rth fringe. Otherwise, we say that this size is 0. We model the size of the rth fringe with the random variable $X_{n;r}^L : \mathcal{L}_n \to \mathbb{N}_0$.

The rth fringes of positive size can then be enumerated by the bivariate generating function

$$H_r(z,v) = \sum_{\substack{\ell \text{ path} \\ \operatorname{rdeg}(\ell) \geq r}} v^{|\rho_L^r(\ell)|} z^{|\ell|}$$

where $|\ell|$ denotes the length of a lattice path.

Deriving a recursion for these generating functions is pretty straightforward: first, observe that for $r = 0$, the exponent of v always coincides with the exponent of z as $\rho_L^0(\ell) = \ell$ for all lattice paths ℓ of length ≥ 1. Thus

$$H_0(z,v) = L(zv) = \frac{4zv}{1-4zv},$$

where $L(z)$ is the generating function counting all paths of length ≥ 1.

The recursion itself follows from the fact that rth fringes of a path ℓ are $(r-1)$th fringes of its reduction $\rho_L(\ell)$. Thus, by the same argument that was used in the proof of Proposition 2.3.1, we have

$$H_r(z,v) = 4H_{r-1}\left(\left(\frac{z}{1-2z}\right)^2, v\right). \tag{2.26}$$

In this recursion the second parameter, v, does not change. This justifies the application of Proposition 2.2.4 in order to rewrite $H_r(z,v)$ by means of the substitution $z = Z(u)$. We obtain

$$H_r(z,v) = \frac{4^{r+1}\frac{u^{2^r}}{(1+u^{2^r})^2}v}{1-\frac{4u^{2^r}}{(1+u^{2^r})^2}v} = \frac{4^{r+1}u^{2^r}v}{(1+u^{2^r})^2 - 4u^{2^r}v}.$$

The generating function $H_r(z,v)$ can now be used to derive the asymptotic behavior of the expectation $\mathbb{E}X_{n;r}^L$ and the variance $\mathbb{V}X_{n;r}^L$ of the size of the rth fringe, where all paths of length n arise with the same probability.

The first few of those generating functions are

$$\begin{aligned}
H_0(z,v) &= 4vz + 16v^2z^2 + 64v^3z^3 + 256v^4z^4 + 1024v^5z^5 + 4096v^6z^6 \\
&\quad + 16384v^7z^7 + 65536v^8z^8 + 262144v^9z^9 + O(z^{10}), \\
H_1(z,v) &= 16vz^2 + 64vz^3 + (64v^2 + 192v)z^4 + (512v^2 + 512v)z^5 \\
&\quad + (256v^3 + 2560v^2 + 1280v)z^6 + (3072v^3 + 10240v^2 + 3072v)z^7 \\
&\quad + (1024v^4 + 21504v^3 + 35840v^2 + 7168v)z^8 \\
&\quad + (16384v^4 + 114688v^3 + 114688v^2 + 16384v)z^9 + O(z^{10}), \\
H_2(z,v) &= 64vz^4 + 512vz^5 + 2816vz^6 + 13312vz^7 + (256v^2 + 58112v)z^8 \\
&\quad + (4096v^2 + 241664v)z^9 + O(z^{10}), \\
H_3(z,v) &= 256vz^8 + 4096vz^9 + O(z^{10}).
\end{aligned}$$

In order to get a better understanding of the behavior of fringe sizes, we investigate the minimum and maximum value of the random variable $X_{n;r}^L$ modeling the size of the rth fringe of a random lattice path of length n.

Proposition 2.3.9.
Let n, $r \in \mathbb{N}_0$. If $r = 0$, then $X_{n;0}^L$ is a deterministic quantity with $X_{n;0}^L = n$. For $r > 0$, the bound

$$[\![n > 0 \text{ and } r = 1]\!] \le X_{n;r}^L \le \left\lfloor \frac{n}{2^r} \right\rfloor$$

holds and is sharp.

Proof. The proof follows the same idea as the proof of Proposition 2.2.5. In particular, the concept of "minimal expansion" of binary trees corresponds to expanding single steps into segments of length 2. Furthermore, an appropriate family of lattice paths can be constructed from the steps $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$ by iteratively expanding the path either minimally, or expanding the first step into a segment of length 3 and the rest minimally. □

Theorem 2.3.10.
Let $r \in \mathbb{N}_0$ be fixed. The expectation and variance of the size of the rth fringe of a random path of length n have the asymptotic expansions

$$\mathbb{E}X_{n;r}^L = \frac{n}{4^r} + \frac{1-4^{-r}}{3} + O(n^3\theta_r^{-n}) \tag{2.27}$$

and

$$\mathbb{V}X_{n;r}^L = \frac{4^r-1}{3\cdot 16^r}n + \frac{-2\cdot 16^r - 5\cdot 4^r + 7}{45\cdot 16^r} + O(n^5\theta_r^{-n}), \tag{2.28}$$

where $\theta_r = \frac{4}{2+2\cos(2\pi/2^r)} > 1$. If additionally $r > 0$, then for the random variables $X_{n;r}^L$ modeling the rth fringe size of lattice paths of length n we have

$$\mathbb{P}\Bigl(\frac{X_{n;r}^L - \mathbb{E}X_{n;r}^L}{\sqrt{\mathbb{V}X_{n;r}^L}} \le x\Bigr) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-w^2/2}\,dw + O(n^{-1/2}),$$

i.e. the random variables $X_{n;r}^L$ are asymptotically normally distributed.

Proof. The generating function $H_r(z, v)$ only sums over all lattice paths with reduction degree $\ge r$. In a first step we show that the number of excluded paths is exponentially small when compared with the number of all paths.

In order to do so, we consider the generating function

$$H_r(z, 1) = \frac{4^{r+1}u^{2^r}}{(1-u^{2^r})^2}.$$

From (2.26) we know that $H_r(z, 1)$ is a meromorphic function, i.e. all its singularities are poles and no square-root singularities can occur. In the u-world the singularities can be expressed as the 2^rth roots of unity. From the proof of Proposition 2.2.3 we also know that on the unit circle,

$$Z(u) = \frac{1}{2+2\operatorname{Re}u}$$

holds, such that with Property (e) of Proposition 2.2.3 we know that the dominant singularity (in terms of $z = Z(u)$) is a simple pole at $z = Z(1) = 1/4$, and the next singularity is a pole of order two at

$$\frac{\theta_r}{4} := Z(e^{-2\pi i/2^r}) = \frac{1}{2+2\cos(2\pi/2^r)},$$

which translates into a contribution of $O(n4^n\theta_r^{-n})$.

Together with the local expansion

$$H_r(z,1) = \frac{1}{1-4z} + O(1)$$

for $z \to 1/4$, and with the fact that $H_r(z,1)$ is meromorphic, we find that

$$[z^n]H_r(z,1) = 4^n + O(n(2+2\cos(2\pi/2^r))^n), \tag{2.29}$$

as claimed. For determining the moments, fringes of size 0 do not yield a contribution (as they are weighted with 0), such that we can use the generating function $H_r(z,v)$.

It is easy to see that $\mathbb{E}X_{n;r}^L$ can be obtained by dividing the coefficient of z^n in $\bigl(\frac{\partial H_r}{\partial v}(z,v)\bigr)\big|_{v=1}$ by the normalization factor. In particular, we find

$$\left(\frac{\partial H_r}{\partial v}(z,v)\right)\bigg|_{v=1} = 4^{r+1}u^{2^r}\frac{(1+u^{2^r})^2}{(1-u^{2^r})^4} = \frac{4^{-r}}{(1-4z)^2} + \frac{1-4^{1-r}}{3(1-4z)} + O(1).$$

Applying singularity analysis to this meromorphic function and dividing by 4^n yields (2.27).

For the variance we compute asymptotic expansions for the second moment by considering the generating function

$$\left(\frac{\partial^2 H_r}{\partial v^2}(z,v) + \frac{\partial H_r}{\partial v}(z,v)\right)\bigg|_{v=1} = \frac{2\cdot 16^{-r}}{(1-4z)^3} + \frac{4^{-r}-4\cdot 16^{-r}}{(1-4z)^2} + \frac{1-20\cdot 4^{-r}+34\cdot 16^{-r}}{15(1-4z)} + O(1).$$

In this case, singularity analysis and normalization leads to a contribution of

$$\frac{n^2}{16^r} + \frac{4^r-1}{16^r}n + \frac{16^r-5\cdot 4^r+4}{15\cdot 16^r} + O(n^5\theta_r^{-n})$$

for the second moment. Subtracting $(\mathbb{E}X_{n;r}^L)^2$ from this expansion results in (2.28).

For the limiting distribution we restrict our model to lattice paths admitting r reductions and study the corresponding random variable $\tilde{X}_{n;r}^L$. By (2.29) this induces an exponentially small error in the sense that

$$\mathbb{P}(\tilde{X}_{n;r}^L \in A) = \mathbb{P}(X_{n;r}^L \in A)\bigl(1 + O(\theta_r^{-n})\bigr)$$

for all $A \subseteq \mathbb{N}_0$.

By singularity perturbation of meromorphic functions (cf. [21, Theorem IX.9]) we immediately find that $\tilde{X}_{n;r}^L$ is asymptotically normally distributed—and as a direct consequence of the exponentially small error observed above, $X_{n;r}^L$ is asymptotically normally distributed as well. □

As we have the generating function $H_r(z,v)$ in an explicit form, the expected value can also be extracted explicitly by means of Cauchy's integral formula.

Proposition 2.3.11.
For given $r \in \mathbb{N}_0$, the expected size of the rth fringe $\mathbb{E}X_{n;r}^L$ of a random path of length n is given by the explicit formula

$$\mathbb{E}X_{n;r}^L = 4^{r+1-n} \sum_{\lambda \geq 1} \frac{2\lambda^3 + \lambda}{3} \left[\binom{2n-1}{n - 2^r\lambda} - \binom{2n-1}{n-2^r\lambda - 1} \right].$$

Proof. Applying Cauchy's integral formula to

$$\mathbb{E}X_{n;r}^L = 4^{-n}[z^n] \left(\frac{\partial H_r}{\partial v}(z,v) \right) \bigg|_{v=1}$$

yields

$$\begin{aligned}
\mathbb{E}X_{n;r}^L &= 4^{-n}[z^n] 4^{r+1} u^{2^r} \frac{(1+u^{2^r})^2}{(1-u^{2^r})^4} = \frac{4^{r+1-n}}{2\pi i} \oint u^{2^r} \frac{(1+u^{2^r})^2}{(1-u^{2^r})^4} \frac{dz}{z^{n+1}} \\
&= \frac{4^{r+1-n}}{2\pi i} \oint u^{2^r} \frac{(1+u^{2^r})^2}{(1-u^{2^r})^4} \frac{(1-u)(1+u)^{2n+2}}{(1+u)^3} \frac{du}{u^{n+1}} \\
&= 4^{r+1-n} [u^n](1-u)(1+u)^{2n-1} u^{2^r} \frac{(1+u^{2^r})^2}{(1-u^{2^r})^4} \\
&= 4^{r+1-n} \sum_{\lambda \geq 1} \frac{2\lambda^3 + \lambda}{3} [u^n](1-u)(1+u)^{2n-1} u^{2^r\lambda},
\end{aligned}$$

where the last step is justified by

$$x \frac{(1+x)^2}{(1-x)^4} = \sum_{\lambda \geq 1} \frac{2\lambda^3 + \lambda}{3} x^\lambda. \tag{2.30}$$

The statement of the theorem then follows by extracting the coefficients by means of the binomial theorem. □

Analogously to our investigations concerning branches in binary trees, we also study the behavior of the overall fringe size, i.e. the sum over the size of the rth fringes for $r \geq 0$. Like the reduction degree, this parameter can also be interpreted as a complexity measure for lattice paths. We will model this quantity with the random variable $X_n^L := \sum_{r \geq 0} X_{n;r}^L$.

A first observation regarding the behavior of X_n^L can be followed directly from Proposition 2.3.9.

Proposition 2.3.12.
Let $n \in \mathbb{N}_0$ and let $w_2(n)$ denote the binary weight, i.e. the number of non-zero digits in the binary expansion of n. Then the bound

$$n + [\![n > 1]\!] \leq X_n^L \leq 2n - w_2(n) \leq 2n - 1$$

holds and is sharp.

Proof. Analogously to the proof of Proposition 2.2.9. □

Furthermore, summing the explicit expressions for $\mathbb{E}X_{n;r}^L$ obtained in Proposition 2.3.11 yields an explicit formula for $\mathbb{E}X_n^L$, the expected fringe size for lattice paths of length n.

Corollary 2.3.13.
The expected fringe size $\mathbb{E}X_n^L$ of a random path of length n can be computed as

$$\mathbb{E}X_n^L = \frac{1}{12 \cdot 4^n} \sum_{k=1}^{n} \big(2k^3(2 - 2^{-\nu_2(k)}) + k(2^{\nu_2(k)+1} - 1)\big)\left[\binom{2n-1}{n-k} - \binom{2n-1}{n-k-1}\right].$$

Proof. Analogously to the proof of Corollary 2.2.10. □

The following theorem quantifies the asymptotic behavior of $\mathbb{E}X_n^L$.

Theorem 2.3.14.
Asymptotically, the behavior of the expected fringe size $\mathbb{E}X_n^L$ for a random path of length n is given by

$$\mathbb{E}X_n^L = \frac{4}{3}n + \frac{1}{3}\log_4 n + \frac{5 + 3\gamma - 11\log 2}{18 \log 2} + \delta(\log_4 n) + O\Big(\frac{\log n}{n}\Big), \tag{2.31}$$

where $\delta(x)$ is a 1-periodic fluctuation of mean zero with Fourier series expansion

$$\delta(x) = \frac{2}{3\sqrt{\pi}\log 2} \sum_{k \neq 0} \Gamma\Big(\frac{3 + \chi_k}{2}\Big)\big(2\zeta(\chi_k - 1) + \zeta(\chi_k + 1)\big)e^{2k\pi i x}.$$

Proof. We follow the strategy from the proof of Theorem 2.2.11. First of all, observe that with the substitution $u = e^{-t}$,

$$H^{(1)}(z) := \sum_{r \geq 0} \Big(\frac{\partial H_r}{\partial v}(z, v)\Big)\Big|_{v=1} = \sum_{r \geq 0} 4^{r+1} u^{2^r} \frac{(1 + u^{2^r})^2}{(1 - u^{2^r})^4} = \frac{4}{3} \sum_{\substack{r \geq 0,\\ \lambda \geq 1}} 4^r (2\lambda^3 + \lambda) e^{-t\lambda 2^r},$$

where we used (2.30). The Mellin transform $h^{(1)}(s) := \mathcal{M}(H^{(1)})(s)$ is then easily determined, we find

$$\begin{aligned} h^{(1)}(s) &= \frac{4}{3} \sum_{\substack{r \geq 0,\\ \lambda \geq 1}} 4^r (2\lambda^3 + \lambda)\lambda^{-s} 2^{-rs}\Gamma(s) = \frac{4}{3}\Gamma(s) \sum_{r \geq 0} (2^{2-s})^r \sum_{\lambda \geq 1} (2\lambda^{3-s} + \lambda^{1-s}) \\ &= \frac{4}{3}\Gamma(s) \frac{2\zeta(s-3) + \zeta(s-1)}{1 - 2^{2-s}}, \end{aligned}$$

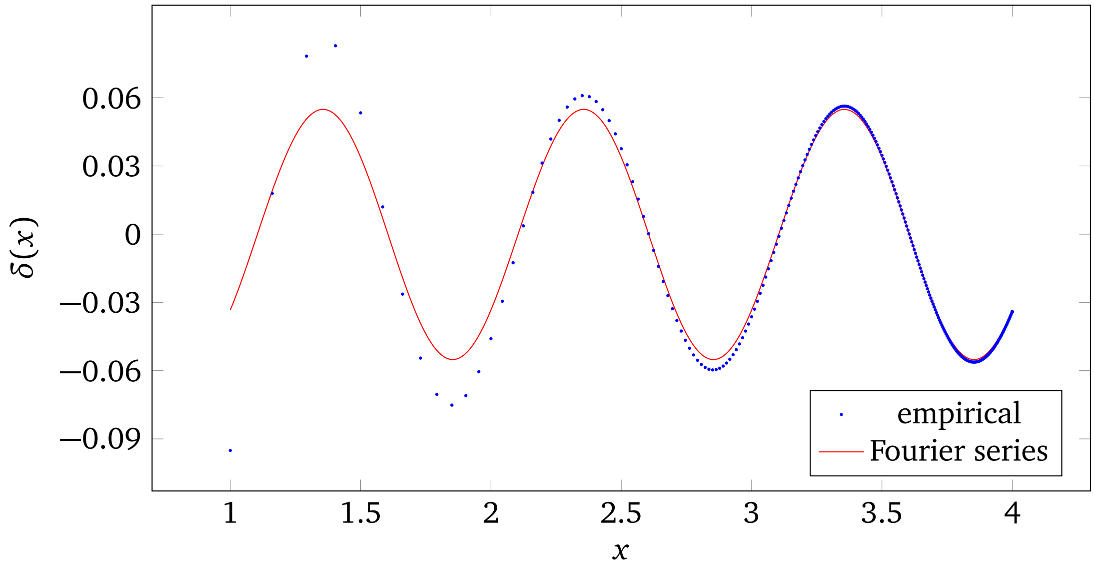


Figure 2.6: Partial Fourier series (20 summands) compared with the empirical values of the function δ from Theorem 2.3.14.

a function that is analytic for $\operatorname{Re}(s) > 4$. Thus, by the Mellin inversion formula, we have

$$H^{(1)}(z) = \frac{1}{2\pi i}\int_{5-i\infty}^{5+i\infty} h^{(1)}(s)t^{-s}\,ds,$$

again valid in a punctured complex neighborhood of $1/4$ with $|\arg(4z-1)| > 2\pi/5$.

With an analogous justification as in the proof of Theorem 2.2.11 and the proof of Theorem 2.3.8 we shift the line of integration to the line $\operatorname{Re}(s) = -3$ and take the contribution from the residue at -2 as the error term (which gives an expansion error of $O(1-4z)$). Analogously to the previous theorems, the remaining integral is absorbed by this error term.

By shifting the line of integration we cross a simple pole at $s = 4$, a pole of order two at $s = 2$, infinitely many simple poles at $s = 2 + \chi_k$, $k \in \mathbb{Z} \setminus \{0\}$, and a simple pole in $s = 0$.

With $P = \{-2, 0, 2, 4\} \cup \{2 + \chi_k \mid k \in \mathbb{Z} \setminus \{0\}\}$ we find that

$$\begin{aligned}\sum_{p\in P} \operatorname{Res}_{s=p}(h^{(1)}(s)t^{-s}) = \frac{64}{3}t^{-4} + \left(\frac{2}{3} + \frac{10}{9\log 2}\right)t^{-2} - \frac{4\log t}{3\log 2}t^{-2} + \frac{4}{135} \\ + \frac{4}{3\log 2}\sum_{k\neq 0}\Gamma(2+\chi_k)(2\zeta(-1+\chi_k) + \zeta(1+\chi_k))t^{-2-\chi_k} + O(t^2).\end{aligned}$$

Local expansion in terms of $z \to 1/4$ and controlling the error of the translation of the residues along the vertical line $\operatorname{Re} s = 2$ analogously to (2.17) then results in

$$\frac{4}{3(1-4z)^2} - \frac{\log(1-4z)}{6(1-4z)\log 2} + \left(\frac{5}{18\log 2} - \frac{35}{18}\right)\frac{1}{1-4z} \\ + \frac{1}{3\log 2}\sum_{k\neq 0}\Gamma(2+\chi_k)(2\zeta(-1+\chi_k)+\zeta(1+\chi_k))(1-4z)^{-1-\chi_k/2} + O(\log(1-4z)),$$

from which the statement of the theorem follows after extracting the coefficient growth of this expansion by means of singularity analysis, dividing by 4^n, and applying the duplication formula for the Gamma function. □

Acknowledgment We thank Michael Fuchs for a hint regarding the central limit theorem for additive tree parameters.

3

Fringe Analysis of Plane Trees Related to Cutting and Pruning

Rooted plane trees are reduced by four different operations on the fringe. The number of surviving nodes after reducing the tree repeatedly for a fixed number of times is asymptotically analyzed. The four different operations include cutting all or only the leftmost leaves or maximal paths. This generalizes the concept of pruning a tree.

The results include exact expressions and asymptotic expansions for the expected value and the variance as well as central limit theorems.

This chapter is an adapted version of [24], which is joint work with Clemens Heuberger, Sara Kropf, and Helmut Prodinger.

3.1 Introduction

Plane trees are among the most interesting elementary combinatorial objects; they appear in the literature under many different names such as ordered trees, planar trees, planted plane trees, etc. They have been analyzed under various aspects, especially due to their relevance in Computer Science. Two particularly well-known quantities are the height, since it is equivalent to the stack size needed to explore binary (expression) trees, and the pruning number (pruning index), since it is equivalent to the register function (Horton-Strahler number) of binary trees. Several results for the height of plane trees can be found in [8, 17, 50], for the register function, we refer to [9, 20, 37], and for results on the connection between the register function and the pruning number to [9, 72].

As pointed out in the introduction in Chapter 1, in this chapter we investigate fully deterministic tree reductions (as opposed to removing edges according to some probabilistic model, see [32, 44, 49]) acting on the fringe of the tree. This means that only (a subset of) leaves (and som adjacent structures) are removed. To be more precise, we consider four different models:

- In one round, all leaves together with the corresponding edges are removed (see Section 3.2).
- In one round, all maximal paths (linear graphs), with the leaves on one end, are removed (see Section 3.3). This process is called pruning.
- A leaf is called an old leaf if it is the leftmost sibling of its parents. This concept was introduced in [6]. In one round, only old leaves are removed (see Section 3.4).
- The last model deals with pruning old paths. There might be several interesting models related to this; the one we have chosen here is that in one round maximal paths are removed, under the condition that each of their nodes is the leftmost child of their parent node (see Section 3.5).

The four tree reductions are illustrated in Figure 3.1. We describe these reductions more formally in the corresponding sections.

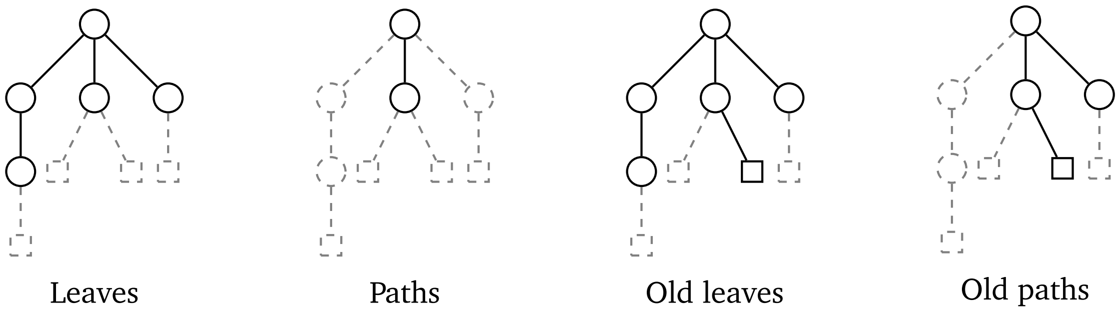


Figure 3.1: Removal of (old) leaves / paths.

The first model is clearly related to the height of the plane tree, and the second one to the Horton-Strahler number via the pruning index [72, 66]. While there are no surprises about the number of rounds that the process takes here, we are interested in how the fringe develops. The number of leaves and nodes altogether in the remaining tree after a fixed number of reduction rounds is the main parameter analyzed in this chapter.

For the sake of simplicity, we will use the same notation for each of the following reduction analyses. In case we need to compare objects from two different sections, we will distinguish them by adding appropriate superscripts.

The random variable $X_{n,r}$ models the tree size after reducing a plane tree of size n (that is chosen uniformly at random among all trees with n nodes) r-times iteratively according to one of our four reductions. If a tree does not "survive" r rounds of reductions, we consider the size of the resulting tree to be 0. In particular, for $r = 0$, the given plane tree is not changed and $X_{n,0} = n$.

As we will see, a key aspect of the analysis of $X_{n,r}$ is the translation of the algorithmic description of the reduction into an operator Φ that acts on the corresponding generating functions.

In Section 3.2, the reduction cutting away all leaves from the tree is discussed. Section 3.2.1 contains all necessary auxiliary concepts required in order to study the r-fold application of this reduction. In Section 3.2.2, we determine the operator Φ acting on the corresponding generating function explicitly and prove some direct consequences. Then, in Section 3.2.3 we carry out the analysis of the behavior of $X_{n,r}$ by computing explicit expressions and asymptotic expansions for the factorial moments of $X_{n,r}$ as well as a central limit theorem.

Section 3.3 is devoted to the study of the reduction that cuts away all paths. As we will see in Section 3.3.1, we can actually obtain all results regarding the behavior of $X_{n,r}$ as consequences of the corresponding results in Section 3.2. In Section 3.3.2, we analyze the asymptotic behavior of the expected number of paths required to construct a plane tree of size n, i.e. the number of paths we can cut away until the tree cannot be reduced any further.

Sections 3.4 and 3.5 are devoted to the analysis of reductions removing only leftmost leaves and leftmost paths from the tree, respectively. In particular, in Section 3.5.3, we study the total number of old paths that can be removed from a tree until it cannot be reduced any further.

The following supplementary SageMath [59] worksheets are available:

- `treereductions.ipynb` for most of the asymptotic computations in Sections 3.2, 3.3, and 3.4,
- `old_paths.ipynb` for most of the asymptotic computations in Section 3.5,
- `factorial_moments_leaves.ipynb` for computation of the factorial moments in Theorem 3.2.13,
- `factorial_moments_old_paths.ipynb` for computation of the factorial moments in Theorem 3.5.6.

Additionally, in order to run these computations yourself, you also need to download the following two utility files:

- `identities_common.py`,
- `conditional_substitution.py`.

All these files including some instructions on how to use them can be found at `https://benjamin-hackl.at/publications/treereductions/`.

3.2 Cutting Leaves

3.2.1 Preliminaries

In this section we investigate the effect of the tree reduction that cuts away all leaves from a given tree. However, before we can do so, we require some auxiliary concepts, which we discuss in this section. Most importantly, we need a generating function counting plane trees with respect to their number of inner nodes and leaves, which is intimately linked to Narayana numbers. The generating function presented in the following proposition is actually well-known (see, e.g. [21, Example III.13]).

Proposition 3.2.1.
The generating function $T(z,t)$ which enumerates plane trees with respect to their internal nodes (marked by the variable z) and leaves (marked by t) is given explicitly by

$$T(z,t) = \frac{1-(z-t)-\sqrt{1-2(z+t)+(z-t)^2}}{2}. \tag{3.1}$$

Proof. This can be obtained directly from the symbolic equation describing the combinatorial class of plane trees $\mathcal{T}$, which is illustrated in Figure 3.2. In particular, □ and ● represent leaves and internal nodes, respectively.

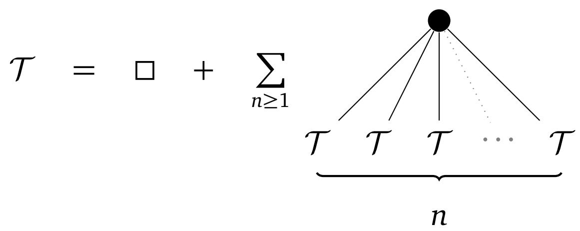


Figure 3.2: Symbolic equation for plane trees.

The symbolic equation translates into the functional equation

$$T(z,t) = t + \frac{zT(z,t)}{1-T(z,t)},$$

which yields (3.1) after solving for $T(z,t)$ and choosing the appropriate branch. □

In the context of plane trees, the so-called *Narayana numbers* count the number of trees with a given size and a given number of leaves (cf. [12]). As these numbers will appear throughout the entire chapter, we introduce them formally and investigate some properties within the following statements.

Definition 3.2.2.
The *Narayana numbers* are defined as

$$N_{n,k} = \frac{1}{n}\binom{n}{k-1}\binom{n}{k}$$

for $1 \leq n$ and $1 \leq k \leq n$, and $N_{0,0} = 1$. All other indices give $N_{n,k} = 0$. Combinatorially, for $n \geq 1$ the Narayana number $N_{n,k}$ corresponds to the number of plane trees with n edges (i.e. $n+1$ nodes) and k leaves. The *Narayana polynomials* are defined as

$$N_n(x) = \sum_{k=1}^{n} N_{n,k} x^{k-1}$$

for $n \geq 1$ and $N_0(x) = 1$, and the *associated Narayana polynomials* are defined as

$$\tilde{N}_n(x) = x \cdot N_n(x)$$

for $n \geq 0$. Note that

$$N_n(1) = \tilde{N}_n(1) = C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the nth Catalan number.

Remark.
The generating function $\frac{1}{z}T(z,z) = \frac{1-\sqrt{1-4z}}{2z}$ enumerates Catalan numbers, see [11, Theorem 3.2], and the generating function $T(z,tz)$ enumerates Narayana numbers

$$T(z,tz) = zt + \sum_{n\geq 2}\sum_{k=1}^{n-1} N_{n-1,k} z^n t^k = \sum_{n\geq 1} z^n \tilde{N}_{n-1}(t). \tag{3.2}$$

We will frequently use this relation in the form

$$T(z,t) = \sum_{n\geq 1} z^n \tilde{N}_{n-1}\left(\frac{t}{z}\right). \tag{3.3}$$

Furthermore [28], it is easily checked that $T(z,tz)$ satisfies the ordinary differential equation

$$(1-2(t+1)z+(1-t)^2z^2)\frac{\partial}{\partial z}T(z,tz) - ((1-t)^2z - t - 1)T(z,tz) = t(1+z-tz).$$

Extracting the coefficient of z^{n+2} then yields the recurrence relation

$$(n+3)\tilde{N}_{n+2}(t) - (2n+3)(t+1)\tilde{N}_{n+1}(t) + n(t-1)^2\tilde{N}_n(t) = 0 \tag{3.4}$$

for $n \geq 0$.

The following proposition gives another useful property of associated Narayana polynomials.

Proposition 3.2.3.
Let $n \geq 0$, then we have the relation

$$t^{n+1}\tilde{N}_n\left(\frac{1}{t}\right) = (1-t)[\![n=0]\!] + \tilde{N}_n(t). \tag{3.5}$$

Proof. This relation follows from extracting the coefficient of z^{n+1} from the identity $T(tz,z) = T(z,tz) + (1-t)z$ with the help of (3.3).

While it is straightforward to prove that the identity is valid by means of algebraic manipulation, we also give a combinatorial proof.

From a combinatorial point of view, both generating functions $T(tz,z)$ and $T(z,tz)$ enumerate plane trees where z marks the tree size, the only difference is that the variable t enumerates inner nodes in $T(tz,z)$ and leaves in $T(z,tz)$. We want to show that for trees of size $n \geq 2$, these two classes are equal, resulting in $T(tz,z) - z = T(z,tz) - tz$.

To construct an appropriate bijection between the class of trees of size n with k leaves and the class of trees of size n with k inner nodes we need to have a closer look at the well-known *rotation correspondence* [21, I.5.3], which is a bijection between plane trees of size n and binary trees with $n-1$ inner nodes. In fact, the leaves in the binary tree are strongly related to the leaves and inner nodes of the original tree:

– Left leaves in the binary tree are only attached to those nodes whose companions in the plane tree have no children, i.e., to those who correspond to leaves in the plane tree.

– Right leaves, on the other hand, are attached to nodes whose companion nodes in the plane tree have no sibling right of them. This means that for every node with children, i.e., for every inner node, there is precisely one rightmost child and thus precisely one right leaf in the binary tree.

The bijection between the two tree classes can now be described as follows: given some tree of size n and k leaves, apply the rotation correspondence in order to obtain a binary tree. Then mirror the binary tree by swapping all left and right children. Transform this mirrored tree back by means of the inverse rotation correspondence, and the result is a plane tree of size n and k inner nodes as mirroring the binary tree swapped the number of left and right leaves in the tree. This proves the proposition. □

Derivatives of the associated Narayana polynomials defined above will occur within the analysis of a reduction model later, which is why we compute some special values in the following proposition.

Proposition 3.2.4.
Evaluating the rth derivative of the associated Narayana polynomials at 1, i.e. $\tilde{N}_n^{(r)}(1)$, gives the number of trees with $n+1$ nodes where precisely r leaves are selected and labeled from 1 to r. In particular, for $n \geq 1$ we have

$$\tilde{N}_n'(1) = \frac{1}{2}\binom{2n}{n}, \qquad \tilde{N}_n''(1) = (n-1)\binom{2n-2}{n-1}.$$

Proof. The combinatorial interpretation follows immediately by rewriting

$$\tilde{N}_n^{(r)}(1) = \sum_{k=1}^{n} N_{n,k} k^{\underline{r}},$$

where we used the notion $k^{\underline{r}} = k(k-1)\cdots(k-r+1)$ for the falling factorial. Explicit values can be obtained by differentiating (3.2) r-times with respect to t, then setting $t = 1$ and extracting the coefficient of z^{n+1}. □

Remark.
By the combinatorial interpretation of Proposition 3.2.4 we find that $\tilde{N}_n'(1) = \frac{1}{2}\binom{2n}{n}$ enumerates the number of leaves, summed over all trees with $n+1$ nodes. At the same time, as there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ such trees, the total number of nodes in these trees is $\binom{2n}{n}$. This implies that exactly half of all nodes in all trees of given size are leaves!

In fact, this interpretation also motivates a second, purely combinatorial proof of the explicit value of $\tilde{N}_n'(1)$: the bijection correspondence maps trees of size $n+1$ to binary trees with n inner nodes. In the proof of Proposition 3.2.3 we already observed that the number of left leaves in the binary tree obtained from the rotation correspondence is equal to the number of leaves in the plane tree.

As binary trees with n inner nodes have $n+1$ leaves, and as there are C_n binary trees with n inner nodes, the total number of leaves in all binary trees with n inner nodes is $\binom{2n}{n}$. By symmetry, there have to be equally many left leaves as right leaves—which proves that there are $\frac{1}{2}\binom{2n}{n}$ left leaves, and thus $\tilde{N}_n'(1) = \frac{1}{2}\binom{2n}{n}$.

In addition to the polynomials related to the Narayana numbers, there is another well-known sequence of polynomials that will occur throughout this chapter.

Definition 3.2.5.
The *Fibonacci polynomials* are recursively defined by

$$F_r(z) = F_{r-1}(z) + zF_{r-2}(z)$$

for $r \geq 2$ and $F_0(z) = 0$, $F_1(z) = 1$.

For many identities involving Fibonacci numbers, there is an analogous statement for Fibonacci polynomials. The identity presented in the following proposition will be used repeatedly throughout this chapter.

Proposition 3.2.6 (d'Ocagne's Identity, [28]).
Let $s, r \in \mathbb{Z}_{\geq 0}$ where $s \geq r$. Then we have

$$F_{r+1}(z)F_s(z) - F_r(z)F_{s+1}(z) = (-z)^r F_{s-r}(z). \tag{3.6}$$

Proof. The left-hand side of (3.6) can be expressed as the determinant of

$$\begin{pmatrix} F_{r+1}(z) & F_r(z) \\ F_{s+1}(z) & F_s(z) \end{pmatrix}.$$

At the same time, for $r, s \geq 1$ we can write

$$\begin{pmatrix} F_{r+1}(z) & F_r(z) \\ F_{s+1}(z) & F_s(z) \end{pmatrix} = \begin{pmatrix} F_r(z) & F_{r-1}(z) \\ F_s(z) & F_{s-1}(z) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & 0 \end{pmatrix}.$$

Combining these two observations yields

$$F_{r+1}(z)F_s(z) - F_r(z)F_{s+1}(z) = \det \begin{pmatrix} F_{r+1}(z) & F_r(z) \\ F_{s+1}(z) & F_s(z) \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ F_{s+1-r}(z) & F_{s-r}(z) \end{pmatrix} \det \begin{pmatrix} 1 & 1 \\ z & 0 \end{pmatrix}^r,$$

which proves the statement. □

Observe that setting $s = r + 1$ in (3.6) yields the identity

$$F_{r+1}(z)^2 - F_r(z)F_{r+2}(z) = (-z)^r, \tag{3.7}$$

which we will make heavy use of later on.

An important tool in the context of plane trees is the substitution $z = u/(1+u)^2$, which allows us to write some expressions in a manageable form. It is easy to check that with this substitution, we can write Fibonacci polynomials as

$$F_r(-z) = \frac{1 - u^r}{(1-u)(1+u)^{r-1}}. \tag{3.8}$$

The fact that this substitution also works for Fibonacci polynomials is not that surprising, as $zF_r(-z)/F_{r+1}(-z)$ is the generating function of plane trees with height $\leq r$ (see [8]).

3.2.2 Leaf-Reduction and the Expansion Operator

The reduction $\rho : \mathcal{T} \setminus \{\Box\} \to \mathcal{T}$ we want to investigate now can be explained very easily. For any tree $\tau \in \mathcal{T} \setminus \{\Box\}$ we obtain the reduced tree $\rho(\tau)$ simply by removing all leaves from τ. Repeated application of ρ to a tree is illustrated in Figure 3.3.

It is easy to see that this operator is certainly not injective: there are many trees that reduce to the same tree. However, it is also easy to see that ρ is surjective, as we can always construct an expanded tree that reduces to any given tree τ by attaching leaves to all leaves of τ.

In fact, the operator ρ^{-1} mapping trees $\tau \in \mathcal{T}$ to the set of preimages is easier to handle from a combinatorial point of view. This is because we can model the expansion of trees in the language of generating functions.

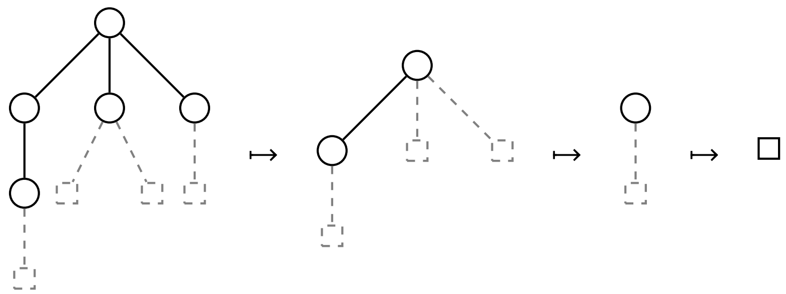


Figure 3.3: Illustration of the “cutting leaves”-operator ρ.

Proposition 3.2.7.
Let $\mathcal{F} \subseteq \mathcal{T}$ be a family of plane trees with bivariate generating function $f(z,t)$, where z marks inner nodes and t marks leaves. Then the generating function of $\rho^{-1}(\mathcal{F})$, the family of trees whose reduction is in $\mathcal{F}$, is given by

$$\Phi(f(z,t)) = (1-t)f\left(\frac{z}{(1-t)^2}, \frac{zt}{(1-t)^2}\right). \tag{3.9}$$

Proof. It is obvious from a combinatorial point of view that the operator Φ has to be linear. Thus we only have to determine how a tree represented by an arbitrary monomial $z^n t^k$, i.e. a tree τ with n inner nodes and k leaves, is expanded.

In order to obtain all possible tree expansions from τ, we perform the following operations: first, all leaves of τ are expanded by appending a nonempty sequence of leaves to each of them. Then, every inner node of τ is expanded by appending (possibly empty) sequences of leaves between two of its children as well as before the first and after the last one.

In terms of generating functions, expanding the leaves of τ corresponds to replacing t by $zt/(1-t)$. Expanding the inner vertices is a bit more involved: by considering that every inner node has precisely one more available position to attach new leaves than it has children we find that there are $2n+k-1$ available positions overall within τ. Therefore we find

$$\Phi(z^n t^k) = z^n \left(\frac{zt}{1-t}\right)^k \frac{1}{(1-t)^{2n+k-1}},$$

which, as Φ is linear, immediately proves (3.9). $\square$

Corollary 3.2.8.
The generating function for plane trees $T(z,t)$ satisfies the functional equation

$$T(z,t) = t + \Phi(T(z,t)). \tag{3.10}$$

Proof. This follows directly from the fact that $\rho : \mathcal{T} \setminus \{\square\} \to \mathcal{T}$ is surjective, i.e. $\rho^{-1}(\mathcal{T}) = \mathcal{T} \setminus \{\square\}$. $\square$

Corollary 3.2.9.
The Narayana numbers satisfy the identity

$$N_{n+k-1,k} = \sum_{\ell=1}^{k} \binom{2n+k-\ell-2}{k-\ell} N_{n-1,\ell}$$

for $n \geq 2$, $k \geq 1$.

Proof. The result follows from extracting the coefficient of $z^n t^k$ from both sides of (3.10). □

Remark.
Note that in [5] there is a very short proof based on Dyck paths for this identity, and actually the argumentation there is strongly related to our tree reduction here: by the well-known glove bijection, it is easy to see that cutting away all leaves of a plane tree translates into removing all peaks within the corresponding Dyck path.

We are now interested in determining a multivariate generating function enumerating plane trees with respect to the tree size as well as the size of the tree after applying the tree reduction ρ a fixed number of times.

Proposition 3.2.10.
Let $r \in \mathbb{N}_0$. The trivariate generating function $G_r(z, v_I, v_L) = G_r^{\mathrm{L}}(z, v_I, v_L)$ enumerating plane trees whose leaves can be cut at least r-times, where z marks the tree size, and v_I and v_L mark the number of inner nodes and leaves of the r-fold cut tree, respectively, is given by

$$G_r(z, v_I, v_L) = \Phi^r(T(zv_I, tv_L))|_{t=z} = \frac{1-u^{r+2}}{(1-u^{r+1})(1+u)} T\left(\frac{u(1-u^{r+1})^2}{(1-u^{r+2})^2} v_I, \frac{u^{r+1}(1-u)^2}{(1-u^{r+2})^2} v_L\right). \tag{3.11}$$

Proof. First, observe that formally, we can obtain the generating function enumerating plane trees that can be reduced at least r-times with respect to their size by considering $\Phi^r(T(z,t))|_{t=z}$. If we additionally track some size parameter like the number of inner nodes or the number of leaves before the expansion by marking their size with v_I and v_L, then we obtain a generating function for plane trees that can be reduced at least r-times where v_I and v_L mark inner nodes and leaves in the original tree and z marks the size of the expanded tree. From a different point of view, z marks the size of the original tree and v_I and v_L mark the number of inner nodes and leaves of the r-fold reduced tree, meaning that we have

$$G_r(z, v_I, v_L) = \Phi^r(T(zv_I, tv_L))|_{t=z},$$

which proves the first equation in (3.11).

As Φ is linear, we are mainly interested in finding a representation for $\Phi^r(z^n t^k)|_{t=z}$. To do so, we consider the strongly related operator

$$\Psi(f(z,t)) := f\left(\frac{z}{(1-t)^2}, \frac{zt}{(1-t)^2}\right).$$

It is easy to prove by induction that iterative application of Φ can be expressed in terms of Ψ via

$$\Phi^r(f(z,t)) = \Psi^r(f(z,t)) \prod_{j=0}^{r-1} (1 - \Psi^j(t)),$$

which means that we can concentrate on the investigation of the linear operator Ψ. Note that Ψ is also multiplicative, meaning that $\Psi^r(z^n t^k) = \Psi^r(z)^n \Psi^r(t)^k$.

Again by induction, it is easy to show that the recurrences

$$\Psi^{r+1}(t) = \frac{z\Psi^r(t)}{\prod_{j=0}^{r}(1-\Psi^j(t))^2} \quad \text{and} \quad \Psi^{r+1}(z) = \frac{z}{\prod_{j=0}^{r}(1-\Psi^j(t))^2}$$

hold for $r \geq 0$. Now define $f_r := \Psi^r(t)|_{t=z}$ and $g_r := \Psi^r(z)|_{t=z}$. We prove by induction that these quantities can be represented by means of Fibonacci polynomials as

$$f_r = \frac{z^{r+1}}{F_{r+2}(-z)^2} \quad \text{and} \quad g_r = \frac{zF_{r+1}(-z)^2}{F_{r+2}(-z)^2}$$

for $r \geq 0$, where the recurrence relations from above, the identity (3.7) as well as the relation

$$\prod_{j=0}^{r-1}(1-f_j) = \frac{F_{r+2}(-z)}{F_{r+1}(-z)}$$

for $r \geq 0$ play integral parts in the proof.

With these explicit representations, we find

$$\Phi^r(z^n t^k)|_{t=z} = \Psi^r(z^n t^k)|_{t=z} \prod_{j=0}^{r-1}(1-f_j) = \frac{z^{n+k(r+1)}F_{r+1}(-z)^{2n-1}}{F_{r+2}(-z)^{2n+2k-1}}. \tag{3.12}$$

Then, using (3.8) and rewriting the right-hand side of (3.12) in terms of u, where $z = u/(1+u)^2$, yields

$$\Phi^r(z^n t^k)|_{t=z} = \frac{1-u^{r+2}}{(1-u^{r+1})(1+u)} \left(\frac{u(1-u^{r+1})^2}{(1-u^{r+2})^2}\right)^n \left(\frac{u^{r+1}(1-u)^2}{(1-u^{r+2})^2}\right)^k.$$

By linearity, we are allowed to apply Φ^r to every summand in the power series expansion of $f(z,t)$ separately—which proves the statement. □

The generating function $G_r(z,v,v)$ tells us how many nodes (marked by v) are still in the tree after r reductions. For the sake of brevity we set $G_r(z,v) := G_r(z,v,v)$. It is completely described in terms of the function $T(z,t)$, although in a non-trivial way. Results about moments and the limiting distribution can be extracted from this explicit form.

With the help of the mathematics software system SageMath [59], the generating function $G_r(z,v)$ can be expanded. For small values of r, the first few summands are

$$G_1(z,v) = vz^2 + (v^2+v)z^3 + (v^3+3v^2+v)z^4 + (v^4+6v^3+6v^2+v)z^5 + O(vz^6),$$

$$G_2(z,v) = vz^3 + (v^2+3v)z^4 + (v^3+5v^2+7v)z^5 + (v^4+7v^3+18v^2+15v)z^6 + O(vz^7),$$
$$G_3(z,v) = vz^4 + (v^2+5v)z^5 + (v^3+7v^2+18v)z^6 + (v^4+9v^3+33v^2+57v)z^7 + O(vz^8).$$

As announced in Section 3.1, we investigate the behavior of the random variable $X_{n,r} = X^{\mathrm{L}}_{n,r}$ that models the number of nodes which are left after reducing a random tree τ with n nodes r-times. In case the r-fold application of ρ to τ is not defined, we consider the resulting tree size to be 0. In terms of the random variable, this means that $X_{n,r} = 0$ for these trees. Note that the tree τ is chosen uniformly at random among all trees of size n. With the help of the generating function $G_r(z,v)$ we are able to express the probability generating function of $X_{n,r}$ as

$$\mathbb{E}v^{X_{n,r}} = \frac{a_{n,r} + [z^n]G_r(z,v)}{C_{n-1}} \tag{3.13}$$

where $a_{n,r}$ is the number of trees of size n which are empty after reducing r-times. We have $a_{n,r} = C_{n-1} - [z^n]G_r(z,1)$.

In addition to $X_{n,r}$, we also consider the random variables $I_{n,r}$ and $L_{n,r}$ that model the number of inner nodes and leaves, respectively, that remain after reducing a random tree with n nodes r times. The generating functions corresponding to $I_{n,r}$ and $L_{n,r}$ are $G_r(z,v,1)$ and $G_r(z,1,v)$, respectively.

The relations $X_{n,r} \stackrel{d}{=} I_{n,r} + L_{n,r}$ and $I_{n,r} \stackrel{d}{=} X_{n,r+1}$ hold by the combinatorial interpretation of the operator Φ.

3.2.3 Asymptotic Analysis

We find explicit generating functions for the factorial moments of the random variables $X_{n,r}$, $I_{n,r}$, and $L_{n,r}$.

Proposition 3.2.11 ([28]).
The dth factorial moments of $X_{n,r}$, $I_{n,r}$ and $L_{n,r}$ are given by

$$\begin{aligned}\mathbb{E}X_{n,r}^{\underline{d}} = \mathbb{E}I_{n,r-1}^{\underline{d}} &= \frac{1}{C_{n-1}}[z^n]\frac{\partial^d}{\partial v^d}G_r(z,1)\Big|_{v=1} \\ &= \frac{1}{C_{n-1}}[z^n]\frac{u^d d!}{(1+u)(1-u^{r+1})^d(1-u)^{d-1}}\tilde{N}_{d-1}(u^r)\end{aligned} \tag{3.14}$$

and

$$\mathbb{E}L_{n,r}^{\underline{d}} = \frac{1}{C_{n-1}}[z^n]\frac{u^{dr+2d}(1-u)d!}{(1+u)(1-u^{r+2})^d(1-u^{r+1})^d}\tilde{N}_{d-1}\Big(\frac{1}{u}\Big) \tag{3.15}$$

where $z = u/(1+u)^2$ for $d \in \mathbb{Z}_{\geq 1}$.

Remark.
For $d \geq 2$, $u^d\tilde{N}_{d-1}(u^{-1})$ can be replaced by $\tilde{N}_{d-1}(u)$ in (3.15), see (3.5).

Proof. We use the abbreviations

$$a := \frac{u(1-u^{r+1})^2}{(1-u^{r+2})^2}, \qquad b := \frac{u^{r+1}(1-u)^2}{(1-u^{r+2})^2}, \qquad c := \frac{1-u^{r+2}}{(1-u^{r+1})(1+u)}.$$

We consider the exponential generating function of $\partial^d/(\partial v)^d G_r(z,v)$ to be a Taylor series and obtain

$$\sum_{d\geq 0} \frac{1}{d!}\frac{\partial^d}{\partial v^d} G_r(z,v) q^d = G_r(z, v+q).$$

By Proposition 3.2.10, extracting the coefficient of q^d yields

$$\frac{\partial^d}{\partial v^d} G_r(z,v)\bigg|_{v=1} = d![q^d]G_r(z,1+q) = d!c[q^d]T(a(1+q),b(1+q)).$$

We have

$$\begin{aligned} T(a(1+q),b(1+q)) &= \frac{1-(a-b)(1+q)-\sqrt{1-2(1+q)(a+b)+(1+q)^2(a-b)^2}}{2} \\ &= \frac{1-(a-b)-(a-b)q}{2} \\ &\quad - \frac{\sqrt{1-2(a+b)+(a-b)^2-2q(a+b-(a-b)^2)+q^2(a-b)^2}}{2}. \end{aligned}$$

By using the fact that

$$1-2(a+b)+(a-b)^2 = \Delta^2 \quad \text{for} \quad \Delta = \frac{(1-u)(1-u^{r+1})}{1-u^{r+2}}$$

and by choosing α and β such that

$$\alpha+\beta = \frac{a+b-(a-b)^2}{\Delta^2}, \qquad \alpha-\beta = \frac{a-b}{\Delta},$$

we obtain

$$\begin{aligned} T(a(1+q),b(1+q)) &= \frac{\Delta}{2}\left(\frac{1}{\Delta}-(\alpha-\beta)-(\alpha-\beta)q-\sqrt{1-2q(\alpha+\beta)+q^2(\alpha-\beta)^2}\right) \\ &= \frac{\Delta(\frac{1}{\Delta}-1-(\alpha-\beta))}{2}+\Delta T(\alpha q,\beta q). \end{aligned}$$

Extracting the coefficient of q^d for $d \geq 1$ yields

$$\frac{\partial^d}{\partial v^d} G_r(z,v)\bigg|_{v=1} = cd!\Delta[q^d]\sum_{d\geq 1}\alpha^d q^d \tilde{N}_{d-1}\left(\frac{\beta}{\alpha}\right) = cd!\Delta\alpha^d \tilde{N}_{d-1}\left(\frac{\beta}{\alpha}\right)$$

where (3.3) has been used.

Noting that

$$\alpha = \frac{u}{(1-u)(1-u^{r+1})} \quad \text{and} \quad \beta = \frac{u^{r+1}}{(1-u)(1-u^{r+1})}$$

completes the proof of (3.14).

For the proof of (3.15), we proceed in the same way and use the identity

$$T(a, b(1+q)) = \frac{1-\Delta-(a-b)}{2} + \Delta T(\alpha' q, \beta' q)$$

for

$$\alpha' = \frac{u^{r+2}}{(1-u^{r+1})(1-u^{r+2})}, \qquad \beta' = \frac{u^{r+1}}{(1-u^{r+1})(1-u^{r+2})}.$$

In fact, we have

$$\begin{aligned} T(a, b(1+q)) &= \frac{1-(a-b-bq)-\sqrt{1-2(a+b+bq)+(a-b-bq)^2}}{2} \\ &= \frac{(1-(a-b))-(-bq)-\sqrt{\Delta^2-2q(b+b(a-b))+b^2q^2}}{2} \\ &= \frac{1-\Delta-(a-b)}{2} + \Delta\frac{1-(\alpha'-\beta')q-\sqrt{1-2q(\alpha'+\beta')+(\alpha'-\beta')^2q^2}}{2} \\ &= \frac{1-\Delta-(a-b)}{2} + \Delta T(\alpha' q, \beta' q) \end{aligned}$$

where α' and β' have been chosen such that

$$\alpha' + \beta' = \frac{b+ba-b^2}{\Delta^2} \text{ and } \alpha' - \beta' = -\frac{b}{\Delta},$$

which implies the values for α' and β' given above.

Thus the qth derivative of the generating function $G_r(z, 1, 1+q)$ is

$$d!\Delta c {\alpha'}^d \tilde{N}_{d-1}\left(\frac{\beta'}{\alpha'}\right),$$

which proves (3.15). □

From the proof of Proposition 3.2.11, we extract the following identities for the modified Narayana polynomials.

Remark.
For $d \in \mathbb{Z}_{\geq 1}$ the power series identities

$$\sum_{n\geq 1} \binom{n}{d} \frac{u^{n-d}(1-ux)^{2n+d-1}(1-u)^{d-1}}{(1-u^2x)^{2n-1}} \tilde{N}_{n-1}\left(\frac{x(1-u)^2}{(1-ux)^2}\right) = \tilde{N}_{d-1}(x) \tag{3.16}$$

$$\sum_{n\geq 1} \frac{u^{n-2d}(1-ux)^{2n-d-1}(1-u)^{2d-1}}{(1-u^2x)^{2n-d-1}d!} \tilde{N}_{n-1}^{(d)}\left(\frac{x(1-u)^2}{(1-ux)^2}\right) = \tilde{N}_{d-1}\left(\frac{1}{u}\right) \tag{3.17}$$

hold, where $\tilde{N}_{n-1}^{(d)}$ denotes the dth derivative of $\tilde{N}_{n-1}$.

Proof. In the proof of Proposition 3.2.11, we showed that

$$\frac{\partial^d}{\partial v^d} cT(av, bv)\Big|_{v=1} = cd!\Delta\alpha^d \tilde{N}_{d-1}\Big(\frac{\beta}{\alpha}\Big).$$

Expanding the left side using (3.3) and evaluating the derivative yields (3.16) (where u^r has been replaced by the independent variable x).

The identity (3.17) is proved in the same way. □

A previous version of the proof of Proposition 3.2.11 was using the power series identities (3.16) and (3.17), which can also be derived by using the Mathematica package Sigma (see [60]).

Corollary 3.2.12.
The expected value of $X_{n+1,r}$ is explicitly given by

$$\mathbb{E}X_{n+1,r} = \frac{1}{C_n}\sum_{\ell\geq 1}\left(\binom{2n}{n+1-\ell(r+1)} - \binom{2n}{n-\ell(r+1)}\right).$$

Proof. Using Proposition 3.2.11 and Cauchy's integral formula, we have

$$\begin{aligned} C_n\mathbb{E}X_{n+1,r} &= [z^{n+1}]\frac{u^{r+1}}{(1+u)(1-u^{r+1})} \\ &= \frac{1}{2\pi i}\oint_{\gamma} \frac{u^{r+1}}{(1+u)(1-u^{r+1})}\frac{dz}{z^{n+2}} \\ &= \frac{1}{2\pi i}\oint_{\tilde{\gamma}} \frac{u^{r+1}(1-u)(1+u)^{2n}}{1-u^{r+1}}\frac{du}{u^{n+2}}, \end{aligned}$$

where γ is a circle around 0 with a sufficiently small radius such that γ', the image of γ under the transformation, is a small contour circling 0 exactly once as well.

Expanding $(1-u^{r+1})^{-1}$ into a geometric series and exchanging integration and summation, we obtain

$$C_n\mathbb{E}X_{n+1,r} = \sum_{\ell\geq 1}[u^{n+1-\ell(r+1)}](1-u)(1+u)^{2n},$$

which implies the result. □

Having determined a closed form for this generating function allows us to analyze the asymptotic behavior of $X_{n,r}$ in a relatively straightforward way.

Theorem 3.2.13.
Let $r \in \mathbb{N}_0$ be fixed and consider $n \to \infty$. Then the expected size and the corresponding variance of an r-fold cut plane tree are given by

$$\mathbb{E}X_{n,r} = \frac{n}{r+1} - \frac{r(r-1)}{6(r+1)} + O(n^{-1}), \tag{3.18}$$

and

$$\mathbb{V}X_{n,r} = \frac{r(r+2)}{6(r+1)^2} n + O(1). \tag{3.19}$$

The factorial moments are asymptotically given by

$$\mathbb{E}X_{n,r}^{\underline{d}} = \frac{1}{(r+1)^d} n^d + \frac{d}{12(r+1)^d}(dr^2 - 4dr - 3r^2 - 6d + 6r + 6)n^{d-1} + O(n^{d-3/2})$$

for $d \geq 1$. Note that all O-constants above depend implicitly on r.

Proof. In a nutshell, we want to extract the growth of the derivatives of the generating functions $\frac{\partial^d}{\partial v^d} G_r(z,1)$, as dividing these quantities by C_{n-1} yields the factorial moments. We want to extract the growth by means of singularity analysis (cf. [18]).

In order to do so, we first need to establish the location of the dominant singularity of these generating functions, which are explicitly given in (3.14).

The singularities of (3.14) are roots of unity in terms of u. Substituting back $u = (1 - \sqrt{1-4z})/(2z) - 1$ maps these roots of unity to real numbers greater or equal to $1/4$ and only $u = 1$ is mapped to $z = 1/4$. Thus $z = 1/4$ is the dominant singularity of (3.14). A more detailed treatment of these analytic properties of the substitution $z = u/(1+u)^2$ can be found in Proposition 2.2.3.

As $N_0(x) = 1$, we obtain the expansion

$$\frac{1}{2(r+1)}(1-u)^{-1} - \frac{1}{4} + \frac{r^2 - r - 3}{24(r+1)}(1-u) + O((1-u)^2)$$

for the function on the right-hand side of (3.14) with $d = 1$. Then, the expansion

$$\begin{aligned}(1-u)^{-\kappa} = 2^{-\kappa}(1-4z)^{-\kappa/2} + 2^{-\kappa}\kappa(1-4z)^{-(\kappa-1)/2} \\ + 2^{-\kappa}\frac{\kappa(\kappa-1)}{2}(1-4z)^{-(\kappa-2)/2} + O((1-4z)^{-(\kappa-3)/2})\end{aligned} \tag{3.20}$$

for fixed $\kappa \in \mathbb{C}$ yields

$$\frac{1}{4(r+1)}(1-4z)^{-1/2} + \frac{r^2-r-3}{12(r+1)}(1-4z)^{1/2} + O((1-4z)^{3/2}) + \text{power series in } (1-4z).$$

By singularity analysis, the nth coefficient, normalized by C_{n-1}, is asymptotically

$$\mathbb{E}X_{n,r} = \frac{n}{r+1} - \frac{r(r-1)}{6(r+1)} + O(n^{-1})$$

using

$$C_{n-1} = 4^{n-1}\frac{1}{n^{3/2}\sqrt{\pi}}\Big(1 + \frac{3}{8}n^{-1} + O(n^{-2})\Big).$$

The higher order factorial moments follow similarly by expanding the function on the right-hand side of (3.14) for general $d > 1$ around $u = 1$ with the help of SageMath, where in particular the explicit values of the derivatives of the Narayana polynomials from Proposition 3.2.4 are required.

Singularity analysis of the resulting expansion yields the expression given in the statement of the theorem. Finally, note that the variance can be computed by using

$$\mathbb{V}X_{n,r} = \mathbb{E}X_{n,r}^{\underline{2}} + \mathbb{E}X_{n,r} - (\mathbb{E}X_{n,r})^2.$$

□

Theorem 3.2.14.
The size $X_{n,r}$ of the tree obtained from a random plane tree with n nodes by cutting it r-times is, after standardization, asymptotically normally distributed for $n \to \infty$ and fixed r, i.e.,

$$\frac{X_{n,r} - \dfrac{n}{r+1}}{\sqrt{\dfrac{r(r+2)}{6(r+1)^2}n}} \xrightarrow{d} \mathcal{N}(0,1).$$

To be more precise, for $x \in \mathbb{R}$ we have

$$\mathbb{P}\Big(\frac{X_{n,r} - n\mu}{\sqrt{\sigma^2 n}} \le x\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt + O(n^{-1/2}),$$

with $\mu = \frac{1}{r+1}$ and $\sigma^2 = \frac{r(r+2)}{6(r+1)^2}$ and where the O-constant depends implicitly on r.

As $I_{n,r-1} \stackrel{d}{=} X_{n,r}$, the same also holds for this random variable.

The rest of this section is devoted to the proof of this central limit theorem. In order to derive the fact that the number of remaining nodes after r reductions is asymptotically normally distributed, we first show that the number of nodes that are deleted after r reductions is asymptotically normally distributed. Then, as the sum of the number of remaining nodes and the number of deleted nodes is equal to the original tree size, we obtain immediately that the number of remaining nodes has to be asymptotically normally distributed as well.

We begin by considering the function $F_r : \mathcal{T} \to \mathbb{N}_0$ which maps a plane tree τ to the number of nodes that are deleted when reducing the tree r times, i.e. the difference between the size of τ and the size of $\rho^r(\tau)$. Let τ_n now denote a plane tree with n nodes.

For the sake of convenience, we consider $F_r(\tau_n)$ to be n if r is larger than the maximal number of reductions that can be applied to τ_n before the tree cannot be reduced further. In particular, this means that $F_r(\square) = 1$ for $r \ge 1$.

It is easy to see that the parameter $F_r(\tau_n)$ is a so-called *additive tree parameter*, meaning that

$$F_r(\tau_n) = F_r(\tau_{i_1}) + \cdots + F_r(\tau_{i_\ell}) + f_r(\tau_n)$$

holds, where $\tau_{i_1}, \ldots, \tau_{i_\ell}$ are the subtrees rooted at the children of the root of τ_n, and $f_r : \mathcal{T} \to \{0,1\}$ is a *toll function* recursively defined by

$$f_r(\tau_n) = \begin{cases} 1, & \text{if } F_{r-1}(\tau_{i_k}) = i_k \text{ for all } k = 1, \ldots, \ell, \\ 0, & \text{otherwise,} \end{cases}$$

for $r \geq 1$ and $f_0(\tau_n) = 0$.

In order to prove asymptotic normality for additive tree parameters, we can use [67, Theorem 2], which requires us to show that the expected value of the toll function is exponentially decreasing in n. This is done in the following lemma.

Lemma 3.2.15.
The expected value of $f_r(\tau_n)$ is exponentially decreasing in n.

Remark.
Of course, $n - F_r(\tau_n)$ is also an additive parameter. However, the expected value of the corresponding growth function is not exponentially decreasing.

Proof. Define

$$q_{n,r} = \mathbb{E}(f_r(\tau_n)) = \mathbb{P}(F_{r-1}(\tau_{i_k}) = i_k \text{ for all } k = 1, \ldots, \ell) = \mathbb{P}(F_r(\tau_n) = n)$$

and the corresponding generating function

$$Q_r(z) = \sum_{n \geq 1} C_{n-1} q_{n,r} z^n.$$

Observe that $F_r(\tau_n) = n$ holds if and only if τ_n has height less than r, as removing all leaves from a tree reduces its height by precisely one. Therefore, the generating function $Q_r(z)$ is the generating function enumerating trees of height less than r.

It is well-known (cf. [8]) that the generating function for plane trees of height less than r can be expressed in terms of Fibonacci polynomials as

$$Q_r(z) = \frac{z F_{r-1}(-z)}{F_r(-z)}.$$

The roots of $F_r(-z)$ are also well-known and can be written as $\alpha_{j,r} = (4\cos^2(j\pi/r))^{-1}$ for $j = 1, \ldots, \lfloor (r-1)/2 \rfloor$.

Thus $Q_r(z)$ is a rational function and its coefficients have the form

$$C_{n-1} q_{n,r} = \sum_j c_{j,r} \alpha_{j,r}^{-n}$$

for constants $c_{j,r}$. We have $|\alpha_{j,r}| > 4$. As

$$C_{n-1} \sim \frac{4^{n-1}}{\sqrt{\pi}\, n^{3/2}},$$

there exists a constant $c \in (0,1)$ such that $q_{n,r} = O(c^n)$. □

Thus, by the strategy discussed above, we find that not only $F_r(\tau_n)$ but also $X_{n,r} = n - F_{n,r}$ is asymptotically normally distributed.

Remark.
Note that the fact that $F_1(\tau_n)$ is asymptotically normally distributed means that the Narayana numbers are asymptotically normally distributed, see for example [11, Theorem 3.13].

As sketched above, Lemma 3.2.15 allows us to apply [67, Theorem 2] in order to prove that $F_r(\tau_n)$, and therefore also $X_{n,r} = n - F_r(\tau_n)$ is asymptotically normally distributed. All that remains to prove is that the speed of convergence is $O(n^{-1/2})$.

We do so by noting that the proof for asymptotic normality in Wagner's theorem is based on [11, Theorem 2.23], where a version of Hwang's Quasi-Power Theorem [31] without quantification of the speed of convergence is used. Replacing this argument with the multi-dimensional quantified version given in [29] then gives us the desired speed of convergence of $O(n^{-1/2})$.

3.3 Cutting Paths

3.3.1 Expansion Operator and Results

Let $\mathcal{P}$ denote the combinatorial class of paths, i.e. trees in which every node is either a leaf or has precisely one child. The tree reduction $\rho\colon \mathcal{T} \setminus \mathcal{P} \to \mathcal{T}$ which we will focus on in this section reduces a tree by cutting away all paths of the tree. This operation is illustrated in Figure 3.4.

Analogously to our approach in Section 3.2.2, we first determine the corresponding expansion operator Φ. In order to do so, we need the generating function for the family of paths $\mathcal{P}$, which is given by $P = P(z,t) = \frac{t}{1-z}$. For the sake of readability, we omit the arguments of P.

Proposition 3.3.1.
Let $\mathcal{F} \subseteq \mathcal{T}$ be a family of plane trees with bivariate generating function $f(z,t)$, where z marks inner nodes and t marks leaves. Then the generating function for $\rho^{-1}(\mathcal{F})$, the family of trees whose reduction is in $\mathcal{F}$, is given by

$$\Phi(f(z,t)) = (1-P) f\left(\frac{z}{(1-P)^2}, \frac{zP^2}{(1-P)^2}\right). \tag{3.21}$$

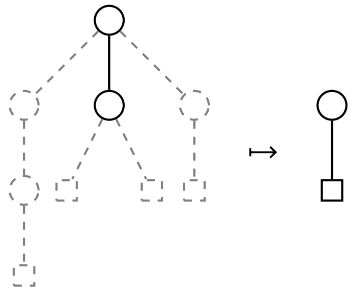


Figure 3.4: Illustration of the "cutting paths"-operator ρ.

Proof. The fact that Φ is a linear operator is obvious from a combinatorial point of view, meaning that we may concentrate on some tree τ with n inner nodes and k leaves, represented by $z^n t^k$.

We follow the proof of Proposition 3.2.7 and observe that all possible tree expansions of τ can be obtained by the following operations: the leaves of τ are expanded by appending a sequence of at least two paths to each of them. Note that appending a single path to a leaf is not allowed, because this would just extend the path ending in that leaf, which causes ambiguity. Then, the inner nodes are expanded as well by appending (possibly empty) sequences of paths to the $2n+k-1$ available positions between, before, and after their children.

Translating this expansion to the language of generating functions yields

$$\Phi(z^n t^k) = z^n \left(\frac{zP^2}{1-P}\right)^k \frac{1}{(1-P)^{2n+k-1}},$$

which proves (3.21). □

Corollary 3.3.2.
The generating function for plane trees $T(z,t)$ satisfies the functional equation

$$T(z,t) = P + \Phi(T(z,t)). \tag{3.22}$$

Proof. Surjectivity of ρ implies $\rho^{-1}(\mathcal{T}) = \mathcal{T} \setminus \mathcal{P}$, which proves the statement after translating this into the language of generating functions with the help of Φ. □

In the following proposition, we determine the generating function $G_r(z, v_I, v_L)$ measuring the effect of applying the path reduction r times on the size of the tree. Most interestingly, we will see that the path connection is in fact strongly related to the leaf reduction from the previous section.

Proposition 3.3.3.
The trivariate generating function $G_r(z, v_I, v_L) = G_r^{\mathrm{P}}(z, v_I, v_L)$ enumerating plane trees whose

paths can be cut at least r-times, where z marks the tree size and v_I and v_L mark the number of inner nodes and leaves of the r-fold cut tree, respectively, is given by

$$\begin{aligned} G_r(z, v_I, v_L) &= \Phi^r(T(zv_I, tv_L))|_{t=z} \\ &= \frac{1-u^{2^{r+1}}}{(1-u^{2^{r+1}-1})(1+u)} T\left(\frac{u(1-u^{2^{r+1}-1})^2}{(1-u^{2^{r+1}})^2} v_I, \frac{u^{2^{r+1}-1}(1-u)^2}{(1-u^{2^{r+1}})^2} v_L \right), \end{aligned}$$

where $z = u/(1+u)^2$.

Proof. By the same reasoning as in the proof of Proposition 3.2.10, the generating function we are interested in is $G_r(z, v_I, v_L) = \Phi^r(T(zv_I, tv_L))|_{t=z}$, meaning that we want to study the iterated application of Φ. To do so, we consider the strongly related operator

$$\Psi(f(z,t)) := f\left(\frac{z}{(1-P)^2}, \frac{zP^2}{(1-P)^2} \right).$$

The relation

$$\Phi^r(f(z,t)) = \Psi^r(f(z,t)) \prod_{j=0}^{r-1} (1 - \Psi^j(P))$$

can be proved easily by induction and enables us to determine the behavior of Φ via Ψ.

First of all, for $r \geq 0$ and $r \geq 1$, the relations

$$\Psi^r(z) = \frac{z}{\prod_{j=0}^{r-1}(1-\Psi^j(P))^2} \quad \text{and} \quad \Psi^r(P) = \frac{z(\Psi^{r-1}(P))^2}{\prod_{j=0}^{r-1}(1-\Psi^j(P))^2 - z}$$

can be proved easily by induction, respectively. Also observe that we can write $\Psi^r(t) = \Psi^r(z)\Psi^{r-1}(P)^2$. Now let $f_r = \Psi^r(z)|_{t=z}$, $g_r = \Psi^r(t)|_{t=z}$, and $h_r = \Psi^r(P)|_{t=z}$. With the help of the identity $\prod_{j=0}^{r}(1+u^{2^j}) = \frac{1-u^{2^{r+1}}}{1-u}$ we are able to prove the explicit formula

$$h_r = \frac{u^{2^{r+1}-1}(1-u)}{1-u^{2^{r+2}-1}} = \frac{z^{2^{r+1}-1}}{F_{2^{r+2}-1}(-z)}, \tag{3.23}$$

where $z = u/(1+u)^2$ and the second equation is a consequence of (3.8). Using (3.23), we immediately find

$$f_r = \frac{u(1-u^{2^{r+1}-1})^2}{(1-u^{2^{r+1}})^2} = \frac{zF_{2^{r+1}-1}(-z)^2}{F_{2^{r+1}}(-z)^2} \quad \text{and} \quad g_r = \frac{u^{2^{r+1}-1}(1-u)^2}{(1-u^{2^{r+1}})^2} = \frac{z^{2^{r+1}-1}}{F_{2^{r+1}}(-z)^2}.$$

Putting everything together yields

$$\begin{aligned} \Phi^r(z^n t^k)|_{t=z} &= \frac{z^{n+(2^{r+1}-1)k} F_{2^{r+1}-1}(-z)^{2n-1}}{F_{2^{r+1}}(-z)^{2n+2k-1}} \\ &= \frac{1-u^{2^{r+1}}}{(1-u^{2^{r+1}-1})(1+u)} \left(\frac{u(1-u^{2^{r+1}-1})^2}{(1-u^{2^{r+1}})^2} \right)^n \left(\frac{u^{2^{r+1}-1}(1-u)^2}{(1-u^{2^{r+1}})^2} \right)^k, \end{aligned}$$

which directly implies the statement. □

The following result shows that there is an intimate connection between the "cutting leaves"-reduction from Section 3.2 and the "cutting paths"-reduction, as can be seen after comparing the statement of Proposition 3.2.10 with the statement of Proposition 3.3.3.

Corollary 3.3.4.
The generating function $G_r(z, v_I, v_L) = G_r^{\mathrm{P}}(z, v_I, v_L)$ measuring the change in size after cutting away all paths from plane trees r times is equal to the generating function $G^{\mathrm{L}}_{2^{r+1}-2}(z, v_I, v_L)$ measuring the change in size after cutting away all leaves from plane trees $2^{r+1}-2$ times.

This connection is now especially important for the analysis of the random variable $X_{n,r} = X^{\mathrm{P}}_{n,r}$ modeling the number of nodes that are left after reducing a random tree τ with n nodes r times by removing all paths. In fact, it follows that

$$X^{\mathrm{P}}_{n,r} \stackrel{d}{=} X^{\mathrm{L}}_{n,2^{r+1}-2},$$

meaning that the asymptotic analysis of the factorial moments of $X^{\mathrm{P}}_{n,r}$ as well as the limiting distribution follow directly from the corresponding results in Section 3.2.3.

Theorem 3.3.5.
Let $r \in \mathbb{N}_0$ be fixed and consider $n \to \infty$. Then expectation and variance of the random variable $X_{n,r} = X^{\mathrm{P}}_{n,r}$ can be expressed as

$$\mathbb{E}X_{n,r} = \frac{n}{2^{r+1}-1} - \frac{(2^r-1)(2^{r+1}-3)}{3(2^{r+1}-1)} + O(n^{-1}), \tag{3.24}$$

and

$$\mathbb{V}X_{n,r} = \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2} n + O(1). \tag{3.25}$$

The factorial moments are asymptotically given by

$$\begin{aligned}\mathbb{E}X^{\underline{d}}_{n,r} = &\frac{n^d}{(2^{r+1}-1)^d} \\ &+ \frac{d}{12(2^{r+1}-1)^d}(4^{r+1}d - 2^{r+4}d - 3\cdot 4^{r+1} + 9\cdot 2^{r+2} + 6d - 18) \\ &+ O(n^{d-3/2}).\end{aligned}$$

Furthermore, $X_{n,r} = X^{\mathrm{P}}_{n,r}$ is asymptotically normally distributed, i.e., for $x \in \mathbb{R}$ we have

$$\mathbb{P}\Big(\frac{X_{n,r} - \mu n}{\sqrt{\sigma^2 n}} \le x\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt + O(n^{-1/2})$$

for $\mu = \frac{1}{2^{r+1}-1}$ and $\sigma^2 = \frac{2^{r+1}(2^r-1)}{3(2^{r+1}-1)^2}$. All O-constants in this theorem depend implicitly on r.

3.3.2 Total number of paths

In the context of this reduction it is interesting to investigate the total number of paths needed to construct a given tree. To determine this parameter we can reduce the tree repeatedly and count the number of leaves. The sum of the number of leaves over all reduction steps is equal to the number of paths, which follows from the observation that leaves mark the endpoints of all paths.

Formally, given the random variables $P_{n,r}$ counting the number of leaves in the rth reduction of a tree of size n, we want to analyze the random variable $P_n := \sum_{r\geq 0} P_{n,r}$.

Proposition 3.3.6.
The expected number of paths needed to construct a uniformly random tree of size n satisfies

$$\mathbb{E}P_n = \frac{1}{C_{n-1}}[z^n]\frac{1-u}{1+u}\sum_{r\geq 1}\frac{u^{2^r}}{(1-u^{2^r})(1-u^{2^r-1})}, \tag{3.26}$$

where $z = u/(1+u)^2$.

Proof. As a consequence of Proposition 3.3.3, the bivariate generating function enumerating plane trees where z marks tree size and v marks the number of leaves after r path reductions can be written as

$$\frac{1-u^{2^{r+1}}}{(1-u^{2^{r+1}-1})(1+u)}T\Big(\frac{u(1-u^{2^{r+1}-1})^2}{(1-u^{2^{r+1}})^2}, \frac{u^{2^{r+1}-1}(1-u)^2}{(1-u^{2^{r+1}})^2}v\Big).$$

By differentiating this generating function once with respect to v and setting $v=1$ afterwards, we obtain an expression where $C_{n-1}\mathbb{E}P_{n,r}$ can be extracted as the coefficient of z^n. By (3.15) with $d=1$ and r replaced by $2^{r+1}-2$, we have

$$\mathbb{E}P_{n,r} = \frac{1}{C_{n-1}}[z^n]\frac{1-u}{1+u}\frac{u^{2^{r+1}}}{(1-u^{2^{r+1}})(1-u^{2^{r+1}-1})}.$$

Summation over $r\geq 0$ and shifting the index of summation by one completes the proof. □

Our strategy for determining an asymptotic expansion for $\mathbb{E}P_n$ as given in (3.26) is based on the Mellin transform.

Theorem 3.3.7.
For $n\to\infty$, the expected number of paths required to construct a uniformly random tree of size n is given by the asymptotic expansion

$$\mathbb{E}P_n = (\alpha-1)n + \frac{1}{6}\log_4 n - \frac{\gamma + 4(\alpha-1)\log 2 + \log 2 + 24\zeta'(-1) + 2}{12\log 2} + \delta(\log_4 n) + O(n^{-1/4}), \tag{3.27}$$

where

$$\delta(x) := \frac{1}{\log 2} \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1+\chi_k)\Gamma(\chi_k/2)\zeta(-1+\chi_k)e^{2k\pi i x} \tag{3.28}$$

with $\chi_k = \frac{2k\pi i}{\log 2}$ is a fluctuation with mean 0 and $\alpha := \sum_{k\geq 1} 1/(2^k-1) \approx 1.606695$, γ is the Euler–Mascheroni constant and ζ is the Riemann zeta function.

Remark.
The constant α appears in the asymptotic analysis of digital search trees (see e.g. [39]).

Proof. In order to obtain an asymptotic expansion from (3.26), we rewrite

$$P(z) = \frac{1-u}{1+u}\sum_{r\geq 1}\frac{u^{2^r}}{(1-u^{2^r})(1-u^{2^r-1})} = \frac{u}{1+u}\sum_{r\geq 1}\left(\frac{u^{2^r-1}}{1-u^{2^r-1}} - \frac{u^{2^r}}{1-u^{2^r}}\right)$$

where $z = u/(1+u)^2$. The main task to obtain an asymptotic expansion of $P(z)$ is to provide a precise analysis of this sum, which we carry out via the Mellin transform. We consider the function

$$f(t) := \sum_{r\geq 1}\frac{e^{-(2^r-1)t}}{1-e^{-(2^r-1)t}} - \sum_{r\geq 1}\frac{e^{-2^r t}}{1-e^{-2^r t}},$$

obtained from substituting $u = e^{-t}$ in the sum above. With

$$A(s) := \sum_{r\geq 1}\frac{1}{2^{rs}}((1-2^{-r})^{-s}-1) = \sum_{\ell\geq 1}\binom{\ell+s-1}{\ell}\frac{1}{2^{s+\ell}-1}$$

we find that the corresponding Mellin transform of this difference of harmonic sums is given by

$$f^*(s) = \Gamma(s)\zeta(s)A(s)$$

with fundamental strip $\langle 1, \infty\rangle$. In order for the inversion formula to be valid, we need to show that $f^*(s)$ decays sufficiently fast along vertical lines in the complex plane. While $\Gamma(s)$ and $\zeta(s)$ are well-known to decay exponentially and grow polynomially along vertical lines, respectively, the Dirichlet series $A(s)$ has to be investigated in more detail.

We want to estimate the summands in

$$A(s) - \frac{s}{2^{s+1}-1} = \sum_{r\geq 1}\frac{1}{2^{rs}}\left((1-2^{-r})^{-s} - 1 - \frac{s}{2^r}\right).$$

To do so, we consider $g(x) = (1-x)^{-s}$ as a function of a real variable. By means of the integral form of the Taylor approximation error we find

$$\begin{aligned}
|g(2^{-r}) - g(0) - g'(0)\cdot 2^{-r}| &= \left|\int_0^{2^{-r}} s(s+1)(1-t)^{-s-2}(2^{-r}-t)\,dt\right| \\
&\leq |s||s+1|2^{-r}\int_0^{2^{-r}} |1-t|^{-\operatorname{Re}s-2}\,dt \\
&\leq |s||s+1|2^{-2r}(1-2^{-r})^{-\operatorname{Re}s-2},
\end{aligned}$$

where the last inequality is valid under the assumption that $\operatorname{Re} s > -2$. Using this estimate, we find

$$\begin{aligned} |A(s)| &\le \left| A(s) - \frac{s}{2^{s+1}-1} \right| + \left| \frac{s}{2^{s+1}-1} \right| \\ &\le \left| \frac{s}{2^{s+1}-1} \right| + |s||s+1| \sum_{r\ge 1} \frac{1}{(2^r-1)^{\operatorname{Re} s+2}}, \end{aligned}$$

where the sum converges for $\operatorname{Re} s > -2$. Therefore, $A(s)$ has polynomial growth in $\operatorname{Im} s$ for $\operatorname{Re} s > -2$ and $\operatorname{Im} s = \frac{2\pi i}{\log 2}\left(k + \frac{1}{2}\right)$, where $k \in \mathbb{Z}$ and $|k| \to \infty$, as well as on vertical lines with $\operatorname{Re} s > -2$ and $\operatorname{Re} s \ne -1$. This implies that $f^*(s)$ decays sufficiently fast, and thus the inversion formula states

$$f(t) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s)\zeta(s)A(s)t^{-s}\, ds, \tag{3.29}$$

which is valid for real, positive $t \to 0$ (and thus $u \to 1^-$ and $z \to (1/4)^-$, as we have $z = u/(1+u)^2$ and $u = e^{-t}$). In order to extract the coefficient growth (in terms of z) with the help of singularity analysis, we require analyticity in a larger region (cf. [18]), e.g. in a complex punctured neighborhood of $1/4$ with[1] $|\arg(z-1/4)| > 2\pi/5$.

Substituting back t for z, we find

$$t = -\log\left(\frac{1-\sqrt{1-4z}}{2z} - 1 \right) = 2\sqrt{1-4z} + \frac{2}{3}(1-4z)^{3/2} + O((1-4z)^{5/2}),$$

which implies

$$|\arg t| = \frac{1}{2}|\arg(1-4z)| + o(1)$$

such that we have the bound $|\arg t| < 2\pi/5$ for $t \to 0$, given that the restriction on the argument in terms of z is satisfied.

With the help of our estimates on $f^*(s)$ that we discussed above, we find that

$$|f^*(s)t^{-s}| = O\left(|\mathrm{Im}(t)|^4 |t|^{-\mathrm{Re}(s)} \exp\left(-\frac{\pi}{10}|\mathrm{Im}(s)| \right) \right) \tag{3.30}$$

for $-3/2 \le \operatorname{Re} s \le 2$ and $\operatorname{Im} s = \frac{2\pi i}{\log 2}\left(k + \frac{1}{2}\right)$, where $k \in \mathbb{Z}$ and $|k| \to \infty$. This is a consequence of combining the quantified growth of $\Gamma(s)$ (see [10, 5.11.3]) and the growth of $\zeta(s)$ (see [69, 13.51]) with the facts that $A(s)$ is of order $O(\mathrm{Im}(s)^2)$ and $\frac{s}{2^{s+1}-1}$ is of order $O(\mathrm{Im}(s))$ for s taking values in the specified region.

We can evaluate (3.29) by shifting the line of integration from $\mathrm{Re}(s) = 2$ to $\mathrm{Re}(s) = -3/2$ and collecting the residues of the poles we cross. This yields

$$f(t) = \sum_{p\in P} \operatorname{Res}_{s=p}(f^*(s)t^{-s}) + \frac{1}{2\pi i} \int_{-3/2-i\infty}^{-3/2+i\infty} f^*(s)t^{-s}\, ds,$$

[1]Note that the bound $2\pi/5$ is somewhat arbitrary: the argument just needs to be less than $\pi/2$.

where $P = \{-1, 1\} \cup \{-1 + \chi_k \mid k \in \mathbb{Z} \setminus \{0\}\}$. For the error term we use the estimate above and find

$$\frac{1}{2\pi i} \int_{-3/2-i\infty}^{-3/2+i\infty} f^*(s) t^{-s}\, ds = O(|t|^{3/2}).$$

Evaluating the residues yields

$$\begin{aligned} f(t) = A(1)t^{-1} + \frac{1}{12\log 2} t \log t + \frac{\log 2 + 2\gamma + 24\zeta'(-1)}{24\log 2} t \\ + \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{\log 2} \Gamma(\chi_k)\zeta(-1+\chi_k) t^{1-\chi_k} + O(|t|^{3/2}). \end{aligned}$$

Note that with $\alpha := \sum_{k\geq 1} 1/(2^k - 1)$, we have $A(1) = \alpha - 1$.

When substituting back in order to obtain an expansion in terms of $z \to 1/4$, we have to carefully check that the error terms within the sum of the residues at χ_k for $k \in \mathbb{Z} \setminus \{0\}$ can still be controlled. Considering that for some exponent κ, we have the expansion

$$t^{-\kappa} = (1-4z)^{-\kappa/2}(1 + O(1-4z))^{-\kappa/2},$$

and thus

$$\begin{aligned} |(1 + O(1-4z))^{-\kappa/2} - 1| &= \left|\exp\left(-\frac{\kappa}{2}\log(1 + O(1-4z))\right) - 1\right| \\ &\leq \left|\frac{\kappa}{2}\right| |\log(1 + O(1-4z))| \exp\left(\left|\frac{\kappa}{2}\right| |\log(1 + O(1-4z))|\right) \\ &= \left|\frac{\kappa}{2}\right| O(1-4z) \exp\left(\left|\frac{\kappa}{2}\right| O(1-4z)\right). \end{aligned}$$

Setting $\kappa = -1 + \chi_k$ shows that the errors that we sum are of order $O(|k|(1-4z)\exp(|k|O(1-4z)))$. Choosing z sufficiently close to $1/4$ ensures that the exponential growth is negligible compared to the exponential decay proved in (3.30).

Finally, it is easy to see that the factor $\frac{u}{1+u}$ can be rewritten as $\frac{1-\sqrt{1-4z}}{2}$. Multiplying our expansion of $f(t)$ with this factor and substituting back yields the expansion

$$\begin{aligned} P(z) = \frac{\alpha - 1}{4}(1-4z)^{-1/2} - \frac{\alpha-1}{4} - \frac{1}{24\log 2}(1-4z)^{1/2}\log(1-4z) \\ + \frac{2\gamma - 2\alpha\log 2 + 5\log 2 + 24\zeta'(1)}{24\log 2}(1-4z)^{1/2} \\ + \frac{1}{\log 2}\sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma(\chi_k)\zeta(-1+\chi_k)(1-4z)^{(1-\chi_k)/2} + O((1-4z)^{3/4}). \end{aligned}$$

Applying singularity analysis, normalizing the result by C_{n-1}, and rewriting the coefficients of the contributions from the poles at $-1 + \chi_k$ via the duplication formula for the Gamma function (cf. [10, 5.5.5]) then proves the asymptotic expansion for $\mathbb{E}P_n$. □

3.4 Cutting Old Leaves

3.4.1 Preliminaries

In this section we consider a slightly more complex reduction: instead of removing all leaves, we just remove all leftmost leaves. Following [6], we call a leaf that is a leftmost child an *old leaf*.

In order to describe the corresponding expansion in the language of generating functions, we need to change our underlying combinatorial model of trees in a way that specifically marks old leaves.

Let $\mathcal{L}$ be the combinatorial class of plane trees where $\blacksquare$ marks old leaves and $\bullet$ marks all nodes that are neither old leaves nor parents thereof. Now, as a first step we determine the bivariate generating function $L(z,w)$ of $\mathcal{L}$.

Proposition 3.4.1.
The generating function $L(z,w)$ enumerating plane trees with respect to old leaves $\blacksquare$ (marked by the variable w) and all nodes $\bullet$ that are neither old leaves nor parents thereof (marked by z) is given by

$$L(z,w) = \frac{1-\sqrt{1-4z-4w+4z^2}}{2}. \tag{3.31}$$

For $n \geq 2$ there are $C_{k-1}\binom{n-2}{n-2k}2^{n-2k}$ plane trees of size n (meaning n nodes overall) with k old leaves.

For example in Figure 3.5, the original tree corresponds to z^3w^3 because it has three old leaves (dashed nodes) and three nodes which are neither old leaves nor parents of old leaves.

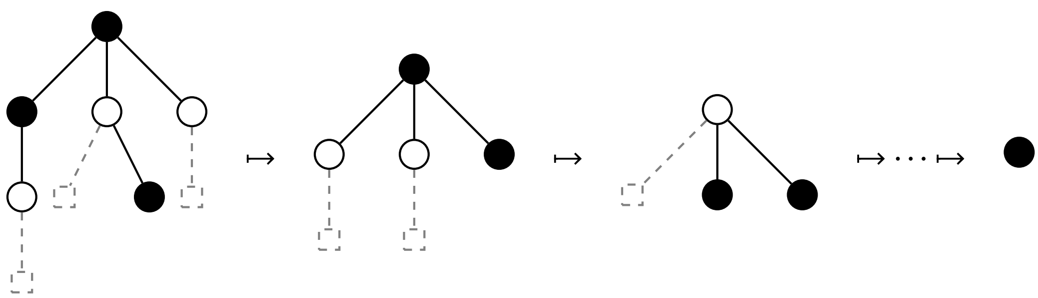


Figure 3.5: Illustration of the "cutting old leaves"-operator ρ.

Proof. We consider the symbolic equation describing the combinatorial class $\mathcal{L}$ of plane trees with respect to old leaves, which is illustrated in Figure 3.6. The functional equation that

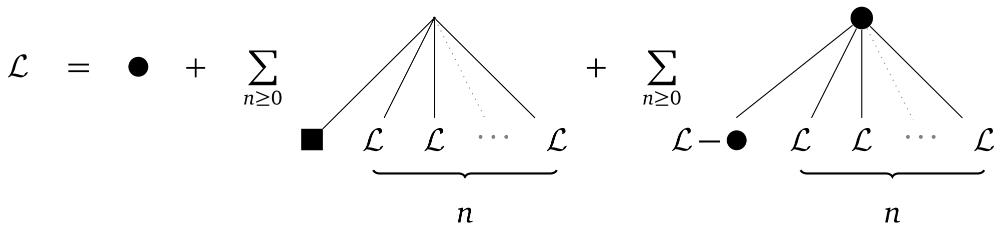


Figure 3.6: Symbolic equation for plane trees w.r.t. old leaves.

can be derived from the symbolic equation by marking ■ with w and ● with z is

$$L(z,w) = z + \frac{w + z(L(z,w) - z)}{1 - L(z,w)}. \tag{3.32}$$

Solving this equation and choosing the correct branch of the root yields (3.31).

To extract coefficients of $L(z,w)$, we rewrite it as

$$L(z,w) = \frac{1}{2}\left(1 - (1-2z)\sqrt{1 - \frac{4w}{(1-2z)^2}}\right) = \frac{1}{2}\left(1 - \sum_{k\geq 0}\binom{1/2}{k}\frac{(-1)^k 4^k w^k}{(1-2z)^{2k-1}}\right) \tag{3.33}$$

$$= z + \sum_{k\geq 1} C_{k-1}\frac{w^k}{(1-2z)^{2k-1}} = z + \sum_{\substack{k\geq 1\\ n\geq 0}} C_{k-1}\binom{n+2k-2}{n} 2^n w^k z^n. \tag{3.34}$$

□

As we will see in the next section, the polynomials defined below will play a similar role for the “old leaves”-reduction as the Fibonacci polynomials played for the “leaves”- and “paths”-reduction.

Definition 3.4.2.
The polynomials $B_r(z)$ are the generating functions of binary trees w.r.t. the number of internal nodes of height $\leq r$ satisfying

$$B_r(z) = 1 + zB_{r-1}(z)^2 \tag{3.35}$$

for $r \geq 1$ and $B_0(z) = 1$.

3.4.2 Expansion Operator and Asymptotic Results

As described in the previous section, we now concentrate on the reduction $\rho\colon \mathcal{L} \to \mathcal{L}$, which removes all old leaves from a tree. Note that $\rho(\bullet) = \bullet$, as the root itself is not an old leaf. We begin our analysis of this reduction by determining the expansion operator Φ.

Proposition 3.4.3.
Let $\mathcal{F} \subseteq \mathcal{L}$ be a family of plane trees with bivariate generating function $f(z,w)$, where z

marks nodes that are neither old leaves nor parents thereof and w marks old leaves. Then the generating function for $\rho^{-1}(\mathcal{F})$, the family of trees whose reduction is in $\mathcal{F}$, is given by

$$\Phi(f(z,w)) = f(z+w,(2z+w)w). \tag{3.36}$$

Proof. Linearity of Φ is obvious from the combinatorial interpretation, meaning that we can focus on the expansion of any tree represented by $z^n w^k$, i.e. a tree with n nodes that are neither old leaves nor parents thereof and k old leaves.

Figure 3.7 illustrates all three possibilities to expand an old leaf ■:

- appending an old leaf to the parent of ■, which turns the original old leaf into ●,
- appending an old leaf to ■ itself, which turns the parent into ●,
- appending old leaves both to ■ and its parent.

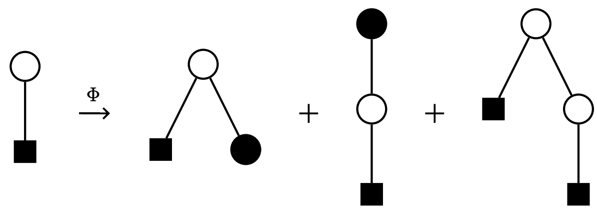


Figure 3.7: All possible expansions of an old leaf.

In terms of generating functions, this means that w is substituted by $2zw+w^2$.

Furthermore, the nodes represented by ● can optionally be expanded by attaching an old leaf to them, otherwise they stay as they are. This option corresponds to the substitution $z \mapsto z+w$.

There are no more operations to expand the tree, so putting everything together yields

$$\Phi(z^n w^k) = (z+w)^n(2zw+w^2)^k,$$

which proves the statement. □

An immediate consequence of the fact that $\rho\colon \mathcal{L} \to \mathcal{L}$ is surjective is the following corollary.

Corollary 3.4.4.
The generating function for plane trees $L(z,w)$ satisfies the functional equation

$$\Phi(L(z,w)) = L(z,w).$$

We now focus on determining the generating function measuring the change in the tree size after repeatedly applying the reduction ρ.

Proposition 3.4.5.
Let $r \in \mathbb{N}_0$. The bivariate generating function $G_r(z,v) = G_r^{\mathrm{OL}}(z,v)$ enumerating plane trees, where z marks the tree size and v marks the size of the r-fold cut tree, is given by

$$G_r(z,v) = \Phi^r(L(zv, wv^2))|_{w=z^2} = L(zB_r(z)v, z(B_{r+1}(z) - B_r(z))v^2),$$

where the $B_r(z)$ are the polynomials enumerating binary trees of height $\le r$ w.r.t. the number of internal nodes.

Proof. First, note that the size of a tree with k old leaves and n nodes that are neither old leaves nor parents thereof is actually $n+2k$, as parents of old leaves are not explicitly marked. This explains why we have to substitute $w = z^2$ in order to arrive at the tree size.

In contrast to the previous sections, the operator Φ is already linear and multiplicative, meaning that we have

$$\Phi^r(z^n w^k) = \Phi^r(z)^n \Phi^r(w)^k.$$

Investigating the repeated application of Φ to z and w leads to the recurrences

$$\Phi^r(z) = \Phi^{r-1}(z) + \Phi^{r-1}(z)^2 - \Phi^{r-2}(z)^2 \quad \text{and} \quad \Phi^r(w) = \Phi^{r+1}(z) - \Phi^r(z)$$

for $r \ge 2$ and $r \ge 0$, respectively. With the recurrence for the polynomials B_r from (3.35) it is easy to prove by induction that

$$\Phi^r(z)|_{w=z^2} = zB_r(z)$$

for $r \ge 0$. Thus, we also find $\Phi^r(w)|_{w=z^2} = z(B_{r+1}(z) - B_r(z))$. Overall, we obtain

$$\Phi^r(z^n w^k)|_{w=z^2} = z^{n+k} B_r(z)^n (B_{r+1}(z) - B_r(z))^k,$$

which, by linearity of Φ, proves the proposition. □

For the next step in our analysis, we turn to the random variable $X_{n,r} = X_{n,r}^{\mathrm{OL}}$ which models the size of the tree that results from reducing a random tree τ with n nodes r-times.

As we have $\rho(\bullet) = \bullet$ (and thus no trees vanish completely), the probability generating function for this random variable is simply

$$\mathbb{E}v^{X_{n,r}} = \frac{[z^n]G_r(z,v)}{C_{n-1}}.$$

While the height polynomials $B_r(z)$ make it very difficult to obtain general results for the factorial moments of $X_{n,r}$, special moments like expectation and variance are no problem, and even a central limit theorem is possible.

Theorem 3.4.6.
Let $r \in \mathbb{N}_0$ be fixed and consider $n \to \infty$. Then the expected tree size after deleting old leaves of a tree with n nodes r-times and the corresponding variance are given by

$$\mathbb{E}X_{n,r} = (2 - B_r(1/4))n - \frac{B_r'(1/4)}{8} + O(n^{-1}), \tag{3.37}$$

and

$$\mathbb{V}X_{n,r} = \Big(B_r(1/4) - B_r(1/4)^2 + \frac{(2 - B_r(1/4))B_r'(1/4)}{2}\Big)n + O(1). \tag{3.38}$$

All O-constants in this theorem depend implicitly on r.

Additionally, the random variable $X_{n,r}$ is asymptotically normally distributed for fixed $r \geq 1$, i.e.

$$\frac{X_{n,r} - \mu n}{\sqrt{\sigma^2 n}} \xrightarrow{d} \mathcal{N}(0,1),$$

where $\mu = (2 - B_r(1/4))$ and $\sigma^2 = \big(B_r(1/4) - B_r(1/4)^2 + \frac{(2-B_r(1/4))B_r'(1/4)}{2}\big)$.

Proof. First of all, we observe that Proposition 3.4.1 and Proposition 3.4.5 combined with the recursion $B_r(z) = 1 + zB_{r-1}(z)^2$ allow us to write the bivariate generating function as

$$G_r(z,v) = \frac{1 - \sqrt{1 - 4zv(B_r(z)(1-v) + v)}}{2}.$$

The asymptotic expansion for the expected value $\mathbb{E}X_{n,r}$ can now be obtained by determining

$$\frac{1}{C_{n-1}}[z^n]\frac{\partial}{\partial v}G_r(z,v)|_{v=1} = \frac{1}{C_{n-1}}[z^n]\frac{z(2 - B_r(z))}{\sqrt{1-4z}}.$$

By means of singularity analysis we find

$$\mathbb{E}X_{n,r} = (2 - B_r(1/4))n - \frac{B_r'(1/4)}{8} - \Big(\frac{3B_r'(1/4)}{16} + \frac{3B_r''(1/4)}{128}\Big)n^{-1} + O(n^{-2}),$$

which proves (3.37). For the second factorial moment we obtain

$$\mathbb{E}X_{n,r}^{\underline{2}} = \frac{1}{C_{n-1}}[z^n]\frac{\partial^2}{\partial v^2}G_r(z,v)|_{v=1} = \frac{1}{C_{n-1}}[z^n]\Big(\frac{2z^2(2 - B_r(z))}{(1-4z)^{3/2}} + \frac{2z(1 - B_r(z))}{(1-4z)^{1/2}}\Big),$$

which yields

$$\begin{aligned}\mathbb{E}X_{n,r}^{\underline{2}} = (2 - B_r(1/4))^2 n^2 + \Big(2B_r(1/4) - B_r(1/4)^2 - 2 + \frac{(2 - B_r(1/4))B_r'(1/4)}{4}\Big)n \\ + \frac{(2 - B_r(1/4))B_r''(1/4)}{64} - \frac{B_r'(1/4)^2}{64} - \frac{B_r(1/4)B_r'(1/4)}{8} + O(n^{-1}).\end{aligned}$$

The variance can now be obtained via $\mathbb{V}X_{n,r} = \mathbb{E}X_{n,r}^{\underline{2}} + \mathbb{E}X_{n,r} - (\mathbb{E}X_{n,r})^2$, which proves (3.38).

In order to show asymptotic normality of $X_{n,r}$ we investigate the random variable $n - X_{n,r}$, which counts the number of nodes that are deleted after reducing some tree r times. Observe that this quantity can be seen as an additive tree parameter F_r defined recursively by

$$F_r(\tau_n) = F_r(\tau_{i_1}) + F_r(\tau_{i_2}) + \cdots + F_r(\tau_{i_\ell}) + f_r(\tau_n) \quad \text{and} \quad F_r(\bullet) = 0$$

where τ_n is some tree of size n, τ_{i_1} up to τ_{i_ℓ} are the subtrees rooted at the children of the root of τ_n, and $f_r : \mathcal{L} \to \{0, 1, \ldots, r-1\}$ is a toll function defined by

$$f_r(\tau_n) = \sum_{j=0}^{r-1} \begin{cases} 1 & \text{if } \rho^j(\tau_n) \text{ has an old leaf attached to its root,} \\ 0 & \text{otherwise,} \end{cases}$$

for $r \geq 1$. Now, as $f_r(\tau_n)$ enumerates the number of old leaves deleted from the root of τ_n after r reductions, $F_r(\tau_n)$ equals the total number of deleted nodes after r reductions.

The fact that r is fixed implies that f_r is not only bounded, but also a so-called *local functional*, meaning that the value of $f_r(\tau_n)$ can already be determined from the first r levels of τ_n. This is because one application of ρ can reduce the distance between the root of the tree and the closest old leaf by at most one. Thus all old leaves that are deleted from the root during r reductions have to be found within the first r levels of τ_n.

As we have now established that f_r is both bounded and a local functional, we are able to apply [33, Theorem 1.13], which proves that $n - X_{n,r}$ is asymptotically normally distributed. Thus $X_{n,r}$ is asymptotically normally distributed as well, which proves the statement. □

Remark.
In [17], the asymptotic behavior of a sequence strongly related to $B_r(1/4)$ was studied: in Section 4, the authors define a sequence f_n such that $f_{r+1} = \frac{1}{2} - \frac{B_r(1/4)}{4}$, in our notation. They prove the asymptotic expansion $f_n = \frac{1}{n + \log n + O(1)}$. This allows us to conclude that the asymptotic behavior of $B_r(1/4)$ can be described as

$$B_r(1/4) = 2 - \frac{4}{r} + \frac{4 \log r}{r^2} + O(r^{-2})$$

for $r \to \infty$.

3.5 Cutting Old Paths

3.5.1 Expansion Operator

As in previous sections, we adapt the “old leaves” reduction to remove all “old paths”. That is, the tree reduction $\rho : \mathcal{L} \to \mathcal{L}$ in this section reduces a tree by removing all paths that end in an old leaf. This operation is illustrated in Figure 3.8, where ■ marks old leaves and ● marks all nodes that are neither old leaves nor parents thereof.

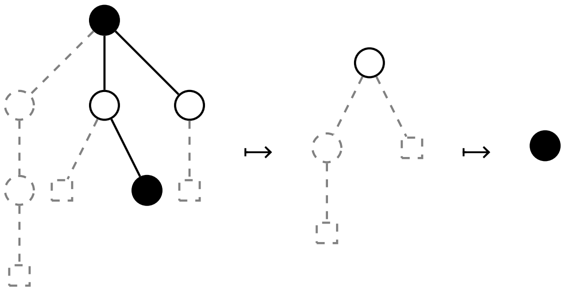


Figure 3.8: Illustration of the “cutting old paths”-operator ρ.

Obviously, we also need the combinatorial class of paths $\mathcal{P}$ for our analysis. The bivariate generating function of $\mathcal{P}$ is given by $P = P(z,w) = \frac{w}{1-z}$, where w and z mark ■ and ●, respectively. Also, we omit the arguments of P for the sake of readability. Now, we determine the shape of the expansion operator Φ.

Proposition 3.5.1.
Let $\mathcal{F} \subseteq \mathcal{L}$ be a family of plane trees with bivariate generating function $f(z,w)$, where z marks nodes that are neither old leaves nor parents thereof and w marks old leaves. Then the generating function for $\rho^{-1}(\mathcal{F})$, the family of trees whose reduction is in $\mathcal{F}$, is given by

$$\Phi(f(z,w)) = f(z+P, zP+P^2). \tag{3.39}$$

Proof. With linearity of the operator Φ being obvious from a combinatorial point of view, we only have to investigate the expansion of any tree represented by $z^n w^k$, i.e. a tree with n nodes that are neither old leaves nor parents thereof and k old leaves.

There are two options to expand an old leaf ■:

- either appending an old path to the parent of ■, which turns the old leaf into ●,
- or an old path is appended to both the parent of ■ and to ■ itself.

Note that just appending an old path to ■ is not a valid expansion as this introduces ambiguity. This is the same argument that we also used in the proof of Proposition 3.3.1. Overall, this means that Φ has to map w to $zP + P^2$.

On the other hand, the nodes represented by ● can optionally be expanded by attaching an old path. Otherwise they stay as they are. Overall, this implies $\Phi(z) = z + P$.

Putting everything together, we immediately arrive at the statement of the Proposition. □

Analogously to the previous reductions, surjectivity of $\rho\colon \mathcal{L} \to \mathcal{L}$ implies the following corollary.

Corollary 3.5.2.
The generating function for plane trees $L(z,w)$ satisfies the functional equation

$$\Phi(L(z,w)) = L(z,w).$$

In order to carry out a detailed analysis of this reduction, we need information about the iterated application of Φ to $L(zv_I, wv_L^2)$, which leads to the generating function $G_r(z, v_I, v_L^2)$ measuring the change in the tree size after r applications of the reduction. The following proposition deals with determining this generating function.

Proposition 3.5.3.
Let $r \in \mathbb{N}_0$. The trivariate generating function $G_r(z, v_I, v_L^2) = G_r^{\mathrm{OP}}(z, v_I, v_L^2)$ enumerating plane trees, where z marks the tree size, v_L marks all old leaves, and v_I marks all nodes that are neither old leaves nor parents thereof, is given by

$$G_r(z, v_I, v_L^2) = \Phi^r(L(zv_I, wv_L^2))|_{w=z^2} = L\left(\frac{u(1-u^{r+1})}{(1+u)(1-u^{r+2})}v_I, \frac{u^{r+2}(1-u)^2}{(1+u)^2(1-u^{r+2})^2}v_L^2\right),$$

where $z = u/(1+u)^2$.

Proof. Observe that the operator Φ is already linear and multiplicative, which is why we can concentrate on finding suitable expressions for $\Phi^r(z)$ and $\Phi^r(w)$.

First of all, for $r \geq 1$ the recurrences

$$\Phi^r(z) = \Phi^{r-1}(z) + \Phi^{r-1}(P), \qquad \Phi^r(w) = \Phi^{r-1}(P)\Phi^r(z)$$

follow immediately from (3.39). Furthermore, the relation

$$\Phi^r(P) = P\prod_{j=1}^{r}\frac{\Phi^j(z)}{1-\Phi^j(z)}$$

can easily be proved by induction. Then, by setting $f_r := \Phi^r(z)|_{w=z^2}$ the recurrences above translate to

$$f_r = f_{r-1} + z\prod_{j=0}^{r-1}\frac{f_j}{1-f_j}.$$

As a next step, we show by induction that f_r can be expressed in terms of Fibonacci polynomials as

$$f_r = \frac{zF_{r+1}(-z)}{F_{r+2}(-z)},$$

where in particular (3.7) was used. As a consequence, we find

$$\Phi^r(P)|_{w=z^2} = f_{r+1} - f_r = \frac{zF_{r+2}(-z)}{F_{r+3}(-z)} - \frac{zF_{r+1}(-z)}{F_{r+2}(-z)} = \frac{z^{r+2}}{F_{r+2}(-z)F_{r+3}(-z)}.$$

This allows us to express $g_r := \Phi^r(w)|_{w=z^2}$ as

$$g_r = \Phi^{r-1}(P)|_{w=z^2} \cdot f_r = \frac{z^{r+2}}{F_{r+2}(-z)^2}.$$

Finally, as we have $\Phi^r(z^n w^k)|_{w=z^2} = f_r^n g_r^k$, substituting $z = u/(1+u)^2$ and using (3.8) completes the proof. □

3.5.2 Analysis of Tree Size and Related Parameters

We investigate the behavior of the random variable $X_{n,r} = X_{n,r}^{\mathrm{OP}}$ which models the number of nodes remaining after reducing a random tree τ with n nodes r-times. The tree τ is chosen uniformly among all trees of size n. Analogously to the "old leaf"-reduction from the previous section, we also have $\rho(\bullet) = \bullet$ for the "old path"-reduction, meaning that no trees vanish completely. For the sake of convenience we set $G_r(z,v) := G_r(z,v,v^2)$, allowing us to write the probability generating function of $X_{n.r}$ as

$$\mathbb{E}v^{X_{n,r}} = \frac{[z^n]G_r(z,v)}{C_{n-1}}.$$

With the help of Proposition 3.5.3, it is easy to obtain expressions for the factorial moments $\mathbb{E}X_{n,r}^{\underline{d}}$ for fixed d by differentiating $G_r(z,v)$ d-times with respect to v and setting $v = 1$ afterwards. General expressions for $d \geq 2$ (coinciding with the value given for $d = 2$) are available but less pleasant.

Lemma 3.5.4 ([28]).
The factorial moments of $X_{n,r}$ are

$$\mathbb{E}X_{n,r} = \frac{1}{C_{n-1}}[z^n]\frac{u(1+u^{r+1})}{(1+u)(1-u^{r+2})},$$
$$\mathbb{E}X_{n,r}(X_{n,r}-1) = \frac{2}{C_{n-1}}[z^n]\frac{(1+u)u^{r+2}}{(1-u)(1-u^{r+2})^2}$$

and

$$\mathbb{E}X_{n,r}^{\underline{d}} = \frac{d!}{C_{n-1}}[z^n]\frac{1-u}{1+u}\left(\frac{u(1+u^{r+1})}{(1-u)(1-u^{r+2})} + \frac{u}{1-u}\sqrt{\frac{1-u^r}{1-u^{r+2}}}\right)^d$$
$$\times \tilde{N}_{d-1}\left(\frac{2u^{2r+2}-u^{r+2}+2u^{r+1}-u^r+2}{(1+u)^2u^r} + \frac{2(1+u^{r+1})(1-u^{r+2})}{u^r(1+u)^2}\sqrt{\frac{1-u^r}{1-u^{r+2}}}\right)$$

for $d \geq 2$.

Proof. The expressions for $d \in \{1,2\}$ can be obtained by differentiation. We consider the general case here.

We use the abbreviations

$$a=\frac{u(1-u^{r+1})}{(1+u)(1-u^{r+2})},\qquad b=\frac{u^{r+2}(1-u)^2}{(1+u)^2(1-u^{r+2})^2},\qquad \Delta=\frac{1-u}{1+u}.$$

By the same argument as in the proof of Proposition 3.2.11, we have

$$\frac{\partial^d}{\partial v^d}G_r(z,v)\bigg|_{v=1}=d![q^d]L(a(1+q),b(1+q)^2).$$

By using (3.31), we rewrite $L(a(1+q),b(1+q)^2)$ as

$$\begin{aligned}L(a(1+q),b(1+q)^2)&=\frac{1-\sqrt{(1-2a-2aq)^2-4b(1+q)^2}}{2}\\&=\frac{1-\sqrt{(1-2a)^2-4b-2q(2a(1-2a)+4b)+q^2(4a^2-4b)}}{2}.\end{aligned}$$

We have

$$(1-2a)^2-4b=\Delta^2,\qquad \sqrt{\frac{4a^2-4b}{\Delta^2}}=\frac{2u}{1-u}\sqrt{\frac{1-u^r}{1-u^{r+2}}}.$$

We choose α and β such that

$$\alpha+\beta=\frac{2a(1-2a)+4b}{\Delta^2},\qquad \alpha-\beta=\sqrt{\frac{4a^2-4b}{\Delta^2}}.$$

This results in

$$\begin{aligned}L(a(1+q),b(1+q)^2)&=\Delta\frac{\frac{1}{\Delta}-1+q(\alpha-\beta)}{2}+\Delta\frac{1-q(\alpha-\beta)-\sqrt{1-2q(\alpha+\beta)+q^2(\alpha-\beta)}}{2}\\&=\Delta\frac{\frac{1}{\Delta}-1+q(\alpha-\beta)}{2}+\Delta T(\alpha q,\beta q).\end{aligned}$$

Using (3.3) to extract the coefficient of q^d for $d\geq 1$ yields

$$\frac{\partial^d}{\partial v^d}G_r(z,v)\bigg|_{v=1}=d!\Delta\left(\frac{\alpha-\beta}{2}[\![d=1]\!]+\alpha^d N_{d-1}\left(\frac{\beta}{\alpha}\right)\right).$$

Inserting everything concludes the proof of the proposition. □

Corollary 3.5.5.
The expected value of $X_{n+1,r}$ is explicitly given by

$$\mathbb{E}X_{n+1,r}=\frac{1}{C_n}\left(\binom{2n}{n}+\sum_{j\geq 0}\left(\binom{2n}{n-(j+1)(r+2)+1}-\binom{2n}{n-j(r+2)-1}\right)\right)$$

Proof. From Lemma 3.5.4, we obtain

$$C_n\mathbb{E}X_{n+1,r}=[z^{n+1}]\frac{(1+u^{r+1})u}{(1+u)(1-u^{r+2})}$$

and proceeding as in Corollary 3.2.12 we obtain the given result. □

By expanding the expressions in Lemma 3.5.4 and using singularity analysis, we obtain the asymptotic growth of the expected value and the variance.

Theorem 3.5.6.
Let $r \in \mathbb{N}$ be fixed and consider $n \to \infty$. Then the expected size and the corresponding variance of an r-fold cut plane tree are given by

$$\mathbb{E}X_{n,r} = \frac{2n}{r+2} - \frac{r(r+1)}{3(r+2)} + O(n^{-1}),$$

and

$$\mathbb{V}X_{n,r} = \frac{2r(r+1)}{3(r+2)^2} n + O(1).$$

For $d \geq 3$, the dth factorial moment is

$$\mathbb{E}X_{n,r}^{\underline{d}} = \frac{2^{d-1}d}{(2d-3)(r+2)^d} n^d + \binom{2d-5}{d-2} \frac{\sqrt{r\pi}\, d}{2^{d-3}(r+2)^{d-1/2}} n^{d-\frac{1}{2}} + O(n^{d-1}).$$

All O-constants in this theorem depend implicitly on r.

Besides the analysis of the tree size, we are also interested in how the numbers of nodes represented by ■ and by ● develop when the tree is reduced repeatedly. Formally, this means that we consider the random variables $X_{n,r}^{\blacksquare}$ and $X_{n,r}^{\bullet}$ counting the number of old leaves and the number of all nodes that are neither old leaves nor parents thereof, respectively. By construction, the relation

$$X_{n,r} = 2 \cdot X_{n,r}^{\blacksquare} + X_{n,r}^{\bullet} \tag{3.40}$$

holds.

The bivariate generating functions corresponding to these random variables can be obtained directly from Proposition 3.5.3. We have

$$G_r^{\blacksquare}(z,v) = G_r(z,1,v), \qquad G_r^{\bullet}(z,v) = G_r(z,v,1).$$

In contrast to $X_{n,r}$, the dth factorial moments for $X_{n,r}^{\blacksquare}$ and $X_{n,r}^{\bullet}$ have simpler expressions.

Proposition 3.5.7 ([28]).
Let $d \in \mathbb{N}$. Then the dth factorial moments of $X_{n,r}^{\blacksquare}$ and $X_{n,r}^{\bullet}$ are given by

$$\mathbb{E}X_{n,r}^{\blacksquare\,\underline{d}} = \frac{(2d-2)^{\underline{d-1}}}{C_{n-1}} [z^n] \frac{1-u}{1+u} \frac{u^{rd+2d}}{(1-u^{r+2})^{2d}} \tag{3.41}$$

and

$$\mathbb{E}X_{n,r}^{\bullet} = \frac{1}{C_{n-1}} [z^n] \frac{u(1-u^{r+1})(1+u^{r+2})}{(1+u)(1-u^{r+2})^2} \tag{3.42}$$

as well as

$$\mathbb{E}X_{n,r}^{\bullet\,\underline{d}} = \frac{1}{C_{n-1}} [z^n] \frac{(1-u^{r+1})^d u^d 2^d d!}{(1-u)^{d-1}(1+u)(1-u^{r+2})^{2d}} \tilde{N}_{d-1}(u^{r+2}) \tag{3.43}$$

for $d > 1$.

Proof of Proposition 3.5.7. As in the proof of Lemma 3.5.4, we use the abbreviations

$$a = \frac{u(1-u^{r+1})}{(1+u)(1-u^{r+2})}, \qquad b = \frac{u^{r+2}(1-u)^2}{(1+u)^2(1-u^{r+2})^2}, \qquad \Delta = \frac{1-u}{1+u}.$$

Then, using (3.33), we get

$$\frac{\partial^d}{\partial v^d} G_r^{\blacksquare}(z,v) = \frac{\partial^d}{\partial v^d} L(a, bv) = -\frac{1-2a}{2}\binom{\frac{1}{2}}{d}\left(1-\frac{4bv}{(1-2a)^2}\right)^{1/2-d} \frac{(-4b)^d}{(1-2a)^{2d}}.$$

Setting $v=1$ and using the fact that

$$(1-2a)^2 - 4b = \Delta^2 \tag{3.44}$$

proves (3.41).

For deriving $\partial^d/(\partial v)^d G_r^{\bullet}(z,v)$, we proceed as in the proof of Proposition 3.2.11. The crucial identity is

$$L(a(1+q), b) = \Delta \frac{\frac{1}{\Delta} - 1 + (\alpha' - \beta')q}{2} + \Delta T(\alpha' q, \beta' q)$$

with

$$\alpha' = \frac{2u(1-u^{r+1})}{(1-u^{r+2})^2(1-u)}, \qquad \frac{\beta'}{\alpha'} = u^{r+2}.$$

This implies (3.42) and (3.43).

In fact, we have

$$\begin{aligned} L(a(1+q), b) &= \frac{1-\sqrt{(1-2a-2aq)^2 - 4b}}{2} \\ &= \frac{1-\sqrt{(1-2a)^2 - 4b - 2q(2a-4a^2) + q^2 4a^2}}{2}. \end{aligned}$$

We use (3.44) and choose α' and β' such that

$$\alpha' + \beta' = \frac{2a-4a^2}{\Delta^2}, \qquad \alpha' - \beta' = \frac{2a}{\Delta}.$$

This results in

$$\begin{aligned} L(a(1+q), b) &= \Delta \frac{\frac{1}{\Delta} - 1}{2} + \Delta \frac{1-\sqrt{1-2q(\alpha'+\beta') + q^2(\alpha'-\beta')^2}}{2} \\ &= \Delta \frac{\frac{1}{\Delta} - 1 + (\alpha'-\beta')q}{2} + \Delta \frac{1-(\alpha'-\beta')q - \sqrt{1-2q(\alpha'+\beta') + q^2(\alpha'-\beta')^2}}{2} \\ &= \Delta \frac{\frac{1}{\Delta} - 1 + (\alpha'-\beta')q}{2} + \Delta T(\alpha' q, \beta' q). \end{aligned}$$

By (3.3), extracting coefficients leads to

$$C_{n-1} \frac{1}{d!} \mathbb{E} X_{n,r}^{\bullet\,\underline{d}} = [z^n] \Delta \left(\frac{\alpha'-\beta'}{2} [\![d=1]\!] + \alpha'^d \tilde{N}_{d-1}\left(\frac{\beta'}{\alpha'}\right)\right)$$

for $d \geq 1$. As

$$\frac{\alpha' + \beta'}{2} = \frac{u(1-u^{r+1})(1+u^{r+2})}{(1-u^{r+2})^2(1-u)},$$

the result follows. $\square$

As in Section 3.2.3, the above proof exhibits some identities:

Remark.
For $d \in \mathbb{Z}_{\geq 1}$, the power series identities

$$\sum_{\substack{n\geq 0\\ k\geq 1}} \frac{u^{n+k}x^k(1-x)^n(1-u)^{2k}}{(1+u)^{n+2k}(1-ux)^{n+2k}} k^{\underline{d}} C_{k-1} \binom{n+2k-2}{n} 2^n = (2d-2)^{\underline{d-1}} \frac{u^d x^d (1-u)}{(1-ux)^{2d}(1+u)} \tag{3.45}$$

and

$$\sum_{\substack{n\geq 0\\ k\geq 1}} \frac{u^{n+k}x^k(1-x)^n(1-u)^{2k}}{(1+u)^{n+2k}(1-ux)^{n+2k}} n^{\underline{d}} C_{k-1} \binom{n+2k-2}{n} 2^n = \frac{(1-x)^d u^d 2^d d!\, \tilde{N}_{d-1}(ux)}{(1-u)^{d-1}(1+u)(1-ux)^{2d}} \tag{3.46}$$

hold.

Proof. We replace u^{r+1} by x in the proof of Proposition 3.5.7 and expand L by (3.34). $\square$

The asymptotic behavior for the factorial moments of $X^{\blacksquare}_{n,r}$ and $X^{\bullet}_{n,r}$ can now be extracted quite straightforward by means of singularity analysis from the representation given in Proposition 3.5.7.

Theorem 3.5.8.
Let $r \in \mathbb{N}_0$ be fixed and consider $n \to \infty$. Then the expected number of old leaves as well as the expected number of nodes that are neither old leaves nor parents thereof in an r-fold "old path"-reduced tree and the corresponding variances are given by the asymptotic expansions

$$\mathbb{E}X^{\blacksquare}_{n,r} = \frac{1}{(r+2)^2} n + \frac{(r+3)(r+1)}{6(r+2)^2} + O(n^{-1}), \tag{3.47}$$

$$\mathbb{E}X^{\bullet}_{n,r} = \frac{2(r+1)}{(r+2)^2} n - \frac{(r^2+3r+3)(r+1)}{3(r+2)^2} + O(n^{-1}),$$

$$\mathbb{V}X^{\blacksquare}_{n,r} = \frac{(r+3)(r+1)}{3(r+2)^4} n + O(1), \tag{3.48}$$

$$\mathbb{V}X^{\bullet}_{n,r} = \frac{2(r^3+4r^2+6r+6)(r+1)}{3(r+2)^4} n + O(1).$$

Additionally, for fixed $d \geq 2$ the behavior of the factorial moments of $X^{\blacksquare}_{n,r}$ and $X^{\bullet}_{n,r}$ is given by

$$\mathbb{E}X^{\blacksquare\,\underline{d}}_{n,r} = \frac{1}{(r+2)^{2d}} + O(n^{d-1}) \tag{3.49}$$

and

$$\mathbb{E}X^{\bullet\,\underline{d}}_{n,r} = \frac{2^d(r+1)^d}{(r+2)^{2d}} n^d + O(n^{d-1}), \tag{3.50}$$

respectively. All O-constants in this theorem depend implicitly on r.

3.5.3 Total number of old paths

Similarly to our approach for counting the total number of paths required to construct a given tree from Section 3.3.2, we can also analyze the number of “old path”-segments within a random tree of size n. Formally, this corresponds to an analysis of the random variable $S_n := \sum_{r\geq 0} X_{n,r}^{\blacksquare}$.

Theorem 3.5.9.
The expected number of “old path” segments within a uniformly random tree of size n is given asymptotically by

$$\mathbb{E}S_n = \left(\frac{\pi^2}{6} - 1\right)n - \frac{\pi^2}{36} - \frac{1}{12} - \frac{\pi^2}{120n} + O(n^{-2}) \tag{3.51}$$

for $n \to \infty$.

Proof. As we have $S_n = \sum_{r\geq 0} X_{n,r}^{\blacksquare}$, we can use (3.41) to write

$$\mathbb{E}S_n = \sum_{r\geq 0} \mathbb{E}X_{n,r}^{\blacksquare} = \frac{1}{C_{n-1}}[z^n]\frac{1-u}{1+u}\sum_{r\geq 0}\frac{u^{r+2}}{(1-u^{r+2})^2}.$$

The main part of this analysis consists of determining an appropriate expansion of the sum in the last equation via the Mellin transform.

By setting $u = e^{-t}$ and by means of expanding via the geometric series, we find

$$\sum_{r\geq 0}\frac{u^{r+2}}{(1-u^{r+2})^2} = \sum_{r,\lambda\geq 0}\lambda u^{\lambda(r+2)} = \sum_{r,\lambda\geq 0}\lambda e^{-t\lambda(r+2)} =: f(t).$$

It is easy to determine the corresponding Mellin transform

$$f^*(s) = \Gamma(s)\zeta(s-1)(\zeta(s)-1)$$

with fundamental strip $\langle 2,\infty\rangle$. The poles of $f^*(s)$ are located at $s \in \{2,1\} \cup -2\mathbb{N}_0$. As this function behaves very nicely along vertical lines because of the exponential decay and the polynomial growth of the gamma function and the zeta function, respectively, we can use the inversion theorem to find

$$f(t) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} f^*(s)t^{-s}\,ds$$

for $t \to 0$. Analyticity in a larger (complex) region can be obtained analogously to the approach in the proof of Theorem 3.3.7.

Shifting the line of integration to $\mathrm{Re}(s) = -5$ and collecting residues, we find

$$f(t) = \sum_{p\in\{2,1,0,-2,-4\}} \mathrm{Res}(f^*(s), s=p)t^{-p} + \frac{1}{2\pi i}\int_{-5-i\infty}^{-5+i\infty} f^*(s)t^{-s}\,ds.$$

As in the proof of Theorem 3.3.7, the integral can be estimated with an error of $O(|t|^5)$. However, for the sake of simplicity, we will use the contribution from the singularity at $s = -4$ as the expansion error. Effectively, we obtain

$$f(t) = \left(\frac{\pi^2}{6} - 1\right)t^{-2} - \frac{1}{2}t^{-1} + \frac{1}{8} - \frac{1}{240}t^2 + O(t^4)$$

for $t \to 0$. Multiplication with the factor $\frac{1-u}{1+u}$, expansion of everything in terms of $z \to 1/4$, carrying out singularity analysis, and normalizing the result by dividing by C_{n-1} yields the result. □

3.6 Future Work

It seems likely that similar results also hold for reductions where one can cut a different structure as long as it is allowed to cut a single leaf. An example is cutting either single leaves or cherries (a root with two children). At least a formulation as an operator as in (3.9) seems possible in general. How much information about the moments and the central limit theorem can be extracted from that may vary (as it varies in this chapter already). Also the case of cutting old structures might be more difficult to handle in general.

4

Growing and Destroying Catalan–Stanley Trees

Stanley lists the class of Dyck paths where all returns to the axis are of odd length as one of the many objects enumerated by (shifted) Catalan numbers. By the standard bijection in this context, these special Dyck paths correspond to a class of rooted plane trees, so-called Catalan–Stanley trees.

This chapter investigates a deterministic growth procedure for these trees by which any Catalan–Stanley tree can be grown from the tree of size one after some number of rounds; a parameter that will be referred to as the age of the tree. Asymptotic analyses are carried out for the age of a random Catalan–Stanley tree of given size as well as for the "speed" of the growth process by comparing the size of a given tree to the size of its ancestors.

This chapter is an adapted version of [27], which is joint work with Helmut Prodinger.

4.1 Introduction

It is well-known that the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ enumerates Dyck paths of length $2n$. In [63], Stanley lists a variety of other combinatorial interpretations of the Catalan numbers, one of them being the number of Dyck paths from $(0,0)$ to $(2n+2,0)$ such that any maximal sequence of consecutive $(1,-1)$ steps ending on the x-axis has odd length. At this point it is interesting to note that there are more subclasses of Dyck paths, also enumerated by Catalan numbers, that are defined via parity restrictions on the length of the returns to the x-axis as well (see, e.g., [4]). The height of the class of Dyck paths with odd-length returns to the origin has already been studied in [53] with the result that the main term of the height is equal to the main term of the height of general Dyck paths as investigated in [8].

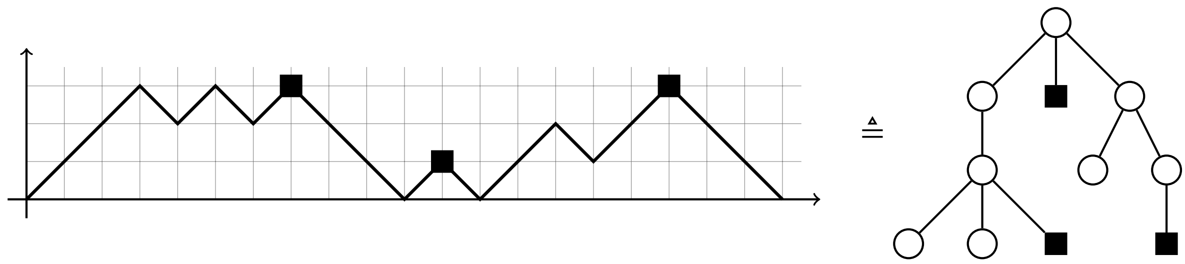


Figure 4.1: Bijection between Dyck paths with odd returns to zero and Catalan–Stanley trees. ■ marks all peaks before a descent to the x-axis and all rightmost leaves in the branches attached to the root, respectively.

By the well-known glove bijection, this special class of Dyck path corresponds to a special class $\mathcal{S}$ of rooted plane trees, where the distance between the rightmost node in all branches attached to the root and the root is odd. This bijection is illustrated in Figure 4.1.

The trees in the combinatorial class $\mathcal{S}$ are the central object of study in this chapter.

Definition 4.1.1.
Let $\mathcal{S}$ be the combinatorial class of all rooted plane trees τ, where the rightmost leaves in all branches attached to the root of τ have an odd distance to the root. In particular, ● itself, i.e., the tree consisting of just the root belongs to $\mathcal{S}$ as well. We call the trees in $\mathcal{S}$ *Catalan–Stanley* trees.

As we have seen in Chapters 2 and 3, there are approaches in which classical tree parameters like the register function for binary trees are analyzed by, in a nutshell, finding a proper way to grow tree families in a way that the parameter of interest corresponds to the age of the tree within this (deterministic) growth process.

Following this idea, the aim of this chapter is to define a "natural" growth process enabling us to grow any Catalan–Stanley tree from ●, and then to analyze the corresponding tree parameters.

In Section 4.2 we define such a growth process and analyze some properties of it. In particular, in Proposition 4.2.5 we characterize the family of trees that can be grown by applying a fixed number of growth iterations to some given tree family. This is then used to derive generating functions related to the parameters investigated in Sections 4.3 and 4.4.

Section 4.3 contains an analysis of the age of Catalan–Stanley trees, asymptotic expansions for the expected age among all trees of size n and the corresponding variance are given in Theorem 4.3.2.

Section 4.4 is devoted to the analysis of how fast trees of given size can be grown by investigating the size of the rth ancestor tree compared to the size of the original tree. This is characterized in Theorem 4.4.2.

Additional resources (i.e., a SageMath [59] worksheet, and instructions on how to use it) can be found at `https://benjamin-hackl.at/publications/catalan-stanley/`.

4.2 Growing Catalan–Stanley Trees

We denote the combinatorial class of rooted plane trees with $\mathcal{T}$, and the corresponding generating function enumerating these trees with respect to their size by $T(z)$. For the sake of readability, we omit the argument of $T(z) = T$ throughout this chapter. By means of the symbolic method [21, Chapter I], the combinatorial class $\mathcal{T}$ satisfies the construction $\mathcal{T} = \bullet \times \mathrm{SEQ}(\mathcal{T})$. It translates into the functional equation

$$T(z) = \frac{z}{1-T(z)} \qquad \Longleftrightarrow \qquad z + T(z)^2 = T(z), \tag{4.1}$$

which will be used throughout the chapter. Additionally, it is easy to see by solving the quadratic equation in (4.1) and choosing the correct branch of the solution, we have the well-known formula $T(z) = \frac{1-\sqrt{1-4z}}{2}$.

Proposition 4.2.1.
The generating function of the combinatorial class $\mathcal{S}$ of Catalan–Stanley trees, where t marks all the rightmost nodes in the branches attached to the root of the tree and z marks all other nodes, is given by

$$S(z,t) = z + \frac{zt}{1-t-T^2}. \tag{4.2}$$

In particular, there is one Catalan–Stanley tree of size 1 and C_{n-2} Catalan–Stanley trees of size n for $n \geq 2$.

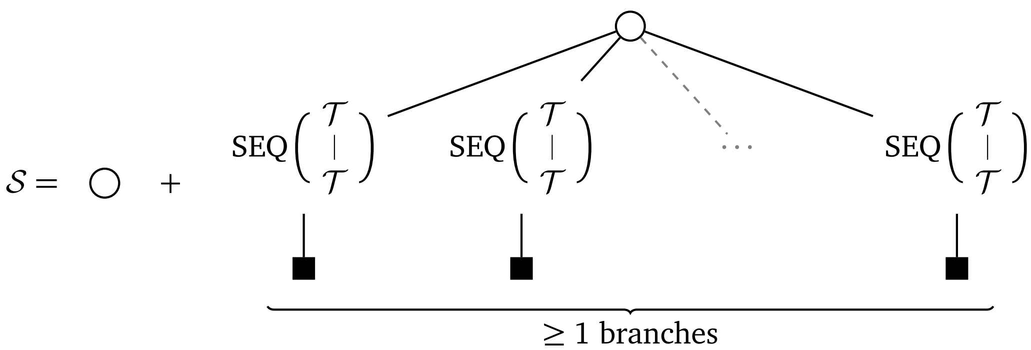


Figure 4.2: Symbolic specification of the combinatorial class $\mathcal{S}$ of Catalan–Stanley trees. Nodes represented by ■ are marked by the variable t, all other nodes are marked by z.

Proof. By using the symbolic method [21, Chapter I], the symbolic representation of $\mathcal{S}$ given in Figure 4.2 translates into the functional equation

$$S(z,t) = z + \frac{z\frac{t}{1-T^2}}{1-\frac{t}{1-T^2}},$$

which simplifies to the equation given in (4.2).

In order to enumerate Catalan–Stanley trees with respect to their size, we consider $S(z,z)$, which simplifies to $z(T+1)$ and thus proves the statement. □

We want to describe how to grow all Catalan–Stanley trees beginning from the tree that has only one node, ●.

We consider the tree reduction $\rho\colon \mathcal{S} \to \mathcal{S}$ that operates on a given Catalan–Stanley tree τ (or just the root) as follows:

Start from all nodes that are represented by t, i.e. the rightmost leaves in the branches attached to the root: if the node is a child of the root, it is simply deleted. Otherwise we delete all children of the grandparent of the node and mark the resulting leaf with t.

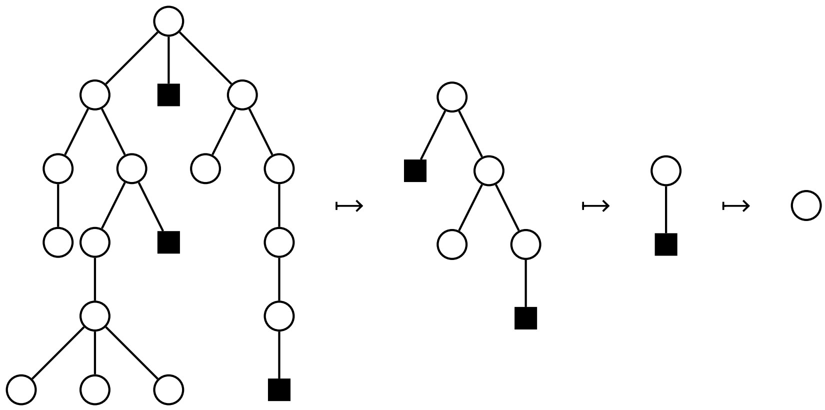


Figure 4.3: Illustration of the reduction operator ρ, ■ marks the rightmost leaves in the branches attached to the root.

This tree reduction is illustrated in Figure 4.3. While the reduction ρ is certainly not injective as there are several trees with the same reduction $\tau \in \mathcal{S}$, it is easy to construct a tree reducing to some given $\tau \in \mathcal{S}$ by basically inserting chains of length 2 before all rightmost leaves in the branches attached to the root. This allows us to think of the operator ρ^{-1} mapping a given tree (or some family of trees) to the respective set of preimages as a *tree expansion operator*. In this context, we also want to define the *age* of a Catalan–Stanley tree.

Definition 4.2.2.
Let $\tau \in \mathcal{S}$ be a Catalan–Stanley tree. Then we define $\alpha(\tau)$, the *age* of τ, to be the number of expansions required to grow τ from the tree of size one, $\bullet$. In particular, we want

$$\alpha(\tau) = r \quad \iff \quad \tau \in (\rho^{-1})^r(\bullet) \text{ and } \tau \notin (\rho^{-1})^{r-1}(\bullet)$$

for $r \in \mathbb{Z}_{\geq 1}$, and we set $\alpha(\bullet) = 0$.

Before we delve into the analysis of the age of Catalan–Stanley trees, we need to be able to translate the tree expansion given by ρ^{-1} into a suitable form so that we can actually use it in our analysis. The following proposition shows that ρ^{-1} can be expressed in the language of generating functions.

Proposition 4.2.3.
Let $\mathcal{F} \subseteq \mathcal{S}$ be a family of Catalan–Stanley trees with bivariate generating function $f(z,t)$, where t marks rightmost leaves in the branches attached to the root and z marks all other nodes. Then the generating function of $\rho^{-1}(\mathcal{F})$, the family of trees whose reduction is in $\mathcal{F}$, is given by

$$\Phi(f(z,t)) = \frac{1}{1-t} f\left(z, \frac{t}{1-t} T^2\right). \tag{4.3}$$

Proof. From a combinatorial point of view it is obvious that the operator Φ has to be linear, meaning that we can focus on determining all possible expansions of some tree represented by the monomial $z^n t^k$, i.e. a tree where the root has k children (and thus k different rightmost leaves in the branches attached to the root), and n other nodes.

In order to expand such a tree represented by $z^n t^k$ we begin by inserting a chain of length two before every rightmost leaf in order to ensure that the distance to the root is still odd. These newly inserted nodes can now be considered to be roots of some rooted plane trees, meaning that we actually insert two arbitrary rooted plane trees before every node represented by t. This corresponds to a factor of $t^k T^{2k}$.

In addition to this operation, we are also allowed to add new children to the root, i.e. we can add sequences of nodes represented by t before or after every child of the root. As observed above, the root has k children and thus there are $k+1$ positions where such a sequence can be attached. This corresponds to a factor of $(1-t)^{-(k+1)}$.

Finally, we observe that nodes that are represented by z are not expanded in any way, meaning that z^n remains as it is.

Putting everything together yields that

$$\Phi(z^n t^k) = \frac{1}{1-t} z^n \left(\frac{tT^2}{1-t}\right)^k,$$

which, by linearity of Φ, proves the statement. $\square$

Corollary 4.2.4.
The generating function $S(z,t)$ satisfies the functional equation

$$\Phi(S(z,t)) = S(z,t).$$

Proof. This follows immediately from the fact that the reduction operator ρ is surjective, as discussed above. □

Actually, in order to carry out a thorough analysis of this growth process for Catalan–Stanley trees we need to have more information on the iterated application of the expansion. In particular, we need a precise characterization of the family of Catalan–Stanley trees that can be grown from some given tree family by expanding it a fixed number of times.

Proposition 4.2.5.
Let $r \in \mathbb{Z}_{\geq 0}$ be fixed and $\mathcal{F} \subseteq \mathcal{S}$ be a family of Catalan–Stanley trees with bivariate generating function $f(z,t)$. Then the generating function enumerating the trees that are r-fold expansions of the trees in $\mathcal{F}$ is given by

$$\Phi^r(f(z,t)) = \frac{1}{1 - t\frac{1-T^{2r}}{1-T^2}} f\left(z, \frac{tT^{2r}}{1 - t\frac{1-T^{2r}}{1-T^2}}\right). \tag{4.4}$$

Proof. By linearity, it is sufficient to determine the generating function for the family of trees obtained by expanding some tree represented by $z^n t^k$. Consider the closely related multiplicative operator Ψ with

$$\Psi(f(z,t)) = f\left(z, \frac{t}{1-t}T^2\right).$$

It is easy to see that we can write the r-fold application of Φ with the help of Ψ as

$$\Phi^r(f(z,t)) = \Psi^r(f(z,t)) \prod_{j=0}^{r-1} \frac{1}{1 - \Psi^j(t)}.$$

As Ψ is multiplicative, we have

$$\Psi^r(z^n t^k) = \Psi^r(z)^n \Psi^r(t)^k,$$

meaning that we only have to investigate the r-fold application of Ψ to z and to t.

We immediately see that $\Psi^r(z) = z$, as Ψ maps z to z itself. For $\Psi^r(t)$, we can prove by induction that the relation

$$\Psi^r(t) = \frac{tT^{2r}}{1 - t\frac{1-T^{2r}}{1-T^2}}$$

holds for $r \geq 0$. Finally, observe that for $j \geq 1$ we have

$$\Psi^j(t) = \frac{\Psi^{j-1}(t)}{1 - \Psi^{j-1}(t)} T^2, \tag{4.5}$$

and thus

$$\Psi^r(t) = \frac{\Psi^{r-1}(t)}{1-\Psi^{r-1}(t)}T^2 = \frac{\Psi^{r-2}(t)}{(1-\Psi^{r-2}(t))(1-\Psi^{r-1}(t))}T^4 = \cdots = \frac{tT^{2r}}{\prod_{j=0}^{r-1}(1-\Psi^j(t))}$$

by iteratively using (4.5) in the numerator. With our explicit formula for $\Psi^r(t)$ from above this yields

$$\prod_{j=0}^{r-1}(1-\Psi^j(t)) = 1 - t\frac{1-T^{2r}}{1-T^2}$$

for $r \geq 1$. Putting everything together we obtain

$$\Phi^r(z^n t^k) = \frac{1}{1-t\frac{1-T^{2r}}{1-T^2}} z^n \Psi^r(t)^k,$$

which proves (4.4) by linearity of Φ^r. □

From this characterization we immediately obtain the generating functions for all the classes of objects we will investigate in the following sections.

Corollary 4.2.6.
Let $r \in \mathbb{Z}_{\geq 0}$. The generating function $F_r^{\leq}(z,t)$ enumerating Catalan–Stanley trees of age less than or equal to r where t marks the rightmost leaves in the branches attached to the root and z marks all other nodes is given by

$$F_r^{\leq}(z,t) = \frac{z}{1-t\frac{1-T^{2r}}{1-T^2}}. \tag{4.6}$$

Proof. As we defined $\rho(\bullet) = \bullet$ we have $\bullet \in \rho^{-1}(\bullet)$, which implies $F_r^{\leq}(z,t)$ is given by $\Phi^r(z)$. □

Corollary 4.2.7.
Let $r \geq 0$. Then the generating function $G_r(z,v)$ enumerating Catalan–Stanley trees where z marks the tree size and v marks the size of the r-fold reduced tree, is given by

$$G_r(z,v) = \Phi^r(S(zv,tv))|_{t=z} = \frac{1}{1-z\frac{1-T^{2r}}{1-T^2}} S\Bigg(zv, \frac{zT^{2r}}{1-z\frac{1-T^{2r}}{1-T^2}}v\Bigg). \tag{4.7}$$

Proof. Observe that the generating function $S(zv,tv)$ enumerates Catalan–Stanley trees with respect to the number of rightmost leaves in the branches attached to the root (marked by t), the number of other nodes (marked by z), and the size of the tree (marked by v). Applying the operator Φ^r to this generating function thus yields a generating function where v still marks the size of the tree, t and z however enumerate the number of rightmost leaves in the branches attached to the root and all other nodes of the r-fold expanded tree, respectively. After setting $t = z$, we obtain a generating function where v marks the size of the original tree and z the size of the r-fold expanded tree—which is equivalent to the formulation in the corollary. □

4.3 Age of Catalan–Stanley Trees

In this section we want to give a proper analysis of the parameter α defined in the previous section. Formally, we do this by considering the random variable D_n modeling the age of a tree of size n, where all Catalan–Stanley trees of size n are equally likely.

Remark.
It is noteworthy that in Chapter 2 it was shown that the well-known register function of a binary tree can also be obtained as the number of times some reduction can be applied to the binary tree until it degenerates. The age of a Catalan–Stanley tree can thus be seen as a "register function"-type parameter as well.

First of all, we are interested in the minimum and maximum age a tree of size n can have.

Proposition 4.3.1.
Let $n \in \mathbb{Z}_{\geq 2}$. Then the bounds

$$1 \leq D_n \leq \left\lfloor \frac{n}{2} \right\rfloor \tag{4.8}$$

hold and are sharp, i.e. there are trees τ, $\tau' \in \mathcal{S}$ of size $n \geq 2$ such that $D_n(\tau) = 1$ and $D_n(\tau') = \lfloor n/2 \rfloor$ hold. The only tree of size 1 is $\bullet$, and it satisfies $D_1(\bullet) = 0$.

Proof. Note that only $\bullet$, the tree of size 1 has age 0, therefore the lower bound is certainly valid for trees of size $n \geq 2$. This lower bound is sharp, as the tree with $n-1$ children attached to the root is a Catalan–Stanley tree and has age 1.

For the upper bound, first observe that given a tree of size $n \geq 3$ the reduction ρ always removes at least 2 nodes from the tree. If the tree is of size 2, then ρ only removes one node. Given an arbitrary Catalan–Stanley tree τ of age r and size n, this means that

$$1 = |\bullet| = |\rho^r(\tau)| \leq |\tau| - 2 \cdot (r-1) - 1 = n - 2r + 1,$$

where $|\tau|$ denotes the size of the tree τ. This yields $r \leq n/2$, and as r is known to be an integer we may take the floor of the number on the right-hand side of the inequality. This proves that the upper bound in (4.8) is valid.

The upper bound is sharp because we can construct appropriate families of trees precisely reaching the upper bound: for even n, the chain of size n is a Catalan–Stanley tree of age $n/2$. For odd $n = 2m + 1$ we consider the chain of size $2m$ and attach one node to the root of it. The resulting tree is a Catalan–Stanley tree of age $m = \lfloor n/2 \rfloor$, and thus proves that the bound is sharp. $\square$

By investigating the generating functions obtained from Corollary 4.2.6 we can characterize the limiting distribution of the age of Catalan–Stanley trees when the size n tends to ∞.

Theorem 4.3.2.
Consider $n \to \infty$. Then the age of a (uniformly random) Catalan–Stanley tree of size n behaves according to a discrete limiting distribution where

$$\begin{aligned}\mathbb{P}(D_n = r) = \Big(&\frac{4(4^r(3r-1)+1)}{(4^r+2)^2} - \frac{4(4^{r+1}(3r+2)+1)}{(4^{r+1}+2)^2}\Big)\\ &-\Big(\frac{6\cdot 64^r(2r^3-5r^2+4r-1)-6\cdot 16^r(16r^3-24r^2+10r-1)+24\cdot 4^r(2r^3-r^2)}{(4^r+2)^4}\\ &-\frac{6\cdot 64^{r+1}(2r^3+r^2)-6\cdot 16^{r+1}(16r^3+24r^2+10r+1)+24\cdot 4^{r+1}(2r^3+5r^2+4r+1)}{(4^{r+1}+2)^4}\Big)n^{-1}\\ &+O\Big(\frac{r^5}{3^r}n^{-2}\Big) \qquad (4.9)\end{aligned}$$

for $r \in \mathbb{Z}_{\geq 1}$. Additionally, by setting

$$\begin{aligned}c_0 &= \sum_{r\geq 1}\frac{4^{r+1}(3r-1)+4}{(4^r+2)^2}\\ &= 2.71825364286795285266483619282193673445854356803 44\ldots,\\ c_1 &= -\sum_{r\geq 1}\frac{6\cdot 64^r(2r^3-5r^2+4r-1)-6\cdot 16^r(16r^3-24r^2+10r-1)+24\cdot 4^r(2r-1)r^2}{(4^r+2)^4}\\ &= -4.2220971510158840823821873477600478080816411210406\ldots,\\ c_2 &= \sum_{r\geq 1}(2r-1)\frac{4^{r+1}(3r-1)+4}{(4^r+2)^2} - c_0^2\\ &= 0.9184560421437479735779714781401949650368895393396 7\ldots,\\ c_3 &= -\sum_{r\geq 1}\frac{(2r-1)}{(4^r+2)^4}\big(6\cdot 64^r(2r^3-5r^2+4r-1)-6\cdot 16^r(16r^3-24r^2+10r-1)\\ &\qquad\qquad + 24\cdot 4^r(2r-1)r^2\big) - 2c_0c_1\\ &= -9.1621753200836274996912436568310268988536534594942\ldots,\end{aligned}$$

the expected age and the corresponding variance are given by the asymptotic expansions

$$\mathbb{E}D_n = c_0 + c_1 n^{-1} + O(n^{-2}), \qquad (4.10)$$

$$\mathbb{V}D_n = c_2 + c_3 n^{-1} + O(n^{-2}). \qquad (4.11)$$

Proof. For the sake of convenience we set $F_r^{\leq}(z) := F_r^{\leq}(z,z)$, where $F_r^{\leq}(z,t)$ is given in (4.6). This univariate generating function now enumerates Catalan–Stanley trees of age $\leq r$ with respect to the tree size.

We begin by observing that $F_r^{\geq}(z)$, the generating function enumerating Catalan–Stanley trees of age $\geq r$ with respect to the tree size is given by

$$F_r^{\geq}(z) = S(z,z) - F_{r-1}^{\leq}(z) = z(1+T) - \frac{z}{1 - z\frac{1-T^{2r-2}}{1-T^2}} = z(1+T)\frac{T^{2r-1}}{1+T^{2r-1}}, \tag{4.12}$$

where the last equation follows after some elementary manipulations and by using (4.1).

Now let $f_{n,r} := [z^n]F_r^{\geq}(z)$ denote the number of Catalan–Stanley trees of size n and age $\geq r$. As we consider all Catalan–Stanley trees of size n to be equally likely, we find

$$\mathbb{P}(D_n = r) = \mathbb{P}(D_n \geq r) - \mathbb{P}(D_n \geq r+1) = \frac{f_{n,r} - f_{n,r+1}}{C_{n-2}}.$$

We use singularity analysis (see [18] and [21, Chapter VI]) in order to obtain an asymptotic expansion for $f_{n,r}$. To do so, we consider z to be in some Δ-domain at $1/4$ (see [21, Definition VI.1]). The task of expanding $F_r^{\geq}(z)$ for $z \to 1/4$ now largely consists of handling the term $\frac{T^{2r-1}}{1+T^{2r-1}}$. Observe that we can write

$$\begin{aligned}\frac{T^{2r-1}}{1+T^{2r-1}} = \frac{1}{1+T^{1-2r}} &= \frac{1}{1+2^{2r-1}(1-\sqrt{1-4z}\,)^{1-2r}},\\ &= \frac{1}{(1+2^{2r-1})\left(1+\frac{2^{2r-1}}{1+2^{2r-1}}\sum_{j\geq 1}\binom{2r+j-2}{j}(1-4z)^{j/2}\right)}\end{aligned}$$

which results in

$$\begin{aligned}\frac{T^{2r-1}}{1+T^{2r-1}} = \frac{2}{4^r+2} &- \frac{2\cdot 4^r(2r-1)}{(4^r+2)^2}(1-4z)^{1/2}\\ &+ \frac{2\cdot 4^r\,(4^r(r-1)-2r)(2r-1)}{(4^r+2)^3}(1-4z)\\ -\frac{2\cdot 4^r(16^r(2r^2-5r+3)-4^{r+2}(r^2-r)+8r^2+4r)(2r-1)}{3(4^r+2)^4}&(1-4z)^{3/2} + O\Big(\frac{r^4}{3^r}(1-4z)^2\Big).\end{aligned}$$

Multiplying this expansion with the expansion of $z(1+T)$ yields the expansion

$$\begin{aligned}F_r^{\geq}(z) = \frac{3}{4(4^r+2)} &- \frac{4^r(3r-1)+1}{2(4^r+2)^2}(1-4z)^{1/2}\\ &+ \frac{16^r(6r^2-7r-1)-2\cdot 4^r(6r^2-5r+7)-12}{4(4^r+2)^3}(1-4z)\\ -\frac{64^r(2r^3-5r^2+r)-2\cdot 16^r(8r^3-12r^2+11r-2)+4^{r+1}(2r^3-r^2-3r)-4}{2(4^r+2)^4}&(1-4z)^{3/2}\\ &+O\Big(\frac{r^4}{3^r}(1-4z)^2\Big).\end{aligned}$$

By means of singularity analysis we extract the nth coefficient and find

$$f_{n,r} = \frac{4^r(3r-1)+1}{4\sqrt{\pi}\,(4^r+2)^2}4^n n^{-3/2} \\ - \left(\frac{3\cdot 64^r(8r^3-20r^2+r+1)-3\cdot 16^r(64r^3-96r^2+100r-19)}{32\sqrt{\pi}\,(4^r+2)^4} \right. \\ \left. + \frac{12\cdot 4^r(8r^3-4r^2-15r)-60}{32\sqrt{\pi}\,(4^r+2)^4}\right)4^n n^{-5/2} + O\Big(\frac{r^5}{3^r}4^n n^{-7/2}\Big).$$

Computing the difference $f_{n,r}-f_{n,r+1}$ and dividing by the Catalan number C_{n-2} then yields the expression for $\mathbb{P}(D_n=r)$ given in (4.9).

The expected value can then be computed with the help of the well-known formula

$$\mathbb{E}D_n = \sum_{r\ge 1}\mathbb{P}(D_n\ge r),$$

which proves (4.10). Finally, the variance can be obtained from $\mathbb{V}D_n = \mathbb{E}(D_n^2)-(\mathbb{E}D_n)^2$, where

$$\mathbb{E}(D_n^2) = \sum_{r\ge 1} r^2\mathbb{P}(D_n=r) = \sum_{r\ge 1}(2r-1)\mathbb{P}(D_n\ge r),$$

which proves (4.11). □

In addition to the asymptotic expansions given in Theorem 4.3.2 we can also determine an exact formula for the expected value $\mathbb{E}D_n$. The key tools in this context are Cauchy's integral formula as well as the substitution $z=\frac{u}{(1+u)^2}$.

Proposition 4.3.3.
Let $n\in\mathbb{Z}_{\ge 2}$. The expected age of the Catalan–Stanley trees of size n is given by

$$\mathbb{E}D_n = \frac{1}{C_{n-2}}\sum_{k\ge 1}(-1)^{k+1}\sigma_0^{\mathrm{odd}}(k)\left(\binom{2n-4-k}{n-3}+\binom{2n-4-k}{n-2}-2\binom{2n-4-k}{n-1}\right), \quad (4.13)$$

where $\sigma_0^{\mathrm{odd}}(k)$ denotes the number of odd divisors of k.

Proof. We begin by explicitly extracting the coefficient $[z^n]F_r^{\ge}(z)$. The expected value can then be obtained by summation over r and division by C_{n-2}.

With the help of the substitution $z=\frac{u}{(1+u)^2}$ we can bring $F_r^{\ge}(z)$ into the more suitable form

$$F_r^{\ge}(z) = \frac{(1+2u)u^{2r}}{(1+u)^3(u^{2r-1}+(1+u)^{2r-1})}.$$

We extract the coefficient of z^n now by means of Cauchy's integral formula. Let γ be a small contour winding around the origin once. Then we have

$$
\begin{aligned}
[z^n]F_r^{\geq}(z) &= \frac{1}{2\pi i}\oint_\gamma \frac{F_r^{\geq}(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i}\oint_{\tilde{\gamma}} \frac{(1+u)^{2n+2}}{u^{n+1}} \frac{(1+2u)u^{2r}}{(1+u)^3(u^{2r-1}+(1+u)^{2r-1})}\frac{1-u}{(1+u)^3}\,du \\
&= [u^{n-2r}](1+2u)(1-u)(1+u)^{2n-2r-3}\frac{1}{1+(\frac{u}{1+u})^{2r-1}} \\
&= [u^{n-2r}](1+u-2u^2)\sum_{j\geq 1}(-1)^{j-1}u^{(2r-1)(j-1)}(1+u)^{2n-4-j(2r-1)} \\
&= \sum_{j\geq 1}(-1)^{j-1}\Bigg(\binom{2n-4-j(2r-1)}{n-3}+\binom{2n-4-j(2r-1)}{n-2} \\
&\qquad\qquad -2\binom{2n-4-j(2r-1)}{n-1}\Bigg),
\end{aligned} \tag{4.14}
$$

where $\tilde{\gamma}$, the integration contour of the second integral, is the transformation of γ under the transformation $z = u/(1+u)^2$ and is also a small contour winding around the origin once.

Now consider the auxiliary sum

$$\vartheta(k) := \sum_{\substack{j,r\geq 1\\ j(2r-1)=k}} (-1)^{j-1}.$$

It is easy to see by distinguishing between even and odd k that with the help of $\sigma_0^{\mathrm{odd}}(k)$, $\vartheta(k)$ can be written as $\vartheta(k) = (-1)^{k-1}\sigma_0^{\mathrm{odd}}(k)$.

Summing the expression from (4.14) over $r \geq 1$, simplifying the resulting double sum by means of the auxiliary sum ϑ, and finally dividing by C_{n-2} then proves (4.13). □

4.4 Analysis of Ancestors

In this section we focus on characterizing the effect of the (repeatedly applied) reduction ρ on a random Catalan–Stanley tree of size n. We are particularly interested in studying the size of the reduced tree. In the light of the fact that all Catalan–Stanley trees can be grown from ● by means of the growth process induced by ρ, we can think of the rth reduction of some tree τ as the rth ancestor of τ.

In order to formally conduct this analysis, we consider the random variable $X_{n,r}$ modeling the size of the rth ancestor some tree of size n, where all Catalan–Stanley trees of size n are equally likely.

Similar to our approach in Proposition 4.3.1 we can determine precise bounds for $X_{n,r}$ as well.

Proposition 4.4.1.
Let $n \in \mathbb{Z}_{\geq 2}$ and $r \in \mathbb{Z}_{\geq 1}$. Then the bounds

$$1 \leq X_{n,r} \leq n - 2(r-1) - 1 \tag{4.15}$$

hold for $r \leq \lfloor n/2 \rfloor$ and are sharp, i.e. there are trees τ, $\tau' \in \mathcal{S}$ of size $n \geq 2$ such that $X_{n,r}(\tau) = 1$ and $X_{n,r}(\tau') = n - 2(r-1) - 1$. For $r > \lfloor n/2 \rfloor$ the variable $X_{n,r}$ is deterministic with $X_{n,r} = 1$.

Proof. Assume that $r \leq \lfloor n/2 \rfloor$. The lower bound is obvious as trees cannot reduce further than to ●, and as the first ancestor of the tree with $n-1$ children attached to the root already is ● the lower bound is valid and sharp.

For the upper bound we follow the same argumentation as in the proof of Proposition 4.3.1 to arrive at

$$1 \leq |\rho^r(\tau)| \leq |\tau| - 2(r-1) - 1 = n - 2r + 1$$

for some Catalan–Stanley tree of size n, which proves that the upper bound is valid. Any tree of size n having the chain of length 2 as its $(r-1)$th ancestor satisfies proves that the upper bound is sharp. This proves (4.15).

In the case of $r > \lfloor n/2 \rfloor$ we observe that as the $\lfloor n/2 \rfloor$th ancestor of any Catalan–Stanley tree of size n already is certain to be ● by Proposition 4.3.1, the rth ancestor is ● as well. □

With the generating function $G_r(z, v)$ enumerating Catalan–Stanley trees with respect to their size (marked by n) and the size of their rth ancestor (marked by v) from Corollary 4.2.7 we can write the probability generating function of $X_{n,r}$ as

$$\mathbb{E}v^{X_{n,r}} = \frac{1}{C_{n-2}}[z^n]G_r(z, v).$$

This allows us to extract parameters like the expected size of the rth ancestor and the corresponding variance.

Theorem 4.4.2.
Let $r \in \mathbb{Z}_{\geq 0}$ be fixed and consider $n \to \infty$. Then the expected value and the variance of the random variable $X_{n,r}$ modeling the size of the rth ancestor of a (uniformly random) Catalan–Stanley tree of size n are given by the asymptotic expansions

$$\mathbb{E}X_{n,r} = \frac{1}{4^r}n + \frac{2 \cdot 4^r - 2r^2 + r - 2}{2 \cdot 4^r} + \frac{(2r+1)(2r-1)(r-3)r}{2 \cdot 4^{r+1}}n^{-1} + O(n^{-3/2}), \tag{4.16}$$

$$\begin{aligned}\mathbb{V}X_{n,r} = \frac{(2^r+1)(2^r-1)}{16^r}n^2 - \frac{\sqrt{\pi}(4^r(3r+1)-1)}{3\cdot 16^r}n^{3/2} \\ + \frac{18\cdot 4^r r^2 + 3\cdot 4^r r - 38\cdot 4^r + 36r^2 - 42r + 38}{18\cdot 16^r}n \\ + \frac{5\sqrt{\pi}(4^r(3r+1)-1)}{8\cdot 16^r}n^{1/2} + O(1).\end{aligned} \tag{4.17}$$

Proof. The strategy behind this proof is to determine the first and second factorial moment of $X_{n,r}$ by extracting the coefficient of z^n in the derivatives $\frac{\partial^d}{\partial v^d}G_r(z,v)|_{v=1}$ for $d \in \{1,2\}$ and normalizing the result by dividing by C_{n-2}.

We begin with the expected value. With the help of SageMath [59] we find for $z \to 1/4$

$$\begin{aligned}\frac{\partial}{\partial v}G_r(z,v)|_{v=1} = \frac{1}{4^{r+2}}(1-4z)^{-1/2} + \frac{3\cdot 4^r - r}{2\cdot 4^{r+1}} - \frac{2\cdot 4^r - 2r^2 + r + 2}{4^{r+2}}(1-4z)^{1/2} \\ - \frac{9\cdot 4^r + 2r^3 - 3r^2 - 5r}{6\cdot 4^{r+1}}(1-4z) + O((1-4z)^{3/2}),\end{aligned}$$

where the O-constant depends implicitly on r. Extracting the coefficient of z^n and dividing by C_{n-2} yields the expansion given in (4.16).

Following the same approach for the second derivative yields the expansion

$$\begin{aligned}\frac{\partial^2}{\partial v^2}G_r(z,v)|_{v=1} = \frac{1}{2\cdot 4^{r+2}}(1-4z)^{-3/2} - \frac{4^r(3r+1)-1}{3\cdot 16^{r+1}}(1-4z)^{-1} \\ + \frac{4^r(18r^2+3r+7)-24r+2}{18\cdot 16^{r+1}}(1-4z)^{-1/2} + O(1),\end{aligned}$$

such that after applying singularity analysis and division by C_{n-2} we obtain the expansion

$$\begin{aligned}\mathbb{E}X_{n,r}^{\underline{2}} = \frac{1}{4^r}n^2 - \frac{\sqrt{\pi}(4^r(3r+1)-1)}{3\cdot 16^r}n^{3/2} + \frac{4^r(18r^2+3r-20)-24r+2}{18\cdot 16^r}n \\ + \frac{5\sqrt{\pi}(4^r(3r+1)-1)}{8\cdot 16^r}n^{1/2} + O(1)\end{aligned}$$

for the second factorial moment $\mathbb{E}X_{n,r}^{\underline{2}}$. Applying the well-known formula

$$\mathbb{V}X_{n,r} = \mathbb{E}X_{n,r}^{\underline{2}} + \mathbb{E}X_{n,r} - (\mathbb{E}X_{n,r})^2$$

then leads to the asymptotic expansion for the variance given in (4.17) and thus proves the statement. □

Besides the asymptotic expansion given in Theorem 4.4.2, we are also interested in finding an exact formula for the expected value $\mathbb{E}X_{n,r}$. We can do so by means of Cauchy's integral formula.

Proposition 4.4.3.
Let $n, r \in \mathbb{Z}_{\geq 1}$. Then the expected size of the rth ancestor of a random Catalan–Stanley tree of size n is given by

$$\mathbb{E}X_{n,r} = \frac{1}{C_{n-2}}\binom{2n-2r-4}{n-2} + 1. \tag{4.18}$$

Proof. We rewrite the derivative $g(z) := \frac{\partial}{\partial v} G_r(z,v)|_{v=1}$ into a more suitable form which makes it easier to extract the coefficients. To do so, we use the substitution $z = u/(1+u)^2$ again, allowing us to express the derivative as

$$g(z) = \frac{u^{2r+2}}{(1-u)(1+u)^{2r+3}} + \frac{(1+2u)u}{(1+u)^3}.$$

Note that as $T = \frac{u}{1+u}$, the summand $\frac{(1+2u)u}{(1+u)^3}$ actually represents $z(1+T)$, implying that the coefficient of z^n in this summand is given by C_{n-2}. Now let γ be a small contour winding around the origin once, so that with Cauchy's integral formula we obtain

$$\begin{aligned}[z^n]g(z) &= \frac{1}{2\pi i}\oint_\gamma \frac{g(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i}\oint_{\tilde{\gamma}} \frac{(1+u)^{2n+2}}{u^{n+1}} \frac{u^{2r+2}}{(1-u)(1+u)^{2r+3}} \frac{1-u}{(1+u)^3}\,du + C_{n-2} \\ &= [u^{n-2r-2}](1+u)^{2n-2r-4} + C_{n-2} = \binom{2n-2r-4}{n-2r-2} + C_{n-2},\end{aligned}$$

where $\tilde{\gamma}$ is the image of γ under the transformation (and is still a small contour winding around the origin once). Dividing by C_{n-2} then proves (4.18). $\square$

5

Ascents in Non-Negative Lattice Paths

Non-negative Łukasiewicz paths are special two-dimensional lattice paths never passing below their starting altitude which have only one single special type of down step. They are well-known and -studied combinatorial objects, in particular due to their bijective relation to trees with given node degrees.

We study the asymptotic behavior of the number of ascents (i.e., the number of maximal sequences of consecutive up steps) of given length for classical subfamilies of general non-negative Łukasiewicz paths: those with arbitrary ending altitude, those ending on their starting altitude, and a variation thereof. Our results include precise asymptotic expansions for the expected number of such ascents as well as for the corresponding variance.

This chapter corresponds to an adapted version of [25] and is joint work with Clemens Heuberger and Helmut Prodinger.

5.1 Introduction

Two-dimensional lattice paths can be defined as sequences of points in the plane $\mathbb{R}^2$ where for any point, the vector pointing to the succeeding point ("step") is from a predefined finite set, the *step set*.

In this chapter, our focus lies on a special class of two-dimensional lattice paths: *non-negative simple Łukasiewicz paths*. A lattice path is said to be *simple* if the horizontal coordinate is the same (e.g. is 1) for all possible steps. In case of a simple path family, we define the *step set* $\mathcal{S}$ as the set of allowed height differences, i.e., the respective y-coordinates between the points of the path. If, additionally, the step set $\mathcal{S} \subseteq \mathbb{Z}$ is integer-valued and contains -1 as the single negative value (meaning that all other values in $\mathcal{S}$ are non-negative), then the corresponding paths are called *simple Łukasiewicz paths*.

If a lattice path starts at the origin and never passes below the horizontal axis, then the path is said to be a *meander* (or non-negative path). And in case such a non-negative path ends on the horizontal axis, it is called an *excursion*.

Lattice path families of this type have been studied intensively, see [2] for a detailed survey on general simple lattice paths, and, for example, [3, 55] for investigations concerning Łukasiewicz paths.

We are interested in analyzing the number of *ascents* in these paths. An *ascent* is an inclusion-wise maximal sequence of up steps (i.e., steps in $\mathcal{S} \setminus \{-1\}$; this might also include the horizontal step corresponding to 0). For an integer $r \geq 1$, if an ascent consists of precisely r steps, then the ascent is said to be an r-ascent. As an example, Figure 5.1 depicts some non-negative Łukasiewicz excursion with emphasized 2-ascents.

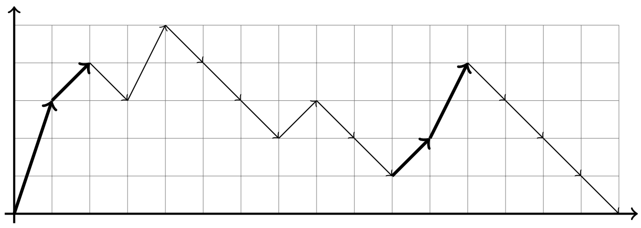


Figure 5.1: Simple Łukasiewicz excursion of length 16 with emphasized 2-ascents where $\mathcal{S} = \{-1, 1, 2, 3\}$.

In this chapter, we give a precise analysis of the number of r-ascents for non-negative simple Łukasiewicz paths of given length, as well as of variants of this class of lattice paths. Our investigation is motivated by [34], where the number of 1-ascents in a special lattice path class related to the classic Dyck paths was analyzed explicitly by elementary methods.

Main Results

Within this chapter, three special classes of non-negative Łukasiewicz paths are of interest:

- *excursions*, i.e., paths that end on the horizontal axis,
- *dispersed excursions*, i.e., excursions where horizontal steps are not allowed except on the horizontal axis,
- *meanders*, i.e., general non-negative Łukasiewicz paths without additional restrictions.

Formally, we conduct our analysis by investigating random variables $E_{n,r}$, $D_{n,r}$, $M_{n,r}$ which model the number of r-ascents in a random excursion, dispersed excursion, and meander of length n, respectively. The underlying probability models are based on equidistribution: within a family, all paths of length n are assumed to be equally likely.

Given $r \in \mathbb{N}$ and considering $n \in \mathbb{N}_0$ with $n \to \infty$, we prove that for excursions we have

$$\mathbb{E}E_{n,r} = \mu n + c_0 + O(n^{-1/2}) \qquad \text{and} \qquad \mathbb{V}E_{n,r} = \sigma^2 n + O(n^{1/2}),$$

for some constants μ, c_0, σ^2 depending on the chosen step set $\mathcal{S}$. The constants are given explicitly in Theorem 5.5.1. Additionally, if n is not a multiple of the so-called *period* of the step set, then the random variable degenerates and we have $E_{n,r} = 0$; see Theorem 5.5.1 for details.

For dispersed excursions, the corresponding computations get rather messy, which is why we restrict ourselves to the investigation of d_n, the number of dispersed excursions of length n, as well as the expected value $\mathbb{E}D_{n,r}$. In particular, for all step sets $\mathcal{S}$ (except for the special case of dispersed Dyck paths with $\mathcal{S} = \{-1, 1\}$), d_n satisfies

$$d_n = c_0 \kappa^n n^{-3/2} + O(\kappa^n n^{-5/2}),$$

with constants c_0 and κ depending on the chosen step set. For the expected number of ascents in this particular lattice path family, we find

$$\mathbb{E}D_{n,r} = \mu n + O(1)$$

for some constant μ depending on $\mathcal{S}$. Explicit values for these constants and more details are given in Theorem 5.5.5.

In the context of meanders we are able to show that for all step sets (with two special exceptions: Dyck meanders with $\mathcal{S} = \{-1, 1\}$, and Motzkin meanders with $\mathcal{S} = \{-1, 0, 1\}$) we have

$$\mathbb{E}M_{n,r} = \mu n + c_0 + O(n^{5/2}\kappa^n) \qquad \text{and} \qquad \mathbb{V}M_{n,r} = \sigma^2 n + O(1),$$

for constants μ, c_0, $\kappa \in (0, 1)$, σ^2 depending on $\mathcal{S}$. Also, the random variable $M_{n,r}$ is asymptotically normally distributed; see Theorem 5.5.7 for explicit formulas for the constants and more details.

In theory, our approach can be used to obtain arbitrarily precise asymptotic expansions for all the quantities above. For the sake of readability we have chosen to only give the main term as well as one additional term, wherever possible.

On a more technical note, in order to deal with general Łukasiewicz step sets in our setting, we make use of a generating function approach (see [21, Chapter I]). In particular, we heavily rely on the technique of *singular inversion* (see [21, Chapter VI.7], [43]), which deals with finding an asymptotic expansion for the growth of the coefficients of generating functions $y(z)$ satisfying a functional equation of the type

$$y = z\,\phi(y)$$

with a suitable function ϕ.

Notation and Special Cases

Throughout this chapter, the step set will be denoted as $\mathcal{S} = \{-1, b_1, \dots, b_{m-1}\}$ with integers $b_j \geq 0$ for all j and $m \geq 1$. The b_j are referred to as up steps—even if the step is a horizontal one.

The so-called characteristic polynomial of the lattice path class, i.e., the generating function corresponding to the set $\mathcal{S}$, is denoted by $S(u) := \sum_{s\in\mathcal{S}} u^s$. The strongly related generating function of the non-negative steps is denoted by $S_+(u) := \sum_{\substack{s\in\mathcal{S}\\ s\geq 0}} u^s$.

In this context, observe that the particular step set $\mathcal{S} = \{-1, 0\}$ corresponds to a, in some sense, pathological family of Łukasiewicz paths. In this case, there is only precisely one non-negative Łukasiewicz path of any given length. The family of meanders and excursions coincides, and also the random variables degenerate in the sense that we have $E_{n,r} = M_{n,r} = [\![n = r]\!]$. Thus, further investigation of this case is not required—which is why we exclude the case $\mathcal{S} = \{-1, 0\}$ from now on.

While in the case of a general step set $\mathcal{S}$ we are forced to deal with implicitly given quantities, for special cases like $\mathcal{S} = \{-1, 1\}$ (Dyck paths), everything can be made completely explicit as we will demonstrate in the course of our investigations.

Structure of this Chapter

In Sections 5.2 and 5.3, we demonstrate two different approaches to determine suitable generating functions required to analyze the number of ascents. The approach in Section 5.2 is fully analytic and fueled by the kernel method and the “adding a new slice”-technique, see [54, Section 2.5]. The other approach in Section 5.3 is a more combinatorial approach based on the inherent relation between Łukasiewicz paths and plane trees with given vertex degrees. Formulas for the respective generating functions are given in Propositions 5.2.2 and 5.3.3.

Then, in Section 5.4 we give a rigorous description of the singular structure of a fundamental quantity, namely a particular inverse function derived in Sections 5.2 and 5.3. Important tools for giving this description are provided in Propositions 5.4.1 and 5.4.2, which are extensions of [21, Theorem VI.6; Remark VI.17].

Section 5.5 contains the actual analysis of ascents for the different lattice path families mentioned above. In particular, in Section 5.5.1 we investigate excursions; the main result is stated in Theorem 5.5.1. Section 5.5.2 deals with the analysis of ascents in dispersed excursions. In this case, the expected number of r-ascents for all but one given step sets is analyzed within Theorem 5.5.5, and the analysis for the remaining one is conducted in Proposition 5.5.6. Finally, Section 5.5.3 contains our results for ascents in meanders. Similarly to the previous section, the analysis for most step sets is given in Theorem 5.5.7, and the remaining cases are investigated in Propositions 5.5.8 and 5.5.9.

The SageMath [59] worksheets used to produce our results can be found at

`https://benjamin-hackl.at/publications/lukasiewicz-ascents/`,

in particular:

- `lukasiewicz-excursions.ipynb` contains the calculations from Section 5.5.1,
- `lukasiewicz-dispersed-excursions.ipynb` for Section 5.5.2,
- `lukasiewicz-meanders.ipynb` for Section 5.5.3, and
- `utilities.py`, which has to be copied into the same folder as the others. This file contains utility code that is used in the notebook files.

5.2 Generating Functions: An Analytic Approach

In this section we will introduce and discuss the preliminaries required in order to carry out the asymptotic analysis of ascents in the different path classes. We begin by taking a closer look at the structure of Łukasiewicz paths.

Of course, the number of excursions of given length n strongly depends on the structure of the step set $\mathcal{S}$. For example, in the case of Dyck paths, i.e., $\mathcal{S} = \{-1, 1\}$, there cannot be any excursions of odd length—Dyck paths are said to be periodic lattice paths.

Definition 5.2.1 (Periodicity of lattice paths).
Let $\mathcal{S}$ be a Łukasiewicz step set with corresponding characteristic polynomial $S(u) = \sum_{s\in\mathcal{S}} u^s$. Then the period of $\mathcal{S}$ (and the associated lattice path family) is the largest integer p for which a polynomial Q satisfying

$$u\,S(u) = Q(u^p)$$

exists. If $p = 1$, then $\mathcal{S}$ is said to be aperiodic, otherwise $\mathcal{S}$ is said to be p-periodic.

Remark.
Observe that if a step set $\mathcal{S}$ has period p, then there are only excursions of length n where $n \equiv 0 \pmod{p}$. This can be seen by considering the generating function enumerating unrestricted paths of length n with respect to their height, i.e., $S(u)^n$. Obviously, the number of excursions of length n is at most the number of unrestricted paths ending at altitude 0, and the latter one can be written as

$$[u^0]\,S(u)^n = [u^n](u\,S(u))^n = [u^n]Q(u^p)^n.$$

Hence, if $n \not\equiv 0 \pmod p$, there are no unrestricted paths ending on the horizontal axis—and thus also no excursions.

With these elementary observations in mind, we can now focus on our main problem: determining a suitable generating function in order to enumerate r-ascents in different classes of nonnegative Łukasiewicz paths. In this context, the well-known *kernel method* will prove to be an appropriate approach.

Proposition 5.2.2.
Let $F(z,t,v)$ be the trivariate ordinary generating function counting non-negative Łukasiewicz paths with step set $\mathcal{S}$ starting at 0, where z marks the length of the path, t marks the number of r-ascents, and v marks the final altitude of the path. Then $F(z,t,v)$ can be expressed as

$$F(z,t,v) = \frac{v - V(z,t)}{v - zL(z,t,v)} L(z,t,v), \tag{5.1}$$

where

$$L(z,t,v) = \frac{1}{1 - z\,S_+(v)} + (t-1)(z\,S_+(v))^r,$$

and where $v = V(z,t)$ is the unique solution of the polynomial equation

$$\big(v - z - z(t-1)(z\,S_+(v))^r\big)(1 - z\,S_+(v)) - z^2\,S_+(v) = 0 \tag{5.2}$$

that satisfies $V(0,1) = 0$. The function $V(z,t)$ is holomorphic in a neighborhood of $(z,t) = (0,1)$.

Proof. Let $\Phi^{(k)}(z,t,v)$ denote the trivariate generating function enumerating non-negative Łukasiewicz paths with respect to the step set $\mathcal{S}$ with precisely k mountains (i.e., with k occurrences of the pattern $\nearrow\searrow$, where $\nearrow$ represents any of the allowed up steps) that end in a down step. The variables z, t, and v mark path length, number of r-ascents, and the final altitude of the path, respectively.

By definition of $\Phi^{(k)}(z,t,v)$, we have $\Phi^{(0)}(z,t,v) = 1$. We now construct $\Phi^{(k+1)}(z,t,v)$ from $\Phi^{(k)}(z,t,v)$ in order to establish a functional identity. Observe that an r-ascent occurs precisely if a sequence of r upsteps followed by at least one downstep is added after a downstep. Then, observe that $L(z,t,v)$ as defined in the statement above is the generating function corresponding to a sequence of up steps (the first summand enumerates sequences of up steps without considering r-ascents; the second summand marks sequences of length r with the variable t). Thus, $L(z,t,v) - 1$ enumerates non-empty sequences of up steps. From there it is easy to see that the generating function $\Phi^{(k)}(z,t,v)(L(z,t,v) - 1)$ enumerates paths with k mountains and an appended non-empty sequence of up steps. The generating function $\Phi^{(k+1)}(z,t,v)$ can then be obtained by attaching another non-empty sequence of down steps. Formally, this can be achieved by substituting v^j (which corresponds to a path ending at altitude j) with

$$v^j \mapsto \sum_{\ell=1}^{j} z^\ell v^{j-\ell} = \frac{z/v}{1-z/v}(v^j - z^j).$$

In particular, this substitution is also applied for $j=0$, i.e., the paths with altitude 0 (which correspond to v^0). In this case the substitution reads $v^0 \mapsto 0$, which means that these paths get eliminated as we are not allowed to take a subsequent down step. Altogether, this gives the recurrence relation

$$\Phi^{(k+1)}(z,t,v) \;=\; \frac{z/v}{1-z/v}\big(\Phi^{(k)}(z,t,v)(L(z,t,v)-1) - \Phi^{(k)}(z,t,z)(L(z,t,z)-1)\big), \quad (5.3)$$

because carrying out the substitution gives the difference between $\Phi^{(k)}(z,t,v)(L(z,t,v)-1)$ and the same term with v replaced by z, multiplied with the factor $\frac{z/v}{1-z/v}$. Observe that summing $\Phi^{(k)}(z,t,v)$ over $k \geq 0$ yields the generating function

$$\Phi(z,t,v) := \sum_{k\geq 0} \Phi^{(k)}(z,t,v)$$

enumerating paths with an arbitrary number of mountains that end on a down step. Thus, summation over $k \geq 0$ in (5.3) proves that $\Phi(z,t,v)$ satisfies the functional equation

$$\Phi(z,t,v)\Big(1-\frac{z}{v-z}(L(z,t,v)-1)\Big) = 1-\frac{z}{v-z}\Phi(z,t,z)(L(z,t,z)-1), \qquad (5.4)$$

where the summand 1 on the right-hand side actually comes from $\Phi^{(0)}(z,t,v)=1$.

Rewriting the left-hand side of this equation after plugging in the definition of $L(z,t,v)$ yields

$$\Phi(z,t,v)\frac{(v-z-z(t-1)(zS_+(v))^r)(1-zS_+(v))-z^2S_+(v)}{(v-z)(1-zS_+(v))}.$$

We proceed in the spirit of the kernel method (see [1, 52]), which basically revolves around the idea of setting v to a suitable root $V(z,t)$ of the polynomial in the numerator such that the numerator (5.2) of the left-hand side disappears.

The existence of such a function is guaranteed by means of the holomorphic implicit function theorem (see, e.g., [35, Section 0.8]). In our case this theorem allows to conclude that in a sufficiently small neighborhood of $(z,t)=(0,1)$ there has to be a unique holomorphic function $V(z,t)$ satisfying $V(0,1)=0$ such that the numerator in the left-hand side of (5.4) disappears by setting $v=V(z,t)$.

At the same time, the denominator cannot vanish: For $t=1$ Equation (5.2) defining $V(z,t)$ can be rewritten as $V(z,1)=zV(z,1)S(V(z,1))$, which allows us to obtain a power series expansion for $V(z,1)$ in the neighborhood of $z=0$. As $V(z,t)$ is holomorphic and $V(z,1)\neq z$, we find $V(z,t) \neq z$ in a small neighborhood of $(z,t) = (0,1)$ by continuity. Thus, the first factor in the denominator does not vanish there. Simultaneously, concerning the second factor, if we had $z S_+(V(z,t)) = 1$, then the kernel equation (5.2) would simplify to $-z$ (which is not identically 0 in a neighborhood of $z = 0$), meaning that $v = V(z,t)$ would no longer be a solution, in contradiction to its definition. Thus, the denominator does not vanish simultaneously with the numerator in a small neighborhood of $(z,t) = (0,1)$.

Thus, we can use this implicitly defined function to find an expression for $\Phi(z,t,z)$—and then, after plugging that into (5.4) and doing some simplifications, we arrive at

$$\Phi(z,t,v) = \frac{v - V(z,t)}{v - zL(z,t,v)}.$$

Then, in order to prove (5.1), recall that $\Phi(z,t,v)$ enumerates all non-negative Łukasiewicz paths ending on a down step $\searrow$. Thus, the generating function enumerating all non-negative Łukasiewicz paths $F(z,t,v)$ can be obtained from $\Phi(z,t,v)$ by appending another (possibly empty) sequence of upsteps, and accounting for another possible r-ascent. This yields

$$F(z,t,v) = \Phi(z,t,v)L(z,t,v)$$

and proves the statement. □

The combinatorial nature of $F(z,t,v)$ allows us to draw an interesting conclusion with respect to the implicitly defined function $V(z,t)$.

Corollary 5.2.3.
Let $V(z,t)$ be the implicitly defined function solving (5.2) from Proposition 5.2.2. Assume that the underlying step set $\mathcal{S}$ has period p. Then $V(z,t)$ is analytic around the origin $(z,t) = (0,0)$ with power series representation

$$V(z,t) = \sum_{j \geq 0} g_j(t) z^{jp+1}, \tag{5.5}$$

where the $g_j(t)$ are polynomials with integer coefficients. Combinatorially, $V(z,t)/z$ is the bivariate generating function enumerating Łukasiewicz excursions with respect to $\mathcal{S}$, where z and t mark the length of the path and the number of r-ascents within, respectively.

Proof. Setting $v = 0$, i.e., ignoring all Łukasiewicz paths not ending on the starting altitude, yields $F(z,t,0) = V(z,t)/z$ as the factor $L(z,t,v)$ in (5.1) then cancels against the denominator. The combinatorial interpretation of the trivariate generating function $F(z,t,v)$ together with the fact that for a p-periodic step set $\mathcal{S}$ there are no Łukasiewicz excursions of length n for $p \nmid n$ proves all the statements above. □

Now, with an appropriate generating function at hand let us discuss our approach for the asymptotic analysis of the number of ascents in a nutshell.

Basically, we set $v = 0$ to obtain a bivariate generating function enumerating Łukasiewicz excursions, and we set $v = 1$ to obtain a generating function enumerating Łukasiewicz meanders. The appropriate generating functions for the factorial moments of $E_{n,r}$ and $M_{n,r}$ (from which expected value and variance can be computed) are then obtained by first differentiating the corresponding generating function with respect to t (possibly more often than once) and then setting $t = 1$ in this partial derivative. The growth of the coefficients of this function can then be extracted by means of singularity analysis.

In particular, this means that in order to compute the asymptotic expansions for the quantities we are interested in, we only need more information on $V(z, 1)$ as well as the partial derivatives $\frac{\partial^\nu}{\partial t^\nu} V(z, t)\Big|_{t=1}$.

Notation.
For the sake of simplicity, and because we will deal with these expressions throughout the entire paper, we omit the second argument in $V(z, t)$ in case $t = 1$, i.e., we set $V(z) := V(z, 1)$, $V_t(z) := V_t(z, 1) = \frac{\partial}{\partial t} V(z, t)|_{t=1}$, $V_z(z) := V_z(z, 1) = \frac{\partial}{\partial z} V(z, t)|_{t=1}$, and so on.

Example 5.2.4 (Explicit $F(z, t, v)$).
In the case of $\mathcal{S} = \{-1, 1\}$ and $r = 1$ the generating function $F(z, t, v)$ can be computed explicitly and we find

$$\begin{aligned} &F(z,t,v) \\ &\quad = \frac{(1+(t-1)vz(1-vz))((1-2vz)(1-(t-1)z^2) - \sqrt{(1-(t+3)z^2)(1-(t-1)z^2)}}{2z(1-(t-1)z^2)(z - v + v^2 z + vz^2(t-1)(1-z))}. \end{aligned} \tag{5.6}$$

In particular, for $v = 0$ (i.e., when we want to get the generating function for 1-ascents in Dyck paths), we find

$$F(z, t, 0) = \frac{1 - (t-1)z^2 - \sqrt{(1-(t+3)z^2)(1-(t-1)z^2)}}{2z^2(1-(t-1)z^2)}.$$

5.3 Generating Functions: A Combinatorial Approach

The combinatorial interpretation of the implicitly defined function $V(z, t)$ solving the kernel equation (5.2) motivates the question whether there is a construction of $F(z, t, v)$ that is derived from the underlying combinatorial structure, instead of finding this structure as a side effect.

The following proposition describes an integral relation enabling a purely combinatorial derivation of the generating function $F(z, t, v)$.

Proposition 5.3.1.

The excursions of Łukasiewicz paths of length n with respect to some step set $\mathcal{S}$ correspond to rooted plane trees with $n+1$ nodes and node degrees contained in the set $1+\mathcal{S}$.

An r-ascent in a Łukasiewicz excursion with respect to the step set $\mathcal{S}$ corresponds to a rooted subtree such that the leftmost leaf in this subtree has height r, and additionally the root node of the subtree is not a leftmost child itself (in the original tree).

Proof. As pointed out in e.g. [2, Example 3], this bijection between rooted plane trees with given node degrees and Łukasiewicz excursions is well known. See [41, Section 11.3] for an approach using words. However, as this bijection and its consequences makes up an integral part of the argumentation within this chapter, we present a short proof ourselves. Furthermore, proving the bijection allows us to find the substructure in the tree corresponding to an r-ascent.

Given a rooted plane tree T consisting of n nodes whose outdegrees are contained in $1+\mathcal{S}$, we construct a lattice path as follows: when traversing the tree in preorder[1], if passing a node with outdegree d, take a step of height $d-1$. The resulting lattice path thus consists of n steps, and always ends on altitude -1, which follows from

$$\sum_{v\in T}(\deg(v)-1) = \sum_{v\in T}\deg(v) - n = (n-1)-n = -1,$$

where $\deg(v)$ denotes the outdegree (i.e., the number of children) of a node v in the tree T. In particular, observe that by taking the first $n-1$ steps of the lattice path, we actually end up with a Łukasiewicz excursion using the steps from $\mathcal{S}$. To see this, first observe that as the last node traversed in preorder certainly is a leaf, meaning that the nth step in the corresponding lattice path is a down step. As the path ends on altitude -1 after n steps, we have to arrive at the starting altitude after $n-1$ steps.

Furthermore, as illustrated in Figure 5.2, adding one to the current height of the constructed lattice path gives the size of the stack remembering the children that still have to be visited while traversing the tree in preorder. Combining the two previous arguments proves that the first $n-1$ steps in the constructed lattice path form a Łukasiewicz excursion.

Similarly, by simply reversing the lattice path construction, a rooted plane tree of size $n+1$ with node degrees in $1+\mathcal{S}$ can be constructed from any Łukasiewicz excursion of length n with respect to $\mathcal{S}$. This establishes the bijection between the two combinatorial families.

Finally, Figure 5.3 illustrates what r-ascents in Łukasiewicz paths are mapped to by means of the bijection above. □

[1] Traversing a tree in preorder corresponds to the order in which the nodes are visited when carrying out a depth-first search on it.

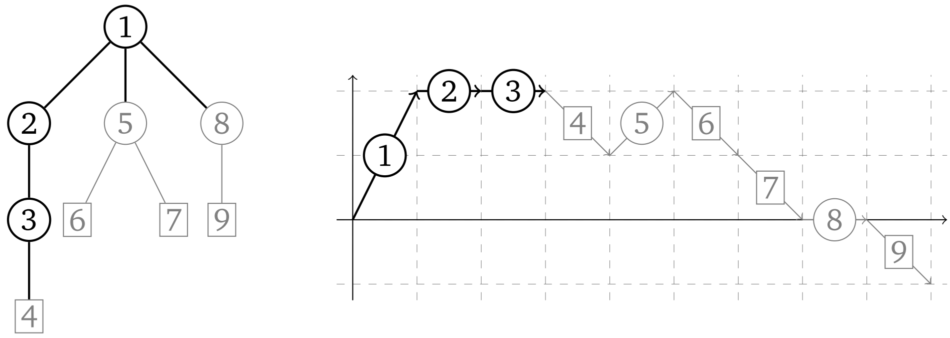


Figure 5.2: Bijection between Łukasiewicz paths and trees with given node degrees. The emphasized nodes and edges indicate the construction of the tree after the first three steps, which illustrates that the height of the Łukasiewicz path is one less than the number of available node positions in the tree.

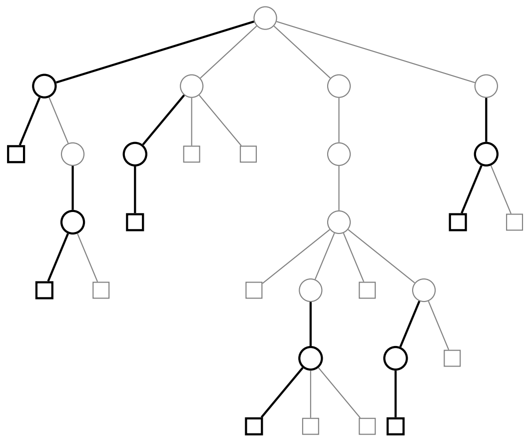


Figure 5.3: Plane tree with 30 nodes bijective to some Łukasiewicz excursion with respect to the step set $\mathcal{S} = \{-1, 0, 1, 2, 3\}$ whose number of 2-ascents is 6. The edges and nodes corresponding to the 2-ascents are emphasized.

In some sense, the bijection from Proposition 5.3.1 can be seen as a generalization of the well-known bijection between Dyck paths and binary trees where the tree is traversed in preorder, internal nodes correspond to up steps and leaves to down steps.

The fact that there is this bijection between Łukasiewicz excursions and these special trees with given node degrees allows us to draw an immediate conclusion regarding the corresponding generating functions.

Corollary 5.3.2 ([28]).
Let $V(z, t)$ be the generating function enumerating rooted plane trees with node degrees in $1 + \mathcal{S}$ where z marks the number of nodes and t marks the number of r-ascents in the corresponding Łukasiewicz excursion. Then $V(z, t)/z$ enumerates Łukasiewicz excursions with respect to $\mathcal{S}$ based on their length (marked by z) and the number of r-ascents (marked

by t).

Additionally, $V(z,t)$ satisfies the equations

$$V(0,t)=0, \qquad V(z,t)=zL(z,t,V(z,t)), \tag{5.7}$$

where

$$L(z,t,v)=\frac{1}{1-z\,S_+(v)}+(t-1)(z\,S_+(v))^r$$

enumerates sequences of up steps. In particular, $V(z):=V(z,1)$, the ordinary generating function enumerating plane trees with node degrees in $1+\mathcal{S}$ with respect to their size, satisfies

$$V(0)=0, \qquad V(z)=zV(z)\,S(V(z)). \tag{5.8}$$

Proof. The first part of this statement is an immediate consequence of the bijection from Proposition 5.3.1. In order to prove (5.7), we observe that $\mathcal{V}$, the combinatorial class of plane trees with vertex outdegrees in $1+\mathcal{S}$, can be constructed combinatorially by means of the symbolic equation

$$\mathcal{V}=\bigcirc\times\mathrm{SEQ}\Big(\bigcirc\times\sum_{\substack{s\in\mathcal{S}\\ s\geq 0}}\mathcal{V}^s\Big).$$

In a nutshell, this constructs trees in $\mathcal{V}$ by explicitly building the path to the leftmost leaf (the first factor in the equation above) in the tree as a sequence of nodes. Apart from a leftmost child, these nodes also have an additional $s\in\mathcal{S}$ branches, $s\geq 0$, where again a tree from $\mathcal{V}$ is attached. Considering that we obtain an r-ascent when using this construction with a sequence of length r, this is precisely what is enumerated by $L(z,t,V(z,t))$. Thus, the symbolic equation directly translates into the functional equation in (5.7). The condition $V(0,t)=0$ is a consequence of the fact that there are no rooted trees without nodes.

Setting $t=1$ in (5.7) leads to (5.8). We also want to give a combinatorial proof of (5.8):

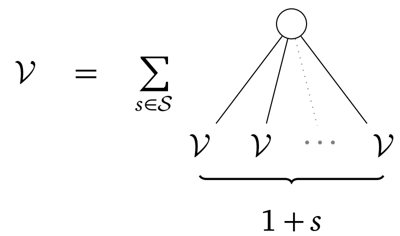


Figure 5.4: Symbolic equation for the family of plane trees $\mathcal{V}$ with outdegrees in $1+\mathcal{S}$. The generating function for $\mathcal{V}$ is $V(z)$, and the root node is enumerated by z.

The implicit equation follows from the observation that a tree with node degrees from $1+\mathcal{S}$ can be seen as a root node (enumerated by z) where $1+s$ for $s\in\mathcal{S}$ such trees are attached. Translating this into the language of generating functions via the symbolic equation illustrated in Figure 5.4, yields

$$V(z) = z \sum_{s \in \mathcal{S}} V(z)^{1+s} = zV(z)\, S(V(z)). \qquad \square$$

The shape of the functional equation (5.8), which is an immediate consequence of the recursive structure of the underlying trees, is rather special. While it is tempting to cancel $V(z)$ on both sides of this equation, it is better to leave it in the present form: on the one hand, $S(u)$ starts with the summand $1/u$—and on the other hand, we require (5.8) to be in this special form $y = z\,\phi(y)$ such that we can use singular inversion to obtain the asymptotic behavior of the coefficients of the generating function $V(z)$. This is investigated in detail in Section 5.4.

The following proposition is the combinatorial counterpart to Proposition 5.2.2.

Proposition 5.3.3.
Let $F(z,t,v)$ be the trivariate ordinary generating function counting non-negative Łukasiewicz paths with step set $\mathcal{S}$ starting at 0, where z marks the length of the path, t marks the number of r-ascents, and v marks the final altitude of the path. Then $F(z,t,v)$ can be expressed as

$$F(z,t,v) = \frac{v - V(z,t)}{v - zL(z,t,v)} L(z,t,v),$$

where $V(z,t)$ and $L(z,t,v)$ are defined as in Corollary 5.3.2.

Proof ([28]). It is not hard to see that by considering a sequence of paths enumerated by $L(z,t,v)$ followed by a single down step (the corresponding generating function for this class is $\frac{1}{1-L(z,t,v)z/v}$), any unrestricted Łukasiewicz path with respect to $\mathcal{S}$ ending on a down step can be constructed.

We want to subtract all paths that pass below the starting altitude in order to obtain the trivariate generating function $\Phi(z,t,v)$ enumerating just the non-negative Łukasiewicz paths. The paths passing below the axis can be decomposed into an excursion enumerated by $V(z,t)/z$ (see Corollary 5.3.2), followed by an (illegal) down step enumerated by z/v, and ending with an unrestricted path again. Thus, the paths to be subtracted are enumerated by

$$\frac{V(z,t)}{z} \frac{z}{v} \frac{1}{1 - L(z,t,v)\frac{z}{v}}.$$

Therefore, we find

$$\Phi(z,t,v) = \frac{v - V(z,t)}{v - zL(z,t,v)}.$$

Keeping in mind that $\Phi(z,t,v)$ only enumerates those non-negative Łukasiewicz paths ending on a down step $\searrow$, the generating function $F(z,t,v)$ enumerating all such paths can be obtained by appending another sequence of upsteps, i.e.,

$$F(z,t,v) = \Phi(z,t,v)L(z,t,v).$$

This proves the statement. □

Now, as we have derived a suitable generating function both via an analytic as well as via a combinatorial approach, we are interested in extracting information like, for example, asymptotic growth rates from $F(z,t,v)$. In order to do so, we need to have a closer look at the function $V(z,t)$, which, as we have already seen in both of the previous approaches, plays a fundamental role in the analysis of ascents.

5.4 Singularity Analysis of Inverse Functions

The aim of this section is, on the one hand, to state and prove an extension of [21, Remark VI.17]. In fact, we simply confirm what is announced in the footnote in [21, p. 405] and give more details. Then, we use these results in order to derive relevant information on the generating function $V(z,t)$ from before.

For the following two propositions, we borrow the notation used in [21, Chapter VI.7].

Proposition 5.4.1.
Let $\phi(u)$ be analytic with radius of convergence $0 < R \le \infty$, $\phi(0) \neq 0$, $[u^n]\,\phi(u) \ge 0$ for all $n \ge 0$ and $\phi(u)$ not affine linear. Assume that there is a positive $\tau \in (0,R)$ such that $\tau\,\phi'(\tau) = \phi(\tau)$. Finally assume that $\phi(u)$ is a p-periodic power series for some maximal p. Denote the set of all pth roots of unity by $G(p)$.

Then there is a unique function $y(z)$ satisfying $y(z) = z\,\phi(y(z))$ which is analytic in a neighborhood of 0 with $y(0) = 0$. It has radius of convergence $\rho = \tau/\,\phi(\tau)$ around the origin. For $|z| \le \rho$, it has exactly singularities at $z = \rho\zeta$ for $\zeta \in G(p)$. For $z \to \rho$, we have the singular expansion

$$y(z) \overset{z\to\rho}{\sim} \sum_{j\ge 0} (-1)^j d_j \left(1 - \frac{z}{\rho}\right)^{j/2}$$

for some computable constants d_j, $j \ge 0$. We have $d_0 = \tau$ and $d_1 = \sqrt{2\,\phi(\tau)/\,\phi''(\tau)}$. Additionally, we have $[z^n]y(z) = 0$ for $n \not\equiv 1 \pmod{p}$.

Proof. Existence, uniqueness, radius of convergence as well as singular expansion around $z \to \rho$ of $y(z)$ are shown in [21, Theorem VI.6].

As ϕ is a p-periodic power series and $\phi(0) \neq 0$, there exists an aperiodic function χ such that $\phi(u) = \chi(u^p)$. From the non-negativity of the coefficients of $\phi(u)$, it is clear that $\chi(u)$ has non-negative coefficients and is analytic for $|u| < R^p$. We consider $\psi(u) := \chi(u)^p$. Then ψ is again analytic for $|u| < R^p$, it has clearly non-negative coefficients, $\psi(0) \neq 0$ and $\psi(u)$ is not an affine linear function. If $[u^m]\chi(u) > 0$ and $[u^n]\chi(u) > 0$ for some $m < n$, then $[u^{pm}]\psi(u) > 0$ as well as $[u^{pm+(n-m)}]\psi(u) > 0$, which implies that ψ is aperiodic.

Finally, we have

$$\tau^p\psi'(\tau^p) = p\tau^p\chi(\tau^p)^{p-1}\chi'(\tau^p) = \tau\,\phi(\tau)^{p-1}\,\phi'(\tau) = \phi(\tau)^p = \chi(\tau^p)^p = \psi(\tau^p).$$

Considering the functional equation $Y(Z) = Z\psi(Y(Z))$, we see that all assumptions of [21, Theorem VI.6] are satisfied; thus it has a unique solution $Y(Z)$ with $Y(0) = 0$ which is analytic around the origin. By the same result, $Y(Z)$ has radius of convergence

$$\frac{\tau^p}{\psi(\tau^p)} = \frac{\tau^p}{\chi(\tau^p)^p} = \left(\frac{\tau}{\phi(\tau)}\right)^p = \rho^p$$

and, as ψ is aperiodic, the only singularity of $Y(Z)$ with $|Z| \le \rho^p$ is $Z = \rho^p$.

We consider the function $\widetilde{y}(z) := z\chi(Y(z^p))$. By definition, it is analytic for $|z| < \rho$ and its only singularities with $|z| \le \rho$ are those z with $z^p = \rho^p$, i.e., $z = \rho\zeta$ for $\zeta \in G(p)$. It is also clear by definition that $[z^n]\widetilde{y}(z) = 0$ for $n \not\equiv 1 \pmod p$. We have $\widetilde{y}(0) = 0$ and

$$z\,\phi(\widetilde{y}(z)) = z\chi((\widetilde{y}(z))^p) = z\chi(z^p\chi(Y(z^p))^p) = z\chi(z^p\psi(Y(z^p))) = z\chi(Y(z^p)) = \widetilde{y}(z)$$

because $z^p\psi(Y(z^p)) = Y(z^p)$ by definition of Y. This implies that $y = \widetilde{y}$. □

While the following proposition is particularly useful in the context of the previous one, it also holds in a slightly more general setting. It gives a detailed description of the singular expansions for p-periodic power series like above.

Proposition 5.4.2.
Let p be a positive integer and let y be analytic with radius of convergence $0 < \rho \le \infty$, where $[z^n]y(z) = 0$ for $n \not\equiv 1 \pmod p$. Assume that $y(z)$ has p dominant singularites located at $\zeta\rho$ for $\zeta \in G(p)$, and that for some $L \ge 0$ and $z \to \rho$, we have the singular expansion

$$y(z) \overset{z\to\rho}{=} \sum_{j=0}^{L-1} d_j\Big(1 - \frac{z}{\rho}\Big)^{-\alpha_j} + O\Big(\Big(1 - \frac{z}{\rho}\Big)^{-\alpha_L}\Big),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_L$ are complex numbers such that $\operatorname{Re}(\alpha_j) \ge \operatorname{Re}(\alpha_{j+1})$ for all $0 \le j < L$.

Then, for $\zeta \in G(p)$, the singular expansion of $y(z)$ for $z \to \zeta\rho$ is given by

$$y(z) \overset{z\to\zeta\rho}{=} \sum_{j=0}^{L-1} \zeta d_j\Big(1 - \frac{z}{\zeta\rho}\Big)^{-\alpha_j} + O\Big(\Big(1 - \frac{z}{\zeta\rho}\Big)^{-\alpha_L}\Big),$$

i.e., the expansion for $z \to \zeta\rho$ can be obtained by multiplying the expansion for $z \to \rho$ with ζ and substituting $z \mapsto \zeta/\rho$. Finally, for the coefficients of $y(z)$ we find

$$[z^n]y(z) = [\![p \mid 1-n]\!][z^n]\Big(p\sum_{j=0}^{L-1} d_j\Big(1 - \frac{z}{\rho}\Big)^{-\alpha_j} + O\Big(\Big(1 - \frac{z}{\rho}\Big)^{-\alpha_L}\Big)\Big), \tag{5.9}$$

which can be made explicit easily by means of singularity analysis (cf. [21, Chapter VI.4]). In particular,

$$[z^n]y(z) = [\![p \mid 1-n]\!]\Big(\sum_{j=0}^{L-1} \frac{pd_j}{\Gamma(\alpha_j)} n^{\alpha_j-1}\rho^{-n} + O\big(n^{\mathrm{Re}(\alpha_0)-2}\rho^{-n}\big) + O\big(n^{\mathrm{Re}(\alpha_L)-1}\rho^{-n}\big)\Big).$$

Proof. As $[z^n]y(z) = 0$ for $n \not\equiv 1 \pmod p$ there is a function χ, analytic around the origin, such that $y(z) = z\chi(z^p)$. Thus, for every $\zeta \in G(p)$, we have

$$y(\zeta z) = \zeta z\chi((\zeta z)^p) = \zeta z\chi(z^p) = \zeta y(z)$$

or, equivalently,

$$y(z) = \zeta y\Big(\frac{z}{\zeta}\Big).$$

Thus the singular expansion for $z \to \zeta\rho$ follows from that for $z \to \rho$ by replacing z with z/ζ and multiplication by ζ.

With the singular expansions at all the dominant singularities located at $\zeta\rho$ for $\zeta \in G(p)$ at hand, we are able to extract the overall growth of the coefficients of $y(z)$ by first applying singularity analysis to every expansion separately, and then summing up all these contributions. When doing so, we use the well-known property of roots of unity that

$$\sum_{\zeta\in G(p)} \zeta^m = p[\![p \mid m]\!] \tag{5.10}$$

for $m \in \mathbb{Z}$ in order to rewrite the occurring sums as $\sum_{\zeta\in G(p)} \zeta^{1-n} = p[\![p \mid 1-n]\!]$. Comparing the resulting asymptotic expansion with (5.9) proves the statement. □

The following result is a consequence of Propositions 5.2.2 (resp. Corollary 5.3.2), 5.4.1, and 5.4.2. It shows that actually we have more than enough information to carry out the asymptotic analysis of the number of ascents, although in general we do not know the function $V(z,t)$ explicitly.

Corollary 5.4.3.
Let $V(z,t)$ be the bivariate generating function from Corollary 5.3.2 and let $V(z) = V(z,1)$.

1. Let $\tau > 0$ be the uniquely determined positive constant satisfying $S'(\tau) = 0$. Then $V(z)$ has radius of convergence $\rho := 1/S(\tau)$ with a square-root singularity for $z \to \rho$. If $\mathcal{S}$ has period p, then the dominant singularities (i.e., singularities with modulus ρ) are located at $\zeta\rho$ with $\zeta \in G(p)$. The corresponding expansions are given by

$$V(z) \overset{z\to\zeta\rho}{=} \zeta\tau - \zeta\sqrt{\frac{2S(\tau)}{S''(\tau)}}\Big(1-\frac{z}{\zeta\rho}\Big)^{1/2} - \zeta\frac{S(\tau)S'''(\tau)}{3S''(\tau)^2}\Big(1-\frac{z}{\zeta\rho}\Big) + O\Big(\Big(1-\frac{z}{\zeta\rho}\Big)^{3/2}\Big). \tag{5.11}$$

2. The evaluation of the partial derivatives $\frac{\partial^\nu}{\partial t^\nu} V(z,t)$ at $t=1$ can be expressed in terms of $V(z)$. For instance, the first partial derivative is given as

$$V_t(z) = -z \frac{(V(z)-z)^r}{V(z)^{r+2} S'(V(z))}. \tag{5.12}$$

Proof. Let $F(z,t,v)$ be given as in the statement of Proposition 5.2.2.

1. The singular expansion of $V(z)$ for $z \to \zeta\rho$ follows from applying Propositions 5.4.1 and 5.4.2 to our given context: Plugging in $t=1$ in (5.2) (or, alternatively, the combinatorial interpretation of $V(z,t)$ as stated in Corollary 5.3.2) proves that $V(z)$ satisfies the functional equation $V(z) = z\,\phi(V(z))$ with $\phi(u) = u\,S(u)$. For this particular $\phi(u)$, the fundamental constant τ is defined as the unique positive real number satisfying $S'(\tau) = 0$. Then, Proposition 5.4.1 yields the singular structure as well as the singular expansion for $z \to \rho$ after checking that $\phi(u)$ satisfies the necessary conditions—and indeed, we have $\phi(0) = 1 \neq 0$, and for step sets other than $\mathcal{S} = \{-1, 0\}$, ϕ is also a nonlinear function. With the computed expansion for $z \to \rho$, we obtain (5.11) from Proposition 5.4.2.

2. As a consequence of $V(z,t)$ being a bivariate generating function where the coefficient of z^n is given by a polynomial in t (see Corollary 5.3.2), and as we know that $V(z) = V(z,1)$ has radius of convergence $\rho = 1/S(\tau)$, we obtain that $V(z,t)$ is analytic in a small neighborhood of $(z,t) = (0,1)$. This allows us to implicitly differentiate the functional equation (5.7) with respect to t. Within the implicit derivative of this equation, the partial derivative $\frac{\partial}{\partial t} V(z,t)$ only occurs linearly, so that we can solve for it.

   Equation (5.12) can now be obtained by setting $t=1$ and using the relation $z\,S(V(z)) = 1$ (see Corollary 5.3.2). Higher-order partial derivatives can be obtained by differentiating again with respect to t before setting $t=1$. □

These observations allow us to employ singularity analysis (see, e.g., [21, Chapter VI]) in order to carry out a precise analysis of the number of r-ascents in certain families of Łukasiewicz paths in the following sections.

We conclude this section with a very useful observation with respect to the nature of the structural constant τ.

Lemma 5.4.4.
Let $\mathcal{S}$ be some[2] Łukasiewicz step set and let τ be the corresponding structural constant, i.e. the unique positive number satisfying $S'(\tau) = 0$. Then $\tau \leq 1$ with equality if and only if $\mathcal{S} = \{-1, 0, 1\}$ or $\mathcal{S} = \{-1, 1\}$.

[2] Recall that we excluded $\mathcal{S} = \{-1, 0\}$ in Section 5.1.

Proof. First, observe that S' is a strictly increasing function. For $u \geq 1$, we have

$$S'(u) \geq S'(1) = -1 + \sum_{\substack{s \in \mathcal{S} \\ s \geq 0}} s \geq 0$$

with equality if and only if $u = 1$ and $\mathcal{S} \in \{\{-1,0,1\},\{-1,1\}\}$. Monotonicity of S' then implies the assertion of the lemma. □

Remark.
The number of Łukasiewicz excursions of length n is trivially bounded from above by $|\mathcal{S}|^n$ which corresponds to all paths with the same step set but without any restrictions. Consequently, the radius of convergence ρ of the generating function of excursions $V(z)/z$ is bounded from below by $\frac{1}{|\mathcal{S}|}$.

Assume that $\mathcal{S} \notin \{\{-1,0,1\},\{-1,1\}\}$. In this case, $\tau < 1$ and $S'(u) > 0$ for $\tau < u < 1$. This implies $|\mathcal{S}| = S(1) > S(\tau) = 1/\rho$, which means that the radius of convergence ρ is strictly larger than the trivial bound $1/|\mathcal{S}|$. In other words, for all but the two simple step sets $\{-1,0,1\}$ and $\{-1,1\}$, the restriction to Łukasiewicz excursions leads to an exponentially smaller number of admissible paths.

The quantity $S'(1)$ is also referred to as the *drift* of the walk (see, e.g., [2, Section 3.2]) and strongly influences the asymptotic behavior of corresponding meanders.

5.5 Analysis of Ascents

5.5.1 Analysis of Excursions

In this section we focus on the analysis of *excursions*, i.e., paths that start and end on the horizontal axis. As mentioned in Section 5.2, on the generating function level, this corresponds to setting $v = 0$ in $F(z,t,v)$ from (5.1). Also note that from this point on it is quite useful to replace $S_+(v) = S(v) - 1/v$ in $F(z,t,v)$.

Recall that $E_{n,r}$ is the random variable modeling the number of r-ascents in a random non-negative Łukasiewicz excursion of length n with respect to some given step set $\mathcal{S}$.

Theorem 5.5.1.
Let $r \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $p \geq 1$ be the period of the step set $\mathcal{S}$. Let τ be the structural constant, i.e., the unique positive solution of $S'(\tau) = 0$. Set $c := \tau\, S(\tau)$.

Then, the expected number of r-ascents in Łukasiewicz paths of length n for $n \equiv 0 \pmod{p}$ as well as the corresponding variance grow with $n \to \infty$ according to the asymptotic expansions

$$\begin{aligned}\mathbb{E}E_{n,r} = \frac{(c-1)^r}{c^{r+2}}n + \frac{(c-1)^{r-2}}{2\tau^2 c^{r+2} S''(\tau)^2}\Big(& S''(\tau)^2\tau^2\big(4c^2-(r+8)c+r+4\big) \\ & - S''(\tau)S(\tau)\big(6c^2-6(r+2)c+r^2+5r+6\big) \\ & - S'''(\tau)c(2c^2-(r+4)c+r+2)\Big) + O(n^{-1/2})\end{aligned} \tag{5.13}$$

and

$$\mathbb{V}E_{n,r} = \Big(\frac{(c-1)^r}{c^{r+2}} + \frac{(2c-2r-3)(c-1)^{2r}}{c^{2r+4}} - \frac{(c-1)^{2r-2}(2c-r-2)^2}{c^{2r+3}\tau^3 S''(\tau)}\Big)n + O(n^{1/2}). \tag{5.14}$$

Additionally, for $n \not\equiv 0 \mod p$, we have $E_{n,r} = 0$. All O-constants depend implicitly on r.

Proof. While the proof for this theorem actually is quite straightforward, it involves some rather computationally expensive operations with asymptotic expansions, which we have carried out with the help of SageMath, see the corresponding worksheet as referenced at the end of Section 5.1.

As discussed in the remark after the definition of periodic lattice paths, for a p-periodic step set $\mathcal{S}$ there are no excursions of length n for $n \not\equiv 0 \mod p$. Thus, the random variable $E_{n,r}$ degenerates to the constant 0 in these cases, allowing us to focus on the case where n is a multiple of p.

Based on the fact that the generating function enumerating r-ascents within Łukasiewicz excursions is given by $F(z,t,0) = V(z,t)/z$, our general strategy for determining asymptotic expansions for the expected number of r-ascents and the corresponding variance is to compute expansions for the first and second factorial moment by normalizing the extracted coefficients of $V_t(z)/z$ and $V_{tt}(z)/z$.

In order to normalize these extracted coefficients, we need to compute an asymptotic expansion for the number of $\mathcal{S}$-excursions of given length. To accomplish this, we could simply use the general framework developed by Banderier and Flajolet in [2, Theorem 3]—however, we choose to analyze this quantity more directly by applying singularity analysis to the singular expansion of $V(z) = V(z,1)$ given in (5.11). With the help of SageMath and Proposition 5.4.2, we immediately find

$$\begin{aligned}[z^n]\frac{V(z)}{z} = {} & p\sqrt{\frac{S(\tau)^3}{2\pi S''(\tau)}}\, S(\tau)^n n^{-3/2} \\ & - \frac{p}{24}\sqrt{\frac{S(\tau)^3}{2\pi S''(\tau)^7}}\big(45\,S''(\tau)^3 + 5\,S(\tau)\,S'''(\tau)^2 \\ & \qquad - 3\,S(\tau)\,S''(\tau)\,S''''(\tau)\big)S(\tau)^n n^{-5/2} \\ & + O(S(\tau)^n n^{-3})\end{aligned} \tag{5.15}$$

for $n \equiv 0 \pmod{p}$.

For the analysis of the expected value, we turn our attention to $V_t(z)/z$. By Corollary 5.4.3, the derivative $V_t(z)$ can be expressed in terms of $V(z)$. Then, as $V(z)$ and $V_t(z)$ are both analytic with radius of convergence ρ, we can use the singular expansion of $V(z)$ together with (5.12) to arrive at a singular expansion of $V_t(z)/z$. By (5.5) we know that Proposition 5.4.2 can be used to obtain the singular expansion of $V_t(z)$ for $z \to \zeta\rho$ with $\zeta \in G(p)$.

This allows us to extract the nth coefficient of $V_t(z)/z$ by means of singularity analysis—and dividing by the growth of the number of excursions of length n yields an asymptotic expansion for $\mathbb{E}E_{n,r}$. This gives (5.13).

In the same manner, by investigating the second derivative $V_{tt}(z)/z$, we can obtain an asymptotic expansion for the second factorial moment, $\mathbb{E}(E_{n,r}(E_{n,r}-1))$. Then, applying the well-known identity

$$\mathbb{V}E_{n,r} = \mathbb{E}(E_{n,r}(E_{n,r}-1)) + \mathbb{E}E_{n,r} - (\mathbb{E}E_{n,r})^2,$$

proves (5.14). □

By means of Theorem 5.5.1 we are immediately able to determine the asymptotic behavior of interesting special cases. We are particularly interested in the most basic setting: $\mathcal{S} = \{-1, 1\}$, i.e., Dyck paths.

Example 5.5.2 (r-Ascents in Dyck paths).
In the case of Dyck paths, we have $u\,S(u) = 1 + u^2$. From there, it is easy to see that $\tau = 1$ and $\rho = 1/2$, and that the family of paths is 2-periodic. By the same approach as in the proof of Theorem 5.5.1, we can determine the expected number and variance of r-ascents in Dyck paths of length $2n$ with higher precision than stated in Theorem 5.5.1, namely as

$$\mathbb{E}D_{2n,r} = \frac{n}{2^{r+1}} - \frac{(r+1)(r-4)}{2^{r+3}} + \frac{(r^2-11r+22)(r+1)r}{2^{r+6}} n^{-1} + O(n^{-2})$$

and

$$\mathbb{V}D_{2n,r} = \left(\frac{1}{2^{r+1}} - \frac{r^2-2r+3}{2^{2r+3}}\right)n - \left(\frac{r^2-3r-4}{2^{r+3}} - \frac{3r^4-20r^3+29r^2-10r-14}{2^{2r+5}}\right) + O(n^{-1/2}).$$

However, as we have a closed expression for $V(z)$, we can do even better. Because of

$$\frac{V(z)}{z} = \frac{1-\sqrt{1-4z^2}}{2z^2},$$

we can also write down the generating function $V_t(z)/z$ for the expected number of r-ascents explicitly. We find the 2-periodic power series

$$\frac{V_t(z)}{z} = \frac{(1-\sqrt{1-4z^2})^r(1+\sqrt{1-4z^2})}{2^{r+1}\sqrt{1-4z^2}}, \tag{5.16}$$

for which, after substituting $Z = z^2$, we can apply Cauchy's integral formula in order to extract the expected values before normalization with the nth Catalan number C_n explicitly.

Considering a contour γ that stays sufficiently close to the origin and winds around it exactly once, and by the integral substitution $Z = \frac{u}{(1+u)^2}$ we obtain

$$\begin{aligned}
[Z^n]\frac{V_t(z)}{z} &= \frac{1}{2\pi i}\oint_\gamma \frac{V_t(z)/z}{Z^{n+1}}\,dZ \\
&= \frac{1}{2\pi i}\oint_{\tilde{\gamma}} \left(\frac{u}{1+u}\right)^r \frac{1}{1-u}\frac{(1+u)^{2n+2}}{u^{n+1}}\frac{1-u}{(1+u)^3}\,du \\
&= \frac{1}{2\pi i}\oint_{\tilde{\gamma}} \frac{(1+u)^{2n-r-1}}{u^{n-r+1}}\,du = [u^{n-r}](1+u)^{2n-r+1} \\
&= \binom{2n-r-1}{n-1},
\end{aligned}$$

where $\tilde{\gamma}$ denotes the image of γ under the transformation; a curve that also stays close to the origin and winds around it once. After normalization, this proves the exact formula

$$\mathbb{E}D_{2n,r} = \frac{1}{C_n}\binom{2n-r-1}{n-1}.$$

5.5.2 Analysis of Dispersed Excursions

Let $\mathcal{S}$ be a Łukasiewicz step set where $0 \notin \mathcal{S}$. In this setting, we define a *dispersed Łukasiewicz excursion* to be an $\mathcal{S}$-excursion where, additionally, horizontal steps "→" can be taken whenever the path is on its starting altitude. Observe that, by our definition of r-ascents, these horizontal steps do not contribute towards ascents, as only the non-negative steps from $\mathcal{S}$ are relevant.

The motivation to study this specific family of Łukasiewicz paths originates from [34], where the authors investigate the total number of 1-ascents in dispersed Dyck paths using elementary methods. Our goal in this section is to find asymptotic expansions for the number of dispersed Łukasiewicz excursions of given length as well as for the expected number of r-ascents in these paths.

We begin our analysis by constructing a suitable bivariate generating function enumerating dispersed Łukasiewicz excursions with respect to their length and the number of r-ascents.

Proposition 5.5.3.
Let $r \in \mathbb{N}$ and $V(z,t)$ as in Proposition 5.2.2 or Corollary 5.3.2. Then the generating function $D(z,t)$ enumerating dispersed $\mathcal{S}$-excursions where z marks the length of the excursion and t marks the number of r-ascents is given by

$$D(z,t) = \frac{1}{z}\frac{V(z,t)}{1-V(z,t)}. \tag{5.17}$$

Proof ([28]). Let $\mathcal{E}$ denote the combinatorial class of $\mathcal{S}$-excursions. The corresponding bivariate generating function is given by $V(z,t)/z$, as proved in Corollary 5.3.2.

By the symbolic method (see [21, Chapter I]), the combinatorial class $\mathcal{D}$ of dispersed excursions can be constructed as

$$\mathcal{D} = (\mathcal{E} \rightarrow)^* \mathcal{E}.$$

Translating this combinatorial construction in the language of (bivariate) generating functions, we find

$$D(z,t) = \frac{1}{1 - \frac{V(z,t)}{z} z} \frac{V(z,t)}{z},$$

and simplification immediately yields (5.17). □

In preparation for the analysis of the generating function $D(z,t)$, we have to investigate the structure of the dominant singularities. In particular, the following lemma states that in most cases, the dominant singularity of $D(z,1)$ comes from the dominant square root singularities on the radius of convergence of $V(z)$.

Lemma 5.5.4.
The radius of convergence of $D(z,1)$ (as well as for the corresponding partial derivatives with respect to t, i.e., $\frac{\partial^\nu}{\partial t^\nu} D(z,t)|_{t=1}$) is given by $\rho = 1/S(\tau)$, where $\tau > 0$ is the structural constant with respect to the step set $\mathcal{S}$.

Proof ([28]). As $V(z)$ is a power series with non-negative coefficients, we have

$$|V(z)| \le V(|z|) \le V(\rho) = \tau \le 1$$

for $|z| \le \rho$ by Lemma 5.4.4. By the same lemma and because we assumed $0 \notin \mathcal{S}$, equality holds only in case of $\mathcal{S} = \{-1, 1\}$. Thus, the denominator $1 - V(z)$ of $D(z,1)$ does not contribute a pole for $|z| < \rho$. □

Lemma 5.5.4 tells us that in the general case of $\tau \neq 1$, the singularities of $D(z,1)$ are of the same type as the singularities of $V(z)$. Therefore, the precise description of the singular structure of $V(z)$ given in Corollary 5.4.3 allows us to carry out the asymptotic analysis.

Recall that $D_{n,r}$ is the random variable modeling the number of r-ascents in a random dispersed Łukasiewicz excursion of length n with respect to some step set $\mathcal{S}$.

Theorem 5.5.5.
Let $p \ge 1$ be the period of the step set $\mathcal{S}$. Assume additionally that for the structural constant τ we have $\tau \neq 1$.

Then d_n, the number of dispersed Łukasiewicz excursions of length n, satisfies

$$d_n = \frac{1}{\sqrt{2\pi}} \frac{p\tau^k(\tau^p(p-k-1)+k+1)}{(1-\tau^p)^2} \sqrt{\frac{S(\tau)^3}{S''(\tau)}} S(\tau)^n n^{-3/2} + O(S(\tau)^n n^{-5/2}) \qquad (5.18)$$

for $n \equiv k \mod p$ and $0 \le k \le p-1$. Furthermore, the expected number of r-ascents grows with $n \to \infty$ according to the asymptotic expansion

$$\mathbb{E}D_{n,r} = \frac{(\tau S(\tau)-1)^r}{(\tau S(\tau))^{r+2}} n + O(1). \tag{5.19}$$

The O-constants depend implicitly on both r as well as on the residue class of n modulo p.

In a nutshell, the proof of this theorem involves a rigorous analysis of the generating functions $D(z,1)$ (for the overall number of dispersed excursions), as well as of $D_t(z,1) = \frac{1}{z}\frac{V_t(z)}{(1-V(z))^2}$ (for the expected number of ascents in these paths). All computations are carried out in the corresponding SageMath worksheet, `lukasiewicz-dispersed-excursions.ipynb`. Furthermore, while our results as stated in (5.18) and (5.19) only list the asymptotic main term, expansions with higher precision are available in the worksheet as well (they just become rather messy very quickly).

Proof. Before we delve into the analysis, let us recall the setting we have to deal with. As the period of the step set $\mathcal{S}$ is p, the function $V(z)$ (and corresponding derivatives with respect to t) has p dominant square root singularities, located at $\zeta/S(\tau)$ with $\zeta \in G(p)$ (see Corollary 5.4.3).

Furthermore, as $\tau \neq 1$, Lemma 5.5.4 tells us that the singular structure of $D(z,1)$ (and its derivatives with respect to t) is directly inherited from $V(z)$, meaning that the singularities of $D(z,1)$ are of the same type as the singularities of $V(z)$.

Thus, after rewriting

$$D(z,1) = \frac{1}{z}\frac{V(z)}{1-V(z)} = \frac{1}{z}\Big(\frac{1}{1-V(z)} - 1\Big),$$

we can use the expansion of $V(z)$ for $z \to \zeta/S(\tau)$ from (5.11) to compute the expansion for $D(z,1)$ with $z \to \zeta/S(\tau)$, yielding

$$D(z,1) \overset{z\to\zeta/S(\tau)}{=} \frac{\tau S(\tau)}{1-\tau\zeta} - \frac{1}{(1-\tau\zeta)^2}\sqrt{\frac{2S(\tau)^3}{S''(\tau)}}\Big(1-\frac{z}{\zeta/S(\tau)}\Big)^{1/2} + O\Big(1-\frac{z}{\zeta/S(\tau)}\Big).$$

By applying singularity analysis we are able to determine the contribution of the singularity located at $\zeta/S(\tau)$ to the overall growth of the coefficients of $D(z,1)$, which can then be obtained by summing up the contributions of all singularities on the radius of convergence. In our case, this translates to summing over all pth roots of unity $\zeta \in G(p)$.

After doing so, we see that in the main term all roots of unity can be grouped together such that we find

$$\frac{1}{\sqrt{2\pi}}\Big(\sum_{\zeta\in G(p)} \frac{\zeta^{-n}}{(1-\tau\zeta)^2}\Big)\sqrt{\frac{S(\tau)^3}{S''(\tau)}}\, S(\tau)^n n^{-3/2}.$$

In fact, when studying these expansions with higher precision, the corresponding sums that occur have the shape

$$\sum_{\zeta\in G(p)} \frac{\zeta^{\ell-n}}{(1-\tau\zeta)^m}$$

for some integers $\ell, m \geq 0$. To find an explicit expression for this sum, we first recall (5.10) as well as another elementary property of roots of unity, namely

$$(1-\zeta\tau)\sum_{j=0}^{p-1}(\tau\zeta)^j = 1-\tau^p.$$

Now let $n \equiv k \mod p$ with $0 \leq k \leq p-1$. Then we can rewrite

$$\sum_{\zeta\in G(p)} \frac{\zeta^{\ell-n}}{(1-\tau\zeta)^m} = \frac{1}{(1-\tau^p)^m}\sum_{\zeta\in G(p)} \zeta^{\ell-k}(1+\tau\zeta+(\tau\zeta)^2+\cdots+(\tau\zeta)^{p-1})^m.$$

By (5.10) we only need to determine those summands in $(1+\tau\zeta+\cdots+(\tau\zeta)^{p-1})^m$ involving τ^j with $0 \leq j \leq m(p-1)$ and $j \equiv k-\ell \mod p$. This can be done easily for explicitly given values of m and ℓ, for example

$$\sum_{\zeta\in G(p)} \frac{\zeta^{-n}}{(1-\tau\zeta)^2} = \frac{p\tau^k(\tau^p(p-k-1)+k+1)}{(1-\tau^p)^2}.$$

Plugging this into the previously obtained asymptotic expansion for the growth of the coefficients of $D(z,1)$ yields (5.18).

For the expected value we focus on the generating function

$$D_t(z,1) = \frac{V_t(z)}{z}\frac{1}{(1-V(z))^2}$$

and proceed similarly to above. Using (5.11) and (5.12) we are again able to compute the singular expansion of $D_t(z,1)$ for $z \to \zeta/S(\tau)$, namely

$$\frac{1}{(1-\tau\zeta)^2}\frac{(\tau S(\tau)-1)^r}{S(\tau)^r\tau^{r+2}\sqrt{2S(\tau)S''(\tau)}}\left(1-\frac{z}{\zeta/S(\tau)}\right)^{-1/2}+O(1).$$

Extracting the contributions of the singularity at $\zeta/S(\tau)$, summing up the contributions of all p singularities, and then finally normalizing the result by dividing by the overall number of dispersed excursions of length n we arrive at (5.19). □

By Lemma 5.4.4, the only family of Łukasiewicz paths that is not covered by Theorem 5.5.5 is $\mathcal{S} = \{-1, 1\}$, the case of dispersed Dyck paths. However, as everything is explicitly given, the analysis is quite straightforward.

Proposition 5.5.6.
Let d_n denote the total number of dispersed Dyck paths of length n, and let $D_{n,r}$ denote the random variable modeling the number of r-ascents in a random dispersed Dyck path of length n.

Then, d_n is given by

$$d_n = \binom{n}{\lfloor n/2 \rfloor} = \sqrt{\frac{2}{\pi}} 2^n n^{-1/2} - \frac{2-(-1)^n}{2\sqrt{2\pi}} 2^n n^{-3/2} + O(2^n n^{-5/2}), \tag{5.20}$$

and the expected number of r-ascents satisfies

$$\begin{aligned} \mathbb{E}D_{n,r} = \frac{n}{2^{r+2}} - \sqrt{\frac{\pi}{2}} \frac{r-2}{2^{r+2}} n^{1/2} + \frac{(r-1)(r-4)}{2^{r+3}} \\ - \sqrt{\frac{\pi}{2}} \frac{(r-2)(2-(-1)^n)}{2^{r+4}} n^{-1/2} + O(n^{-1}). \end{aligned} \tag{5.21}$$

Proof. In the case where $\tau = 1$, the zero in the denominator of $\frac{1}{1-V(z)}$ combines with the square root singularity from $V(z)$ itself. From Example 5.5.2 we know that

$$V(z) = \frac{1-\sqrt{1-4z^2}}{2z} \quad \text{and} \quad V_t(z) = z \frac{(1-\sqrt{1-4z^2})^r (1+\sqrt{1-4z^2})}{2^{r+1}\sqrt{1-4z^2}}.$$

The number d_n of dispersed Dyck paths of length n can be read off as the coefficients of

$$D(z,1) = \frac{1}{z} \frac{V(z)}{1-V(z)} = \frac{1}{2z}\left(\sqrt{\frac{1+2z}{1-2z}} - 1\right) = \frac{1}{\sqrt{1-4z^2}} + \frac{1}{2z}\left(\frac{1}{\sqrt{1-4z^2}} - 1\right).$$

This proves (5.20), where the asymptotic part can be obtained by means of singularity analysis. The explicit formula for d_n in (5.20) is also stated in [34, Lemma 2].

For the expected number of r-ascents, we consider

$$D_t(z,1) = \frac{1}{z} \frac{V_t(z)}{(1-V(z))^2} = \frac{(1-\sqrt{1-4z^2})^r (1-4z^2+\sqrt{1-4z^2}(1-2z^2))}{2^{r+1}(1+2z)(1-2z)^2}.$$

Just as before, the coefficients of this function can also be extracted by means of singularity analysis; the dominant singularities can be found at $z = \pm 1/2$. Extracting the coefficients and dividing by d_n yields (5.21). □

This completes our analysis of r-ascents in dispersed Łukasiewicz excursions.

5.5.3 Analysis of Meanders

In this section we study ascents in meanders, i.e., non-negative Łukasiewicz paths without further restriction. The corresponding generating function can be obtained from (5.1) by setting $v = 1$, which allows arbitrary ending altitude of the path.

In accordance to the results from [2, Theorem 4], the behavior of meanders depends on the sign of the drift (i.e., the quantity $S'(1)$). The following theorem handles the case of positive drift (which, in our setting, is equivalent to $\tau \neq 1$).

Recall that $M_{n,r}$ is the random variable modeling the number of r-ascents in a random non-negative Łukasiewicz path of length n with respect to some given step set $\mathcal{S}$.

Theorem 5.5.7.
Let $\tau > 0$ be the structural constant, i.e., the unique positive solution of $S'(\tau) = 0$, and assume that $\tau \neq 1$.

Then, with $\xi = 1/S(1)$, the expected number of r-ascents in Łukasiewicz meanders of length n as well as the corresponding variance grow with $n \to \infty$ according to the asymptotic expansions

$$\mathbb{E}M_{n,r} = \mu n + \frac{(S(1)-1)^r(2S(1)-1-r)}{S(1)^{r+2}} + \frac{(S(1)-1)^r V_z(\xi)}{S(1)^{r+1}(1-V(\xi))} - \frac{V_t(\xi)}{1-V(\xi)} + O\Big(n^{5/2}\Big(\frac{S(\tau)}{S(1)}\Big)^n\Big), \tag{5.22}$$

and

$$\mathbb{V}M_{n,r} = \sigma^2 n + O(1), \tag{5.23}$$

where μ and σ^2 are given by

$$\mu = \frac{(S(1)-1)^r}{S(1)^{r+2}} \quad \text{and} \quad \sigma^2 = \frac{(S(1)-1)^r}{S(1)^{r+2}} + \frac{(S(1)-1)^{2r}(2S(1)-3-2r)}{S(1)^{2r+4}}.$$

Moreover, for $n \to \infty$, $M_{n,r}$ is asymptotically normally distributed, i.e., for $x \in \mathbb{R}$ we have

$$\mathbb{P}\Big(\frac{M_{n,r}-\mu n}{\sqrt{\sigma^2 n}} \le x\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt + O(n^{-1/2}).$$

All O-constants depend implicitly on r.

Proof. Just as in the analysis of excursions and dispersed excursions, the first quantity we require is m_n, the total number of meanders of length n associated to $\mathcal{S}$. Setting $v = t = 1$ in (5.1) and simplification yields

$$F(z,1,1) = \frac{1-V(z)}{1-zS(1)}.$$

From Corollary 5.4.3 we know that $V(z)$ has radius of convergence $1/S(\tau)$. As $S(u)$ is strictly convex for $u > 0$ and τ solves $S'(\tau) = 0$, this means that $S(\tau) < S(u)$ for all $u > 0$ with $u \neq \tau$. Hence, as $1/S(1) < 1/S(\tau)$, the dominant singularity of $F(z,1,1)$ is the simple pole located at $\xi := 1/S(1)$. Extracting coefficients yields

$$m_n = [z^n]F(z,1,1) = (1-V(\xi))S(1)^n + O(n^{-3/2}S(\tau)^n). \tag{5.24}$$

The error term can directly be deduced from the fact that the next relevant singularity of $F(z,1,1)$ is of square-root type with modulus $1/S(\tau)$ (coming from $V(z)$). Observe that (5.24) could also have been obtained by applying [2, Theorem 4] for our given step set $\mathcal{S}$.

In order to determine the expectation $\mathbb{E}M_{n,r}$, we differentiate $F(z,t,1)$ with respect to t, set $t = 1$, and then extract the coefficients of the resulting generating function. By construction, these coefficients are $m_n \cdot \mathbb{E}M_{n,r}$, allowing us to obtain an asymptotic expansion after normalizing the result.

Carrying out the computations leads to

$$\frac{\partial}{\partial t}F(z,t,1)\Big|_{t=1} = \frac{(cz)^r(cz-1)^2(1-V(z))}{(1-z\,S(1))^2} - \frac{V_t(z)}{1-z\,S(1)},$$

where, for the sake of brevity, we define $c := S(1)-1 = S_+(1)$. Determining the growth of the coefficients then gives

$$\frac{c^r(1-V(\xi))}{S(1)^{r+2}}n\,S(1)^n + \left(\frac{(1-V(\xi))c^r(2S(1)-1-r)+c^r\,S(1)V_z(\xi)}{S(1)^{r+2}} - V_t(\xi)\right)S(1)^n \\ + O(n^{-1/2}\,S(\tau)^n),$$

which, after dividing by the expansion of m_n, proves (5.22).

Analogously, for the second partial derivative of $F(z,t,1)$ with respect to t, we find

$$\frac{\partial^2}{\partial t^2}F(z,t,1)\Big|_{t=1} = -\frac{2z(cz)^{2r}(cz-1)^3(1-V(z))}{(1-z\,S(1))^3} - \frac{2(cz)^r(cz-1)^2V_t(z)}{(1-z\,S(1))^2} - \frac{V_{tt}(z)}{1-z\,S(1)},$$

allowing us to determine the asymptotic growth of the unnormalized second factorial moment, $m_n\mathbb{E}(M_{n,r}(M_{n,r}-1))$. Dividing by m_n and computing the variance by means of $\mathbb{V}M_{n,r} = \mathbb{E}(M_{n,r}(M_{n,r}-1)) + \mathbb{E}M_{n,r} - (\mathbb{E}M_{n,r})^2$ then yields (5.23).

In order to prove that $M_{n,r}$ is asymptotically normally distributed [28], we observe that

$$F(z,t,1) = \frac{(1-V(z,t))(1+(t-1)(1-cz)(cz)^r)}{1-z\,S(1)-(t-1)(1-cz)(cz)^r}$$

has a unique simple pole at $z = 1/S(1)$ for $t = 1$ and, by Rouché's theorem for $|z| < \rho$, it has a single pole for sufficiently small $|t-1|$. Then, in order to apply the theorem on singularity pertubation for meromorphic functions [21, Theorem IX.9], which proves the normal limiting distribution, all that remains to show is that the main term $\sigma^2 \geq 0$ of the asymptotic expansion of the variance does not vanish.

Setting $\sigma^2 = 0$ is equivalent to the equation

$$(3+2r-2S(1))(S(1)-1)^r = S(1)^{r+2},$$

where all occurring quantities are integer-valued. As we have $\gcd(S(1)-1, S(1)) = 1$, this equation can only be valid if $S(1) = 2$. However, in this case, the equation reduces to $(2r-1) = 2^{r+2}$, which is impossible for parity reasons. □

Remark (Computation of constants).
Within (5.22) and (5.23), the asymptotic expansions for the expected number and variance of r-ascents in meanders, constants of the type $V(\xi)$, $V_z(\xi)$, $V_t(\xi)$, where $\xi = 1/S(1)$, occur. For higher precision than in the theorem, also higher derivatives (as well as mixed derivatives) occur.

Although the function $V(z,t)$ is only given implicitly, by the following observation all of those constants can actually be computed. By taking the functional equation (5.8) and rewriting it as

$$\frac{1}{z} = S(V(z)),$$

we can see that for $z = 1/S(1)$ we obtain the relation

$$S(1) = S\left(V\left(\frac{1}{S(1)}\right)\right).$$

Then, because we know that $V(1/S(1)) > 0$ and that $S(u)$ is strictly convex for $u > 0$, the constant can be determined as the unique positive solution of $S(u) = S(1)$ satisfying $u \neq 1$.

For determining the value of the constants involving derivatives, we make use of the same approach as used in Corollary 5.4.3. By means of implicit differentiation we are able to rewrite any derivative of the form $\frac{\partial^{\nu_1+\nu_2}}{\partial t^{\nu_1}\partial z^{\nu_2}} V(z,t)|_{t=1}$ in terms of $V(z)$, allowing us to express all constants in terms of $V(\xi)$.

By Lemma 5.4.4, Theorem 5.5.7 covers all step sets except for $\mathcal{S} = \{-1,1\}$ and $\mathcal{S} = \{-1,0,1\}$. In these cases, we have a similar situation to what we had in Section 5.5.2: the square root singularity coming from $V(z)$ combines with the zero in the denominator.

The following propositions close this gap.

Proposition 5.5.8.
The expected number of r-ascents in the Łukasiewicz meanders of length n associated to $\mathcal{S} = \{-1,1\}$ as well as the corresponding variance grow with $n \to \infty$ according to the asymptotic expansions

$$\mathbb{E}M_{n,r} = \frac{n}{2^{r+2}} + \frac{\sqrt{2\pi}(r-2)}{2^{r+3}} n^{1/2} - \frac{r^2-r-8}{2^{r+3}} + \frac{\sqrt{2\pi}((2-(-1)^n)(r-2)}{2^{r+5}} n^{-1/2} + O(n^{-1}), \tag{5.25}$$

and

$$\mathbb{V}M_{n,r} = \frac{2^{r+3} - r^2(\pi-2) + 4r(\pi-3) - 4\pi + 10}{2^{2r+5}} n + \frac{\sqrt{2\pi}(2^{r+2}(r-2) - r^3 + 3r^2 - 2r + 4)}{2^{2r+5}} n^{1/2} + O(1). \quad (5.26)$$

Proof. The analysis of ascents in this case is pretty much straightforward, as[3]

$$V(z) = \frac{1-\sqrt{1-4z^2}}{2z},$$

and therefore all generating functions involved in this analysis are given explicitly.

The total number of meanders can now either be obtained by following [2, Theorem 4], or simply by extracting the coefficients of

$$F(z,1,1) = -\frac{1-2z-\sqrt{1-4z^2}}{2z(1-2z)},$$

where the dominant singularities are located at $z = \pm 1/2$. In any case, we find that for $n \to \infty$ there are

$$\sqrt{\frac{2}{\pi}} 2^n n^{-1/2} - \sqrt{\frac{2}{\pi}} \frac{2-(-1)^n}{4} 2^n n^{-3/2} + \sqrt{\frac{2}{\pi}} \frac{13-12(-1)^n}{32} 2^n n^{-5/2} + O(2^n n^{-7/2}) \quad (5.27)$$

meanders with steps $\mathcal{S} = \{-1, 1\}$ of length n.

Then, by plugging in

$$V_t(z) = \frac{z(1-\sqrt{1-4z^2})^r(1+\sqrt{1-4z^2})}{2^{r+1}\sqrt{1-4z^2}}$$

and the formula for $V(z)$ into

$$\frac{\partial}{\partial t} F(z,t,1)\Big|_{t=1} = \frac{z^r(1-z)^2(1-V(z))}{(1-2z)^2} - \frac{V_t(z)}{1-2z},$$

we have an explicit representation of the generating function for the expected number of r-ascents before normalization. Extracting the coefficients by means of singularity analysis (the location $z = \pm 1/2$ of the dominant singularities is known from above) and then dividing by (5.27) yields (5.25).

For computing the variance, we proceed similarly: we determine the asymptotic behavior of the second factorial moment $\mathbb{E}(M_{n,r}(M_{n,r}-1))$ by extracting the coefficients of $\frac{\partial^2}{\partial t^2} F(z,t,1)|_{t=1}$ and normalizing the result by dividing by (5.27). Then, (5.26) follows from

$$\mathbb{V}M_{n,r} = \mathbb{E}(M_{n,r}(M_{n,r}-1)) + \mathbb{E}M_{n,r} - (\mathbb{E}M_{n,r})^2. \qquad \square$$

[3] See also Example 5.5.2.

Proposition 5.5.9.
The expected number of r-ascents in the Łukasiewicz meanders of length n associated to $\mathcal{S} = \{-1, 0, 1\}$ as well as the corresponding variance grow with $n \to \infty$ according to the asymptotic expansions

$$\mathbb{E}M_{n,r} = \frac{2^r}{3^{r+2}} n + \frac{\sqrt{3\pi}(r-4)2^{r-2}}{3^{r+2}} n^{1/2} - (3r^2 - r - 96)\frac{2^{r-4}}{3^{r+2}} + \frac{\sqrt{3\pi}(r-4)2^{r-6}}{3^r} n^{-1/2} + O(n^{-1}) \quad (5.28)$$

and

$$\mathbb{V}M_{n,r} = \frac{3^{r+2}2^{r+4} - 2^{2r}(3r^2(\pi-2) - 8r(3\pi-10) + 48\pi - 144)}{16 \cdot 3^{2r+4}} n + \frac{\sqrt{3\pi}(72(r-4)6^r - 2^{2r}(3r^3 - 9r^2 - 28r - 32))}{32 \cdot 3^{2r+4}} n^{1/2} + O(1). \quad (5.29)$$

Proof. The asymptotic expansions for expectation and variance can be obtained with an analogous approach as in the proof of Proposition 5.5.8. In this case, we have

$$V(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z},$$

and the dominant singularity of $F(z, 1, 1)$ (as well as for the corresponding derivatives with respect to t) is located at $z = 1/3$. □

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