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Asymptotic Analysis of Lattice Paths and Related Structures

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Abstract

While classical combinatorics is mostly "just" about enumerating discrete objects, the field of *Analytic Combinatorics* is about the precise analysis of the corresponding asymptotic behavior. A broad spectrum of mathematical disciplines is involved in such an "asymptotic analysis"—most prominently, results from classical combinatorics, complex analysis, and probability theory are used.

The central (discrete) objects of study within this thesis are *lattice paths* and *trees*. After giving an introduction to some central ideas and methods from Analytic Combinatorics, we discuss various special classes of lattice paths and trees. The analyses of these objects are powered by several different ideas ranging from simple consequences of the fundamental analytic framework up to (new) approaches that are specifically tailored for the given problem structure. The results of these novel approaches have also been submitted for publication in an international journal.

Zusammenfassung

Während sich die klassische Kombinatorik normalerweise "nur" mit dem Abzählen diskreter Objekte beschäftigt, interessieren wir uns im Rahmen der *analytischen Kombinatorik* für präzise Analysen des entsprechenden asymptotischen Verhaltens. Für eine solche asymptotische Analyse werden Resultate aus einem breiten Spektrum von mathematischen Disziplinen, wie beispielsweise der klassischen Kombinatorik, der Funktionentheorie, und auch der Wahrscheinlichkeitstheorie verwendet.

Innerhalb dieser Masterarbeit spielen *Gitterpunktpfade* sowie *Bäume* eine zentrale Rolle. Nachdem wir zunächst einige fundamentale Ideen und Methoden aus der analytischen Kombinatorik vorstellen, widmen wir uns danach der asymptotischen Analyse von diversen speziellen Klassen von Gitterpunktpfaden und Bäumen. Die für diese Analysen verwendeten Ansätze reichen hierbei von einfachen Konsequenzen des zugrundeliegenden analytischen Grundgerüsts bis hin zu (neuen) Ansätzen, die speziell auf den jeweiligen Problemtyp zugeschnitten sind. Die daraus entstandenen neuen Ergebnisse wurden auch bei einem internationalen Journal zur Publikation eingereicht.

Contents

Introduction

This thesis revolves around various central aspects of *Analytic Combinatorics*, the discipline in which the *asymptotic behavior* of discrete objects is investigated and determined with the help of analytic methods.

In Chapter 1 some of the most common tools from Analytic Combinatorics are discussed. In particular, Section 1.2 introduces the so-called *Symbolic Method*, which essentially is a specification language for combinatorial structures. As soon as a structure is specified with the help of this language, the construction can be "translated" into the world of *generating functions*. In Section 1.3 we discuss two common frameworks that can be used in order to extract the asymptotic behavior of some combinatorial structure from the associated generating function. Our brief introduction to the general analytic framework is concluded in Section 1.4, where we introduce some related concepts from probability theory.

Then, with this basic framework at hand we consider so-called *lattice paths* in Chapter 2. In Section 2.2 and Section 2.3 we revisit some well-known results on the exact and asymptotic behavior of these special lattice paths. Section 2.4 discusses previously unknown results: among others, a recent conjecture of Zhao that was posed in the Journal of Number Theory is proved. This section is joint work with *Clemens Heuberger, Helmut Prodinger,* and *Stephan Wagner*. A standalone version of it can be found in [19], which has been submitted for publication in an international journal. Note that these deliberations are supported by source code that can be found in Appendix A.

In Chapter 3 we discuss the asymptotic behavior of *trees*. Because of their special recursive structure, the analytic methods introduced in Chapter 1 cannot be applied directly. However, they can be generalized and extended such that trees can be investigated with the same machinery as well. This is discussed and illustrated in Section 3.2. Finally, we discuss some well-known results on the number of deepest nodes in a tree in Section 3.3.

1 Preliminaries

1.1 Introduction

The mathematical discipline of *Analytic Combinatorics* is a fascinating area of research. Combinatorics (or, to be more precise, *Enumerative Combinatorics*) is the mathematical field of study in which *discrete* objects are enumerated. However, sometimes we would like to have more information than a answer from Enumerative Combinatorics provides. For example, consider the following question:

Assume that we are playing a very simple game (without participation fee), in which we can either win or loose one Euro in every round. If we play $2n$ rounds, how many possible "zero-sum game series" are there? That is, how many possible outcomes are there such that we neither loose nor earn any money in the end?

In terms of Enumerative Combinatorics, the answer can be obtained easily: if we neither loose nor earn any money, then obviously we have to win equally many rounds as we loose. This means that the question of counting all possible "zero-sum game series" reduces to counting the number of game series in which exactly n out of $2n$ games are won. This number is exactly given by the binomial coefficient $\binom{2n}{n}$. Alternatively, the problem could also be reduced to counting binary words (i.e. words over the alphabet $\{0, 1\}$) of length $2n$ that have equally many zeros as ones.

From the Enumerative Combinatorics point of view this problem is solved and completely answered. Yet—as stated above—there are still open questions. Actually, we have no idea how the number of "zero-sum game series" behaves if n gets large: is this logarithmic, polynomial, or even exponential growth?

In order to answer this, we have to analyze the quantity $\binom{2n}{n}$ asymptotically—and this is where *analytic methods* come into play. In general, a thorough investigation of the analyticity of the associated *generating function*[1] yields the asymptotic behavior (and in many cases even the limiting distribution, i.e. the stochastic pendant of the asymptotic behavior) of the objects we are interested in.

[1]The notion of *generating functions* is an extremely powerful tool: essentially, these objects represent an entire infinite sequence.

In this chapter, preliminaries for the application of methods from Analytic Combinatorics to discrete structures like, for example, *lattice paths* and *trees* are discussed. In Section 1.2, the notion of *combinatorial classes* will be introduced, which will also direct us to the concept of *generating functions*, which were also mentioned above.

Section 1.3 introduces some analytic concepts and tools like the framework of *Singularity Analysis* as well as the *Mellin transform*. These introductions are accompanied by the discussion of some classical examples.

Of course, there is a strong relation between counting objects, investigating asymptotic structures, and probability theory. The nature of this relation will be explored in Section 1.4. In this brief introduction to some probability theoretic concepts we will also encounter one of the central objects of this thesis: *lattice paths*. Furthermore, the role of generating functions in probability theory with respect to the asymptotic analysis of discrete structures shall be stressed explicitly.

As the results and methods from this chapter are generally well-known, we will refrain from giving proofs and refer to the respective literature instead: when it comes to *Analytic Combinatorics*, P. Flajolet's and R. Sedgewick's impressive book [17] is a standard reference. Additionally, there are some excellent concise overviews in [6].

1.2 Combinatorial Classes

The importance of *generating functions* cannot be underestimated (see, for example, [18, Section 5.4]). In this section, we introduce the framework of "combinatorial classes", which will help us understand naturally how generating functions and operations on them can be interpreted. This section is mainly based on [17, Chapter I].

Definition 1.2.1 (Combinatorial class).
Let $\mathcal{A}$ be a set and define a *size function* $|\cdot| : \mathcal{A} \to \mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$ we have $a_n := \#\{a \in \mathcal{A} : |a| = n\} < \infty$. Then the tuple $(\mathcal{A}, |\cdot|)$ is called a *combinatorial class*.

In a nutshell, combinatorial classes are sets where each element has a certain "size". The number of elements of size n in a combinatorial class $\mathcal{A}$ is denoted as a_n, and the sequence $(a_n)_{n\in\mathbb{N}_0}$ is referred to as the *counting sequence* of the combinatorial class $\mathcal{A}$.

Based on the counting sequence, we may now also define the *generating function* of such a sequence.

Definition 1.2.2 (Ordinary generating function).
Let $(a_n)_{n\in\mathbb{N}_0}$ be a sequence. Then the *ordinary generating function* (OGF) of this sequence is

the *formal power series*

$$A(z) := \sum_{n\geq 0} a_n z^n.$$

Furthermore, $[z^n]A(z)$ shall denote the coefficient of z^n in $A(z)$, i.e. $[z^n]A(z) = a_n$.

Remark (Polynomials and formal power series).
Algebraically, polynomials over a unitary commutative ring R are defined as sequences $(a_n)_{n\in\mathbb{N}_0} \subseteq R$ where almost all coefficients are 0. Formal power series are defined similarly, only the restriction on the coefficients is dropped, meaning that formal power series can be identified with the set of sequences over R. In both cases, the "indeterminate" corresponds to the sequence $(0, 1, 0, \ldots)$, and the multiplication is defined by means of convolution.

Although formal power series are defined algebraically as elements of such a formal power series ring (denoted as $R[\![z]\!]$), we will often treat them like analytic objects and investigate, for example, their radius of convergence.

Remark (Notational convention for combinatorial classes).
In this thesis, we stick to the following convention: unless stated otherwise, combinatorial classes are written in calligraphic letters (e.g. $\mathcal{A}$), the corresponding generating functions are written with capital Latin letters ($A(z)$ in this case), and the associated counting sequence is denoted with lowercase Latin letters, i.e. $A(z) = \sum_{n\geq 0} a_n z^n$.

Example 1.2.3.
Consider the combinatorial class $\mathcal{W}$ of words over the alphabet $\{a, b, c\}$ where the size of a word within the combinatorial class is its length. That is, if ε denotes the *empty word* (which is the unique word of length 0), then we have

$$\mathcal{W} = \{\varepsilon, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, \ldots\}.$$

For the number of objects of size n (denoted as w_n), we observe $w_0 = 1$, $w_1 = 3$, $w_2 = 9$. In general, we have $w_n = 3^n$: for each of the n letters in a word of length n we have 3 possible choices, therefore there are 3^n words of length n.

Based on the explicit formula for w_n, we may also give the *generating function*

$$W(z) = \sum_{n\geq 0} w_n z^n = \sum_{n\geq 0} 3^n z^n = \sum_{n\geq 0} (3z)^n = \frac{1}{1-3z}.$$

In this case, the generating function can be interpreted as a complex function analytic in a neighborhood of 0 (the power series $\sum_{n\geq 0}(3z)^n$ has radius of convergence $\frac{1}{3}$). △

Example 1.2.4.
As another example, consider the combinatorial class $\mathcal{P}$ of *permutations* (i.e. bijective functions on a finite domain). To be precise, we consider bijective functions $\varphi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ where $n \in \mathbb{N}_0$. The size of a permutation within $\mathcal{P}$ is the size of its domain.

It is a well-known fact that there are $n!$ permutations of size n (there are n choices for the first object, $n-1$ for the second, and so on ...), which directly implies $p_n = n!$. The corresponding generating function is

$$P(z) = \sum_{n\geq 0} p_n z^n = \sum_{n\geq 0} n! z^n.$$

Analytically, this function is only defined for $z = 0$ with $P(0) = p_0 = 1$, and the radius of convergence of the power series is 0. Nevertheless, in this case we resort to the definition of $P(z)$ as a formal power series. $\triangle$

Remark.
It is not very surprising that in Example 1.2.3 we could observe a relation between the radius of convergence of the generating functions and the growth of the counting sequence: the magnitude of growth of the counting sequence grows with shrinking radius of convergence.

Within analytic combinatorics, the technique of *Singularity Analysis* makes use of this profound connection between the location and type of the singularities of the generating function, and the asymptotic behavior of the counting sequence. We will introduce the most basic aspects of Singularity Analysis in Section 1.3.1.

Definition 1.2.5 (Operations on combinatorial classes).
Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes with size functions $|\cdot|_{\mathcal{A}}$ and $|\cdot|_{\mathcal{B}}$, respectively. Then we define the following operations on these classes:

- Sum of combinatorial classes: $\mathcal{A} + \mathcal{B} := \{(1, a) \mid a \in \mathcal{A}\} \cup \{(2, b) \mid b \in \mathcal{B}\}$ where $|(1, a)| = |a|_{\mathcal{A}}$ and $|(2, b)| = |b|_{\mathcal{B}}$.
- Product of combinatorial classes: $\mathcal{A} \times \mathcal{B} := \{(a, b) \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ where $|(a, b)| = |a|_{\mathcal{A}} + |b|_{\mathcal{B}}$.
- Kleene-Closure or Sequence construction: assuming that there is no object of size 0 in $\mathcal{A}$, we set $\mathcal{A}^* := \{\varepsilon\} + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \cdots$, where $|\varepsilon| = 0$ in $\mathcal{A}^*$.
- Powerset construction: $\operatorname{PSet}(\mathcal{A})$ is the class of all finite subsets of $\mathcal{A}$.
- Multiset construction: $\operatorname{MSet}(\mathcal{A})$ is the class containing all finite multisets[2] consisting of elements from $\mathcal{A}$.

For the powerset and multiset construction, the size is given as the sum of sizes of the elements in the powerset and multiset, respectively.

From a combinatorial point of view, the operations from the previous definition arise naturally when studying combinatorial structures. For example, words over an alphabet $\mathcal{A}$ are

[2]Multisets are sets where elements are allowed to occur multiple times. A multiset corresponds to the class of all permutations of a finite sequence.

elements from $\mathcal{A}^*$. Another prominent example is the class of integer partitions $\mathcal{P}$, which can be seen as $\mathrm{MSet}(\mathcal{I})$, the multiset construction over the combinatorial class of the positive integers $\mathcal{I} = \{1, 2, 3, \ldots\}$.

However, these operations are only useful if we are also able to construct the respective counting sequences efficiently. The following theorem will establish this link: it translates the operations on combinatorial classes to operations on their generating functions.

Theorem 1.2.6 (Operations on generating functions).
Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be combinatorial classes with ordinary generating functions $A(z)$, $B(z)$, and $C(z)$, respectively. Then the operations on combinatorial classes from Definition 1.2.5 translate into the following operations on their generating functions:

- If $\mathcal{C} = \mathcal{A} + \mathcal{B}$, then $C(z) = A(z) + B(z)$, and especially $c_n = a_n + b_n$ for all $n \in \mathbb{N}_0$.
- If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, then $C(z) = A(z) \cdot B(z)$, and especially $c_n = \sum_{j \le n} a_j b_{n-j}$.
- If $\mathcal{B} = \mathcal{A}^*$, then $B(z) = \frac{1}{1-A(z)}$.
- If $\mathcal{B} = \mathrm{PSet}(\mathcal{A})$, then we have
$$B(z) = \prod_{k \ge 1} (1 + z^k)^{a_k} = \exp\Big(\sum_{k \ge 1} \frac{(-1)^{k-1}}{k} A(z^k) \Big).$$
- If $\mathcal{B} = \mathrm{MSet}(\mathcal{A})$, then we have
$$B(z) = \prod_{k \ge 1} (1 - z^k)^{-a_k} = \exp\Big(\sum_{k \ge 1} \frac{1}{k} A(z^k) \Big).$$

Proof. See [17, Theorem I.1]. □

Basically, this theorem powers the *Symbolic Method*. It is used to find the generating function of a combinatorial class by exploiting the structure of the class using the operations from Definition 1.2.5. This shall be illustrated by the following examples.

Example 1.2.7 (Symbolic Method: words over an alphabet).
As a first example, we consider words over the alphabet $\mathcal{A} = \{a, b, c\}$. The size of the combinatorial objects (i.e. the words) shall be their length. In this sense, the alphabet $\mathcal{A}$ is a combinatorial class as well—and its generating function is given by $A(z) = 3z$. As stated above, from a combinatorial point of view, words over an alphabet can be interpreted as finite sequences of letters. Therefore, the combinatorial class of words over $\mathcal{A}$ is given by $\mathcal{W} := \mathcal{A}^*$, which yields $W(z) = \frac{1}{1-A(z)} = \frac{1}{1-3z}$.

Expanding this function yields
$$W(z) = 1 + 3z + (3z)^2 + (3z)^3 + \cdots = \sum_{n \ge 0} 3^n z^n.$$
Therefore, there are 3^n words of length n over an alphabet with three letters. △

Example 1.2.8 (Symbolic Method: binary words without consecutive ones).
We consider the combinatorial class $\mathcal{A}$ of all words over the alphabet $\{0,1\}$ ("binary words") where consecutive ones are not allowed. It is easy to see that the class $\mathcal{A}$ may be constructed as

$$\mathcal{A} = \{0,10\}^* \times \{\varepsilon, 1\},$$

and therefore the respective generating function is given by

$$A(z) = \frac{1}{1-(z+z^2)} \cdot (1+z) = \frac{1+z}{1-z-z^2}.$$

One easily checks that this generating function generates the shifted Fibonacci sequence $(f_{n+2})_{n\geq 0}$ (see A000045 in [34] for the sequence of Fibonacci numbers). The class $\mathcal{A}$ is strongly related to the concept of so-called non-adjacent forms (NAFs), which are (in their simplest incarnation) digit expansions over the digit set $\{-1,0,1\}$, where consecutive non-zero digits are not allowed. △

Example 1.2.9 (Symbolic Method: rooted plane trees).
We discuss yet another combinatorial class constructed by such elementary operations: the class of *rooted plane trees*[3] $\mathcal{T}$. The class $\mathcal{T}$ may be described recursively, as trees are nothing else than a vertex with a sequence (because of the "left-to-right" order) of subtrees; see also Figure 1.1.

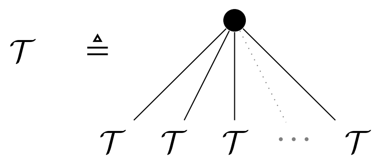

Figure 1.1: Rooted plane trees – symbolic equation

Therefore, symbolically, we may write $\mathcal{T} = \{\bullet\} \times \mathcal{T}^*$, which translates into the equation $T(z) = z \cdot \frac{1}{1-T(z)}$ for the corresponding generating function. From that point, simple algebra yields $T(z) = \frac{1}{2}(1 \pm \sqrt{1-4z})$, where the correct sign has yet to be determined. We will return to the analysis of rooted plane trees in Section 3.2. △

Concerning the applications of combinatorial structures, it is not only desirable to analyze the growth of the structure, but also the asymptotic behavior of other parameters. A concrete example of such a parameter can be given in the context of the second example from before: consider the *weight* of a binary word without consecutive ones, i.e. the number of ones in

[3]See the introduction of Chapter 3 for an introduction to the terminology of trees.

such a word with fixed length[4]. Technically, this can be done by embedding the idea of generating functions in a multivariate setting.

Example 1.2.10 (Average weight, bivariate generating function).
Let $a_{k,n}$ denote the number of binary words without consecutive ones of length n with weight k. Then

$$A(z,u) := \sum_{k,n\geq 0} a_{k,n} u^k z^n$$

is the corresponding *bivariate generating function*[5]. As we are interested in the *average weight*, we investigate

$$\overline{a}_n := \frac{0\cdot a_{0,n} + 1\cdot a_{1,n} + 2\cdot a_{2,n} + \cdots}{a_{0,n} + a_{1,n} + a_{2,n} + \cdots} = \frac{\sum_{k\geq 0} k\cdot a_{k,n}}{\sum_{k\geq 0} a_{k,n}} = \frac{\sum_{k\geq 0} k\cdot a_{k,n}}{f_{n+2}},$$

where f_n denotes the n-th Fibonacci number.

Fortunately, the quantity $\sum_{k\geq 0} k\cdot a_{n,k}$ can be determined by manipulating the bivariate generating function: observe that we have

$$A_u(z,u) := \frac{\partial A}{\partial u}\bigg|_{(z,u)} = \sum_{k,n\geq 0} k\cdot a_{k,n} u^{k-1} z^n \qquad \Longrightarrow \qquad A_u(z,1) = \sum_{k,n\geq 0} k\cdot a_{k,n} z^n.$$

In Example 1.2.8 we have seen that the combinatorial class of these binary words can be given as $\mathcal{A} = \{0,10\}^* \times \{\varepsilon, 1\}$. The bivariate generating functions of $\{0,10\}$ and $\{\varepsilon,1\}$ are $z + uz^2$ and $1 + uz$, where u and z correspond to the weight and the length of a word, respectively.

It can be shown that the statement of Theorem 1.2.6 directly translates into the multivariate setting. Thus, the bivariate generating function is given by

$$A(z,u) = \frac{1}{1-(z+uz^2)}\cdot(1+uz) = \frac{1+uz}{1-z-uz^2}.$$

This yields $A_u(z,1) = \frac{z}{(1-z-z^2)^2}$, which is strongly related to the generating function $\frac{z^2}{(1-z-z^2)^2}$ of the sequence of Fibonacci numbers convoluted with themselves, enumerated by A001629 in [34]. This means that $A_u(z,1)$ enumerates the same sequence, but with offset 1.

By partial fraction decomposition (or Singularity Analysis) the generating functions $A_y(1,z)$ and $A(1,z)$ yield that $\sum_{k\geq 0} k\cdot a_{n,k}$ and $\sum_{k\geq 0} a_{n,k} = f_{n+2}$ asymptotically behave like $\frac{n}{5}\varphi^{n+1}$ and $\frac{1}{\sqrt{5}}\varphi^{n+2}$, respectively, where $\varphi = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio. Therefore, we obtain

$$\overline{a}_n = \frac{\sum_{k\geq 0} k\cdot a_{n,k}}{\sum_{k\geq 0} a_{n,k}} \sim n\cdot\frac{1}{\varphi\sqrt{5}} \qquad \text{for } n\to\infty.$$

[4] For NAFs, this is of great interest: practically, they are used for efficient scalar multiplication in the area of Elliptic Curve Cryptography. Assume we want to compute nP for $n\in\mathbb{N}$ and a point P on the elliptic curve, then the weight of the NAF expansion of n determines the number of “expensive” point additions.

[5] In a nutshell, *multivariate generating functions* are generating functions with a separate variable for every parameter that is investigated.

Note that "$\sim$" stands for "asymptotically equal to". This and more tools for working with asymptotic expressions are discussed in detail in Section 1.3. $\triangle$

This concludes our introduction to the idea of combinatorial classes. Note that the ideas from this section can be adopted in order to deal with *labeled* structures as well (see, for example, [17, Chapter II]).

1.3 Analytic methods

In this section, we will discuss several analytic preliminaries and techniques that are needed and used throughout this thesis. First and foremost, as we will encounter many *asymptotic expressions*, i.e. expressions that describe the behavior of some function when the argument tends to some fixed value (for asymptotic considerations, this will primarily be ∞ or 0). The motivation for the following definition is due to Bachmann and Landau (see Knuth's remark [29] for more background on the history).

Definition 1.3.1 (Asymptotic notations, [17, A.2]).
Let S be a set equipped with some topology, and let $s^* \in S$ be an inner point (i.e. there are infinitely many elements of S in every neighborhood of s^*). Furthermore, let f and g be two functions from $S \setminus \{s^*\}$ to $\mathbb{R}$ or $\mathbb{C}$, and assume that $g(s) \neq 0$ for all $s \in S \setminus \{s^*\}$.

(a) *Big-Oh notation*: if $\frac{f(s)}{g(s)}$ stays bounded for $s \to s^*$, then we write $f(s) \overset{s\to s^*}{=} O(g(s))$. To be more precise, we say that f is "Big-Oh" of g if there is a neighborhood U of s^* as well as a constant $C > 0$ such that $|f(s)| \leq C|g(s)|$ for all $s \in U \setminus \{s^*\}$.

(b) *Little-Oh notation*: we write $f(s) \overset{s\to s^*}{=} o(g(s))$ if $\frac{f(s)}{g(s)}$ tends to 0 for $s \to s^*$. This is equivalent to the condition that for every $\varepsilon > 0$ we find a neighborhood U_ε of s^* such that $|f(s)| \leq \varepsilon|g(s)|$ for all $U_\varepsilon \setminus \{s^*\}$.

(c) *Asymptotic equivalence*: If $\frac{f(s)}{g(s)}$ converges to 1 for $s \to s^*$, then f is said to be asymptotically equivalent to g for $s \to s^*$; we write $f(s) \overset{s\to s^*}{\sim} g(s)$.

(d) *Omega- and Theta notation*: Where the Big-Oh notation states that the growth of $f(s)$ is bound by the growth of $g(s)$, the Omega notation states the exact opposite: we write $f(s) \overset{s\to s^*}{=} \Omega(g(s))$ if $g(s) \overset{s\to s^*}{=} O(f(s))$. In particular, this means that f is of order at least g near s^*.

And finally, if both $f(s) \overset{s\to s^*}{=} O(g(s))$ and $f(s) \overset{s\to s^*}{=} \Omega(g(s))$ hold, then we say that f is of order exactly g near s^* and write $f(s) \overset{s\to s^*}{=} \Theta(g(s))$.

Remark.
The standard scenario is that we analyze the behavior of a function near ∞, so in this thesis we will omit the $s \to \infty$ and just write $f(s) = O(g(s))$ and so on. Very often, we also use

the Big-Oh notation within an equation like $f(s) = g(s) + O(h(s))$, meaning that the growth of the difference $f(s) - g(s)$ is bounded by the growth of $h(s)$.

Furthermore, if we write

$$f(s) \sim \sum_{\ell=-L}^{\infty} a_\ell s^{-\ell},$$

this shall translate to

$$f(s) = \sum_{\ell=-L}^{R-1} a_\ell s^{-\ell} + O(s^{-R})$$

for all integers $R > -L$, even if the series does not converge.

Later on, we have to deal with specific multivariate expansions; these shall be treated analogously to above. In particular, the notation

$$f(r,s) \sim \sum_{\ell=-L}^{\infty} \sum_{j=0}^{J(\ell)} b_{\ell j} \frac{r^j}{s^\ell}$$

is to be understood as

$$f(r,s) = \sum_{\ell=-L}^{R-1} \sum_{j=0}^{J(\ell)} b_{\ell j} \frac{r^j}{s^\ell} + O(r^{J(R)} s^{-R}).$$

Over the course of this thesis, sometimes we encounter series that are related to the theta series $\theta(\tau) = \sum_{n\in\mathbb{Z}} \exp(i\pi\tau n^2)$ (for example, see Theorem 2.4.8). In order to find another representation of these series, the following tool is used.

Theorem 1.3.2 (Poisson summation formula).
Let $f \in L^1(\mathbb{R})$. Then the series $\sum_{n\in\mathbb{Z}} f(n)$ converges, and we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

where $\hat{f}(x) := \int_{-\infty}^{\infty} f(t) \exp(-2\pi i x t)\, dt$ is the Fourier transform of f.

Proof. See [44, Chapter VII, Corollary 2.6] for a discussion of this theorem in the n-dimensional Euclidean space $\mathbb{R}^n$. □

In the following two sections we will introduce two key ideas from asymptotic analysis. Singularity Analysis, on the one hand, is a very powerful framework that essentially allows to extract information on the asymptotic growth of some counting sequence out of (appropriate) singularities of the generating function. The Mellin transform, on the other hand, is an integral transform which will also provide some sort of translation between the asymptotic growth of the counting sequence and the poles of a transformation of the generating function.

1.3.1 Singularity Analysis

The aim of this section is to introduce a powerful method which determines the asymptotic growth of a counting sequence by analyzing the location of its poles. Basically, the framework of *Singularity Analysis* (which originated in [16]) consists of two equally-important parts. On the one hand, there is the *basic transfer* that provides a precise asymptotic analysis for the counting sequences of generic "building blocks" like $F(z) = (1 - z/\rho)^{-\alpha}$. And on the other hand, the so-called *transfer theorem* allows to control the error that is made when applying the basic transfer to local expansions of more complicated generating functions.

Recall that in Example 1.2.3 and 1.2.4 (as well as in the consecutive remark) we observed that the radius of convergence of a generating function $F(z)$ is connected to the asymptotic behavior of the corresponding sequence in the sense that a radius of convergence of ρ corresponds to an exponential growth of ρ^{-n}. This connection can be explained easily: Assume that $F(z)$ has a singularity[6] at $\rho \in \mathbb{C} \setminus \{0\}$. Then let $G(z) := F(\rho z)$, and by elementary properties of the Taylor expansion we find

$$[z^n]F(z) = \rho^{-n}[z^n]F(\rho z) = \rho^{-n}[z^n]G(z),$$

where $G(z)$ has a singularity at $z = 1$. We will see that (under suitable assumptions on the type of singularity) the remaining growth is of at most polynomial order. Combined with what we observed above, this means that the location and the structure of a singularity determine the exponential and the (at most) polynomial growth factor of the corresponding counting sequence, respectively.

Ultimately, this also means that we are only interested in those singularities closest to the origin (which we will call *dominant singularities*): the coefficient growth induced by singularities of greater modulus decays exponentially compared to the growth induced by the dominant singularities. If there are multiple dominant singularities, then under some technical conditions on the area of analyticity of the generating function[7], the respective contributions from every singularity can be summed up.

Now let us turn to the core of the Singularity Analysis framework as introduced in [16, §2] (where also the proofs to the following statements can be found).

Theorem 1.3.3 (Basic transfer: standard scale).
Let α be an arbitrary complex number in $\mathbb{C} \setminus -\mathbb{N}_0$. Then the coefficients in the function

[6] Singularity Analysis is not suited for the case of singularities in the origin. In general, this case corresponds to a sequence with super-exponential growth.

[7] To be precise, analyticity on a "indented disk" around the origin (the indentations exclude the singularities), a so-called Δ-domain is required. See Figure 1.2 for an illustration.

$F(z) = (1-z)^{-\alpha}$ admit the following asymptotic expansion in descending powers of n:

$$[z^n]F(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\Big(1 + \sum_{k=1}^{\infty} \frac{e_k(\alpha)}{n^k}\Big), \tag{1.1}$$

where the $e_k(\alpha)$ are computable polynomials in α of degree $2k$. In particular, we find

$$[z^n]F(z) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\Big(1 + \frac{\alpha(\alpha-1)}{2n} + \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24n^2} + O(n^{-3})\Big)$$

Remark.
The asymptotic scale introduced by this theorem handles the most basic case encountered when analyzing singularities of generating functions, thus the name "standard scale". The ideas of the respective proof, however, can be generalized in order to control a more general situation (logarithmic and iterated logarithmic scales). For example, we have

$$[z^n](1-z)^{-\alpha}\Big(\frac{1}{z}\log\frac{1}{1-z}\Big)^{\beta} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^{\beta}.$$

By exploiting Cauchy's integration formula, the asymptotic analysis of these "building blocks" can be transferred in order to control the error made when analyzing local expansions of generating functions.

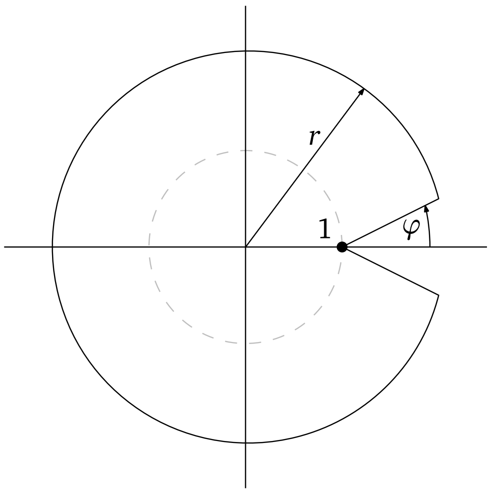

Figure 1.2: $\Delta(r, \varphi)$-region

Theorem 1.3.4 (Asymptotic transfer, standard scale).
Assume that the function $F(z)$ is analytic on the $\Delta(r, \varphi)$-region (subset of $\mathbb{C}$) illustrated in Figure 1.2 except in $z = 1$. Furthermore, let $\alpha \in \mathbb{R}$. Then the following statements hold:

(a) *"Big Oh transfer"*: if $F(z) = O((1-z)^{-\alpha})$ for $z \to 1$ in $\Delta(r, \varphi)$, then $[z^n]F(z) = O(n^{\alpha-1})$ for $n \to \infty$.

(b) *"Little Oh transfer"*: if $F(z) = o((1-z)^{-\alpha})$ for $z \to 1$ in $\Delta(r, \varphi)$, then $[z^n]F(z) = o(n^{\alpha-1})$ for $n \to \infty$.

(c) *"Transfer of asymptotic equality"*: if $F(z) \sim (1-z)^{-\alpha}$ in a neighborhood of $z = 1$ (within $\Delta(r, \varphi)$), then $[z^n]F(z) \sim n^{\alpha-1}/\Gamma(\alpha)$.

In a nutshell, the transfer theorem allows to transfer the error made when expanding the generating function locally (in order to analyze the type of the present singularity) *directly* to the coefficients. Also, note that there are analogous transfer theorems for the other asymptotic scales.

There are many more facets to this framework. For example, within Chapter 3 we will introduce the necessary concepts in order to deal with implicitly defined generating functions of a certain type by means of Singular Analysis. The method can also be extended to deal with more complicated generating functions that include, e.g., polylogarithms. Nevertheless, we leave those generalizations for the reader to find in the literature: see, for example, [40, Sections 2.17, 2.18] for an overview including some typical examples, and [17, Chapter VI] for an extensive treatment including technical details.

1.3.2 Mellin transform

The second central concept from Analytic Combinatorics we introduce is the so-called *Mellin transform*. Other than Singularity Analysis, the technique featuring Mellin transforms does not investigate the poles of the generating function itself, but rather the poles of the corresponding *Mellin transform*.

This approach is particularly useful when the function of interest is a so-called *harmonic sum*—which, in fact, occurs quite often. The following deliberations are primarily based on [15].

We begin with the definition of the Mellin transform.

Definition 1.3.5 (Mellin transform).
Let $f : (0, \infty) \to \mathbb{R}$ be a locally integrable function. Then the *Mellin transform* of f is defined by

$$f^*(s) = \mathcal{M}(f)(s) := \int_0^\infty x^{s-1} \cdot f(x)\, dx$$

for $s \in \mathbb{C}$ such that the integral exists. The largest open set where the Mellin transform exists is called the *fundamental strip*, and it is easy to see that the fundamental strip has to have the form $\langle \alpha, \beta \rangle := \{z \in \mathbb{C} \mid \alpha < \operatorname{Re} s < \beta\}$ for $\alpha, \beta \in \mathbb{R}$.

Of course, the allowed choices for $s \in \mathbb{C}$ strongly depend on the growth of the function f itself—but in general, if the real part of s is too large, then $x^{s-1} f(x)$ grows unbounded for

$x \to \infty$; and if it is too small, then $x^{s-1}f(x)$ tends to ∞ for $x \to 0^+$. Furthermore, note that the Mellin transform is an analytic function on its fundamental strip.

Example 1.3.6.
A striking example for the Mellin transform of a function is $f(x) = e^{-x}$. By definition, we have

$$\mathcal{M}(e^{-x})(s) = \int_0^\infty x^{s-1}e^{-x}\, dx =: \Gamma(s),$$

where the fundamental strip of this transform is $\langle 0, \infty \rangle$. We will use the exponential function throughout this section in order to illustrate the properties of the Mellin transform. $\triangle$

The following lemma summarizes some important properties of the Mellin transform that are immediate consequences of the definition.

Lemma 1.3.7 (Functional properties).
Let $f : (0, \infty) \to \mathbb{R}$ be a locally integrable function with Mellin transform $\mathcal{M}(f) \equiv f^*$ with fundamental strip $\langle \alpha, \beta \rangle$. Then the following holds:

(a) The operator $\mathcal{M}$ is linear.

(b) For $\mu \in \mathbb{R}_{>0}$, the *scaled function* $g(x) := f(\mu x)$ has Mellin transform $g^*(s) = \mu^{-s}f^*(s)$ with the same fundamental strip $\langle \alpha, \beta \rangle$.

(c) For $\rho \in \mathbb{R}^\times$, the function $g(x) = f(x^\rho)$ has Mellin transform $g^*(s) = \frac{1}{|\rho|}f^*\left(\frac{s}{\rho}\right)$ with fundamental strip $\langle \rho\alpha, \rho\beta \rangle$ or $\langle \rho\beta, \rho\alpha \rangle$.

(d) For a finite index set I and scalars $(\lambda_k)_{k\in I} \subseteq \mathbb{R}$, $(\mu_k)_{k\in I} \subseteq \mathbb{R}_{>0}$ we find

$$\mathcal{M}\left(\sum_{k\in I} \lambda_k f(\mu_k x)\right)(s) = \left(\sum_{k\in I} \lambda_k \mu_k^{-s}\right) f^*(s).$$

Proof. While (a) follows from the linearity of the integral, (b) and (c) are consequences of the transformation rules for integrals, and (d) is implied directly by (a) and (b). $\square$

In many applications, the generating function related to the combinatorial class of interest can be expressed a a sum of the shape $f(x) = \sum_{k\geq 0} \lambda_k g(\mu_k x)$ where $\lim_{k\to\infty} \mu_k \in \{0, \infty\}$. These sums are so-called *harmonic sums*[8], and under some technical conditions we can extend the linearity of the Mellin transform to infinitely many summands. This also requires a little knowledge of Dirichlet sums.

Definition 1.3.8 (Dirichlet sum).
Consider the sequences $(\lambda_k)_{k\in\mathbb{N}_0}$ and $(\mu_k)_{k\in\mathbb{N}_0}$ where $\lim_{k\to\infty} \mu_k = \infty$. Then the complex function $\Lambda(s)$ with

$$\Lambda(s) := \sum_{k\geq 0} \lambda_k \mu_k^{-s}$$

[8] Without loss of generality, we may assume $\mu_k \to \infty$ for $k \to \infty$—otherwise, we investigate $f(1/x)$ instead.

is said to be a *Dirichlet series*, where $s \in \mathbb{C}$ is chosen such that the series converges.

Remark.
For a survey on the general theory of Dirichlet series, we refer to [21]. In particular, we recall the result that a Dirichlet series is guaranteed to have a (complex) half-plane of absolute convergence of the form $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > \sigma_a\}$ for some $\sigma_a \in \mathbb{R}$.

A generalized form of linearity for the Mellin transform then states as follows:

Lemma 1.3.9.
Assume that $g\colon (0,\infty) \to \mathbb{R}$ is a locally integrable function whose Mellin transform $g^*(s)$ exists on a strip $\langle \alpha, \beta \rangle$. Furthermore, let $G(x) := \sum_{k\geq 0} \lambda_k g(\mu_k x)$ where $(\lambda_k)_{k\in\mathbb{N}_0} \subseteq \mathbb{R}$, $(\mu_k)_{k\in\mathbb{N}_0} \subseteq \mathbb{R}_{>0}$ and $\lim_{k\to\infty} \mu_k = \infty$ and assume that the half-plane of absolute convergence of $\Lambda(s) := \sum_{k\geq 0} \lambda_k \mu_k^{-s}$ has a non-empty intersection Δ with the strip $\langle \alpha, \beta \rangle$. Then the function $G(x)$ is defined for all $x \in (0,\infty)$, and the corresponding Mellin transform $G^*(s)$ is well-defined on Δ and factors as

$$G^*(s) = \Lambda(s) \cdot g^*(s).$$

Proof. See [15, Lemma 2]. □

Now we shed some light on the framework that makes the Mellin transform interesting from an asymptotic point of view. As stated in the introduction of this section, there is a relation between the so-called *singular expansion* of the Mellin transform and the asymptotic growth of the base function.

Definition 1.3.10 (Singular expansion).
Let the complex-valued function f be meromorphic on $\Omega \subseteq \mathbb{C}$ and let P denote the set of isolated poles. Let $\sum_{j=-r}^{\infty} c_j (z - z_0)^j$ be the Laurent series of f at $z_0 \in P$. Furthermore, let $P_{z_0}(z) := \sum_{j=-r}^{-1} c_j (z - z_0)^j$ denote the principal part of the Laurent series. Then the singular expansion of $f(z)$ in Ω is given by $\sum_{z_0 \in P} P_{z_0}(z)$, which we write as

$$f(z) \asymp \sum_{z_0 \in P} P_{z_0}(z).$$

Example 1.3.11.
First, let us consider an easy example. By partial fraction decomposition it becomes clear that we have

$$\frac{1}{z^2(z+1)} \asymp \frac{1}{z+1} + \frac{1}{z^2} - \frac{1}{z}.$$

A more challenging example is the gamma function $\Gamma(s)$. It is well-known [11, §5.2(i)] that the gamma function has simple poles for $s \in -\mathbb{N}_0$ with $\operatorname{Res}(\Gamma(s), s = -n) = \frac{(-1)^n}{n!}$. Thus, the singular expansion of $\Gamma(s)$ is given by

$$\Gamma(s) \asymp \sum_{n\geq 0} \frac{(-1)^n}{n!} \frac{1}{s+n}. \qquad \triangle$$

Given some simple conditions with respect to the growth of the functions involved, the following two theorems power the translation between the asymptotic growth of some function and the singular expansion of the respective Mellin transform.

Theorem 1.3.12 (Direct mapping theorem, [15, Theorem 3]).
Let $f : (0,\infty) \to \mathbb{R}$ have a Mellin transform $f^*(s)$ with nonempty fundamental strip $\langle \alpha, \beta \rangle$.

(a) Assume that for $x \to 0^+$, the asymptotic expansion

$$f(x) \overset{x\to 0^+}{=} \sum_{(k,\ell)\in I} c_{k,\ell} x^k \log^\ell x + O(x^\gamma)$$

holds, where the index set I is chosen such that $\ell \geq 0, -\gamma < -k \leq \alpha$. Then there exists a meromorphic continuation of $f^*(s)$ to the strip $\langle -\gamma, \beta \rangle$ where the transformation admits the singular expansion

$$f^*(s) \asymp \sum_{(k,\ell)\in I} c_{k,\ell} \frac{(-1)^\ell \ell!}{(s+k)^{\ell+1}} \qquad \text{for } s \in \langle -\gamma, \beta \rangle.$$

(b) Similar to (a), but this time we assume that the asymptotic expansion

$$f(x) \overset{x\to\infty}{=} \sum_{(k,l)} c_{k,\ell} x^k \log^\ell x + O(x^\gamma)$$

holds, where I is chosen such that $\ell \geq 0$ and $\beta \leq -k < -\gamma$. Then there exists a meromorphic continuation of $f^*(s)$ to $\langle \alpha, -\gamma \rangle$. The singular expansion of $f^*(s)$ can be expressed as

$$f^*(s) \asymp -\sum_{(k,\ell)\in I} c_{k,\ell} \frac{(-1)^\ell \ell!}{(s+k)^{\ell+1}} \qquad \text{for } s \in \langle \alpha, -\gamma \rangle.$$

Basically, the direct mapping theorem translates from an asymptotic expansion of a function $f(x)$ at 0 or ∞ to poles of the respective Mellin transform. The terms in the asymptotic expansion near 0 induce poles of the Mellin transform *left* of the fundamental strip; terms in the asymptotic expansion near ∞ induce poles *right* of the fundamental strip.

Theorem 1.3.13 (Converse mapping theorem, [15, Theorem 4]).
Let $f : (0,\infty) \to \mathbb{R}$ be a continuous function with Mellin transform $f^*(s)$ with non-empty fundamental strip $\langle \alpha, \beta \rangle$.

(a) *Continuation to the left.* Assume $f^*(s)$ has a meromorphic continuation on $\langle -\gamma, \beta \rangle$ which is analytic on the vertical line $\mathrm{Re}(s) = -\gamma$ with finite singular expansion (i.e. finite index set I)

$$f^*(s) \asymp \sum_{(k,\ell)\in I} d_{k,\ell} \frac{1}{(s+k)^{\ell+1}}.$$

Furthermore, assume that there is an $\eta \in (\alpha, \beta)$ such that $f^*(\sigma + it) \overset{t\to\infty}{=} O(|t|^{-r})$ for some $r > 1$, uniformly for $-\gamma \le \sigma \le \eta$. Then the function $f(x)$ admits the asymptotic expansion

$$f(x) \overset{x\to 0}{=} \sum_{(k,\ell)\in I} d_{k,\ell} \frac{(-1)^\ell}{\ell!} x^k \log^\ell x + O(x^\gamma).$$

(b) *Continuation to the right.* Similarly, assume that $f^*(s)$ has a meromorphic continuation on $\langle \alpha, -\delta \rangle$ which is analytic on the vertical line $\mathrm{Re}(s) = -\delta$ with finite singular expansion as above. Additionally, if there is an $\eta \in (\alpha, \beta)$ such that $f^*(\sigma + it) = O(|t|^{-r})$ for $r > 1$ uniformly for $\eta \le \sigma \le -\delta$, then $f(x)$ admits the asymptotic expansion

$$f(x) \overset{x\to\infty}{=} -\sum_{(k,\ell)\in I} d_{k,\ell} \frac{(-1)^\ell}{\ell!} x^k \log^\ell x + O(x^\delta).$$

Example 1.3.14.
Take $f(x) = e^{-x}$ and the corresponding Mellin transform, $f^*(s) = \Gamma(s)$. From the last example we already know the singular expansion of $\Gamma(s)$, namely $\Gamma(s) \asymp \sum_{n\ge 0} \frac{(-1)^n}{n!} \frac{1}{s+n}$. Once again, it is well-known (cf. [11, 5.11.9]) that the gamma function decays exponentially fast along vertical lines of the complex plane. Thus, the converse mapping theorem may be applied.

In this case, a summand $\frac{(-1)^n}{n!} \frac{1}{s+n}$ in the singular expansion of the transformation can be translated into a factor of $\frac{(-1)^n}{n!} x^n$. Actually, this is not very surprising—recall that the Taylor series of e^{-x} around 0 is given by $\sum_{n\ge 0} \frac{(-x)^n}{n!} = \sum_{n\ge 0} \frac{(-1)^n}{n!} x^n$.

This nicely illustrates the profound relation between the asymptotic expansion of a function and the singular expansion of the corresponding Mellin transform. △

Remark (Harmonic sums and converse mapping).
In order to guarantee that the converse mapping strategy can be applied to harmonic sums of the form $G(x) = \sum_{k\ge 0} \lambda_k g(\mu_k x)$, we need some additional growth estimates for the Dirchlet series $\Lambda(s) = \sum_{k\ge 0} \lambda_k \mu_k^{-s}$ and the transformed base function $g^*(s)$. In particular, we require $\Lambda(s)$ to grow slowly (at most polynomially) and $g^*(s)$ has to decay faster than polynomially along certain vertical lines of the complex plane. For a more detailed treatment see [15, Theorem 5].

1.4 Probability theory

There is a deep connection between the asymptotic analysis of parameters of combinatorial objects and certain tools from probability theory. For instance, it is common practice to consider combinatorial objects over a suitable probability space such that the respective probability distribution of the parameters can be studied. In essence, this is also what we did in Example 1.2.10: there, we considered binary words to be uniformly distributed, i.e. words of equal length occur with the same probability. This enabled us to consider the “average weight”, which is nothing else than the expected value of a random variable mapping a binary word without consecutive ones to its weight.

The intention of this section is to give a short introduction into some central concepts from probability theory that are used within analytic combinatorics. For a rigorous introduction to probability theory see [13] and [14], or, for a less extensive introduction from a measure-theoretic point of view, see [4].

We begin by recalling some elementary definitions.

Definition 1.4.1 (Random variable, Stochastic process).
A Borel measurable function X from some probability space $(\Omega, \mathcal{F}, \mu)$ to $\mathbb{R}$ is called *(real-valued) random variable*. The corresponding pushforward measure on $\mathbb{R}$ is denoted by $\mathbb{P}_X$, that is $\mathbb{P}_X(A) = \mu(X^{-1}(A))$ for Borel sets $A \subseteq \mathbb{R}$.

Furthermore, a *(real-valued) stochastic process* $(X_t)_{t\in T}$ is a family of real-valued random variables. If the index set T is countable (especially for $T = \mathbb{N}_0$), the process $(X_t)_{t\in T}$ is called discrete-time process or stochastic chain.

Remark.
If it is clear from the context which random variable induces the pushforward measure $\mathbb{P}_X$, we simply write $\mathbb{P}$. For example, for sets like $\{\omega \in \Omega \mid X(\omega) \leq x\} =: \{X \leq x\}$, we write $\mathbb{P}_X(X \leq x) = \mathbb{P}(X \leq x)$.

Example 1.4.2 (Random walks).
A common example for stochastic processes are *random walks*: for $n \in \mathbb{N}_0$, a stochastic chain $(S_k)_{0\leq k\leq n}$ is called *simple symmetric random walk* on $\mathbb{Z}$ of length n starting at 0, if we have $\mathbb{P}(S_0 = 0) = 1$, as well as

$$\mathbb{P}(S_k = j - 1 \mid S_{k-1} = j) = \mathbb{P}(S_k = j + 1 \mid S_{k-1} = j) = \frac{1}{2} \quad \text{for } k \geq 1.$$

In Figure 1.3, the state diagram for simple symmetric random walks on $\mathbb{Z}$ is illustrated.

There are some popular modifications to the idea of random walks, for example the introduction of *reflecting or absorbing barriers*. If the process runs into a reflective barrier, it has to return to its previous state with probability 1. If it runs into an absorbing barrier, it cannot leave the respective state again.

For example, a symmetric random walk on $\mathbb{Z}$ has a reflective barrier in 0 if we have $\mathbb{P}(S_k = 1 \mid S_{k-1} = 0) = 1$. Such a random walk only visits states from $\mathbb{N}_0$—and may thus be considered as a random walk on $\mathbb{N}_0$. We will investigate several classes of random walks (or, to be precise, their realizations: *lattice paths*) over $\mathbb{N}_0$ and $\mathbb{Z}$ thoroughly in Chapter 2. △

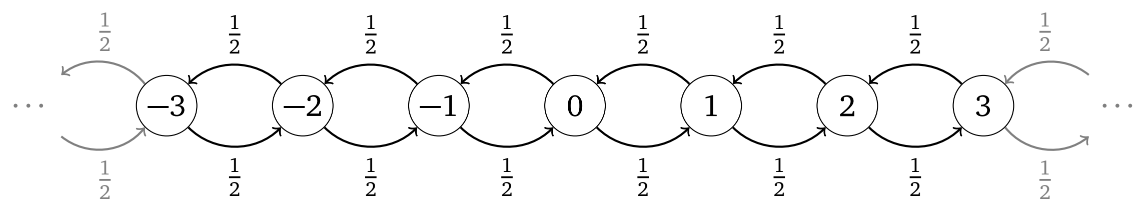

Figure 1.3: State diagram for symmetric random walks on $\mathbb{Z}$

Definition 1.4.3 (Expectation).
The *expectation* or *expected value* of a real-valued random variable X is defined as the Lebesgue integral

$$\mathbb{E}X := \int x \, d\mathbb{P}(x).$$

Furthermore, for a real Borel-measurable function $g\colon \mathbb{R} \to \mathbb{R}$, the expected value of $g(X)$ is defined to be

$$\mathbb{E}g(X) := \int g(x) \, d\mathbb{P}(x).$$

In a combinatorial context, we often study random variables with values in $\mathbb{N}_0$. For these variables, we may also define a special type of generating function: *probability generating functions*.

Definition 1.4.4 (Probability generating functions).
Let X be a random variable with values in $\mathbb{N}_0$. Then define the *probability generating function (PGF)* of X as

$$P_X(z) := \sum_{j \geq 0} \mathbb{P}(X = j) \cdot z^j.$$

Lemma 1.4.5 (Properties of PGFs).
Let X and Y be random variables with values in $\mathbb{N}_0$. Then the following properties hold:

(a) The PGF of $X+Y$ is given as the product of the PGFs of X and Y: $P_{X+Y}(z) = P_X(z) \cdot P_Y(z)$.

(b) The expected value $\mathbb{E}X$ is given by $P_X'(1)$.

(c) For the second derivative we have $P_X''(1) = \mathbb{E}[X(X-1)] = \mathbb{E}X^2 - \mathbb{E}X$.

Proof. These statements follow directly from the definition of the expected value and the probability generating function. □

As we will see later, probability generating functions will play an important role when it comes to investigating the asymptotic behavior of the modeled parameter. However, before we can discuss this in more detail, we need to introduce the concept of limiting distributions and limit laws.

1.4.1 Limiting distributions

For the contents of this thesis, we will mainly focus on the distinction between central and local limit laws, as well as asymptotic normality. For a rigorous treatment of the concept of limiting distributions (from a combinatorial point of view), we refer to [17, Chapter IX].

Let $\mathcal{A}$ be an arbitrary combinatorial class whose elements also have another integer-valued parameter χ that we want to investigate. For example, think of $\mathcal{A}$ as the set of binary words without consecutive ones, and let $\chi(\alpha)$ denote the weight of a word $\alpha \in \mathcal{A}$. For fixed size n, we can describe the behavior of the elements with respect to the parameter χ by means of a probability distribution:

Assume that all elements in $\mathcal{A}$ with the same size are equally likely, that is for $\alpha \in \mathcal{A}$ with $|\alpha| = n$ we have

$$\mathbb{P}(\alpha \mid |\alpha| = n) = \frac{1}{a_n}.$$

Then let $\chi_n : \{\alpha \in \mathcal{A} \mid |\alpha| = n\} \to \mathbb{N}_0$ denote the random variable that maps elements of size n to their respective value of χ. Trivially, this discrete random variable can be characterized by

$$\mathbb{P}(\chi_n = k) = \frac{|\{\alpha \in \mathcal{A} \mid \chi(\alpha) = k, |\alpha| = n\}|}{a_n}.$$

Naturally, we are interested in the asymptotic behavior of the sequence $(\chi_n)_{n\in\mathbb{N}_0}$ of random variables. From a probability theoretic point of view, *central* and *local limit laws* provide information on the asymptotic distribution of such a random variable. *Central limit laws* state that the sequence $(F_n)_{n\in\mathbb{N}_0}$ of associated cumulative distribution functions converges to some cumulative distribution function F pointwise at each point of continuity. This is also known as *convergence in distribution*.

In this case, the distribution related to F is called *limiting distribution*. Note that we do not necessarily actually investigate the parameter χ_n itself: for example, when considering a continuous limiting distribution, the CDFs of the scaled random variables $(\chi_n - \mathbb{E}\chi_n)/\sqrt{\operatorname{Var}(\chi_n)}$ are investigated in general. *Local limit laws*, on the other hand, describe the asymptotic behavior of the probabilities $\mathbb{P}(\chi_n = k)$.

Like in general probability theory, the normal distribution plays a very special role when it comes to limiting distributions. In this context, the framework of *quasi-powers* (which, in some sense, can be seen as an analogue or even a generalization of the central limit theorem) is very well-known. Essentially, this framework allows us to conclude whether the parameter of a combinatorial structure follows an asymptotic normal distribution simply from checking whether the respective probability generating function has a certain shape.

The following statement is due to H. K. Hwang (cf. [24]):

Theorem 1.4.6 (Hwang's Quasi-Power theorem).
Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of real-valued random variables such that their *moment generating function* can be written as

$$M_n(s) := \mathbb{E}e^{X_n s} = \exp(H_n(s))\Big(1 + O\Big(\frac{1}{\kappa_n}\Big)\Big)$$

uniformly for $|s| < \rho$ for some fixed $\rho > 0$ with $H_n(s) := u(s)\beta_n + v(s)$ such that the following properties hold:

(1) The functions u, v, and M_n are analytic for $|s| < \rho$,

(2) The sequences $(\beta_n)_{n\in\mathbb{N}_0}$ and $(\kappa_n)_{n\in\mathbb{N}_0}$ converge to ∞, $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\kappa_n = \infty$,

(3) The *variability condition*: $u''(0) \neq 0$.

Then the scaled random variable $(X_n - u'(0)\beta_n)/\sqrt{u''(0)\beta_n}$ has an asymptotic normal distribution, that is

$$\mathbb{P}\Big(\frac{X_n - u'(0)\beta_n}{\sqrt{u''(0)\beta_n}} \leq x\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt + O\Big(\frac{1}{\sqrt{\beta_n}} + \frac{1}{\kappa_n}\Big),$$

and for the expected value and variance of X_n we have

$$\mathbb{E}X_n = u'(0)\beta_n + v'(0) + O\Big(\frac{1}{\kappa_n}\Big), \qquad \mathrm{Var}(X_n) = u''(0)\beta_n + v''(0) + O\Big(\frac{1}{\kappa_n}\Big).$$

Proof. This slightly modified version of Hwang's original statement ([24, Theorem 1]) can be found in [17, Lemma IX.1] (including a proof). □

Remark.
This remarkable theorem strongly builds upon the Berry-Esseen inequality (cf. [14, p. 538, (3.13)]), which provides an estimate for the maximal difference between the cumulative distribution function of two random variables.

Furthermore, Heuberger generalized Hwang's theorem to the two-dimensional case in [22].

A slightly weaker version of the classical central limit theorem (because we require the first three moments to exist) can now also be seen as a corollary of Hwang's Quasi-Power theorem:

Corollary 1.4.7 (Central limit theorem, [17, Theorem IX.6]).
Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of identical and independently distributed random variables such that their respective first, second, and third moments exist. Then the standardized sum $(\sum_{j=1}^{n} X_j - n\mathbb{E}X_1)/\sqrt{n\,\mathrm{Var}(X_1)}$ is asymptotically normally distributed with mean 0 and variance 1. In particular, we have

$$\mathbb{P}\Big(\frac{\sum_{j=1}^{n} X_j - n\mathbb{E}X_1}{\sqrt{n\,\mathrm{Var}(X_1)}} \leq x\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt + O\Big(\frac{1}{\sqrt{n}}\Big).$$

Proof. Let $S_n = \sum_{j=1}^{n} X_n$. Then, the moment generating function of S_n is given by

$$M_n(s) = \mathbb{E}e^{S_n s} = \prod_{j=1}^{n} \mathbb{E}e^{X_j s} = M(s)^n = \exp(n \log M(s)),$$

where $M(s)$ denotes the moment generating function of X_1. In this case, there is no approximation error $O(1/\kappa_n)$, as the moment generating function could be given exactly. In the notation of Hwang's theorem, we find $H_n(s) = n \cdot \log M(s)$, that is $\beta_n = n$, $u(s) = \log M(s)$, and $v(s) = 0$. The other requirements of the Quasi-Power theorem are also easily checked, and the statement of the central limit theorem follows. □

As we already mentioned in the introduction above, another simple consequence of the Quasi-Power theorem can be used to ensure asymptotic normality simply by investigating the sequence of probability generating functions. In fact, this consequence also illuminates why the corresponding framework is called "Quasi-Power framework".

Corollary 1.4.8 ([17, Theorem IX.8]).
Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of random variables with probability generating functions $(p_n(u))_{n\in\mathbb{N}}$ that can be written as

$$p_n(u) = A(u)B(u)^{\beta_n}\left(1 + O\left(\frac{1}{\kappa_n}\right)\right)$$

such that the functions $p_n(u)$, $A(u)$, and $B(u)$ are analytic in a neighborhood of $u = 1$. Assume that additionally, the following properties hold:

(1) For $u = 1$, B evaluates to 1: $B(1) = 1$,

(2) The sequences $(\beta_n)_{n\in\mathbb{N}_0}$, $(\kappa_n)_{n\in\mathbb{N}_0}$ converge to ∞, $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \kappa_n = \infty$,

(3) The *variability condition*: $B''(1) + B'(1) - B'(1)^2 \neq 0$.

Then $(X_n)_{n\in\mathbb{N}_0}$ is asymptotically normally distributed with an error of $O(1/\sqrt{\beta_n} + 1/\kappa_n)$. Furthermore, we have

$$\mathbb{E}X_n = B'(1)\beta_n + O(1), \qquad \operatorname{Var}(X_n) = (B''(1) + B'(1) - B'(1)^2)\beta_n + O(1).$$

Proof. Note that between the moment and the probability generating function the following relation holds: if $P(u)$ is the probability generating function of a $\mathbb{N}_0$-valued random variable, then $P(e^u)$ is the moment generating function. Therefore, the Quasi-Power theorem may be applied with $u(s) = \log B(e^s)$ and $v(s) = \log A(e^s)$. □

In other words, this corollary states that if the sequence of probability generating functions behaves *roughly* like a sequence of powers of a constant function ("*Quasi-Powers*"), then under some technical conditions, asymptotic normality holds.

Up to now, we primarily concentrated on the asymptotic analysis under the assumption that intrinsic properties of the investigated sequence of random variables (like the respective moment or the probability generating functions) are known. However, in a combinatorial context we have to construct these generating functions from the multivariate generating function obtained by the symbolic method (like illustrated in Example 1.2.10). The following theorem (cf. [17, Theorem IX.9]) allows us to skip this explicit construction:

Theorem 1.4.9 (Meromorphic Pertubation).
Let $F(z,u)$ be a bivariate generating function, and X_n be the $\mathbb{N}_0$-valued random variable defined by[9]

$$\mathbb{P}(X_n = k) = \frac{[u^k z^n]F(z,u)}{[z^n]F(z,1)}.$$

Assume that additionally, the following properties hold:

(1) For a suitable neighborhood of y around 1 and $|z| \leq r$ for a $r > 0$, the generating function can be decomposed as $F(z,u) = \frac{A(z,u)}{C(z,u)}$ such that the functions $A(z,u)$ and $C(z,u)$ are analytic in this region. Furthermore, $C(z,1)$ has a unique simple zero ρ with $|\rho| < r$ and $A(\rho,1) \neq 0$.

(2) We have $\frac{\partial C}{\partial z}(\rho,1) \neq 0$. Therefore, by the implicit function theorem, there is a unique analytic function $\rho(u)$ in a suitable neighborhood of $u = 1$ such that $C(\rho(u),u) = 0$ and $\rho(1) = \rho$.

(3) The *variability condition*: $\sigma^2 = \frac{-(\rho''(1)+\rho'(1))\rho+\rho'(1)^2}{\rho^2} \neq 0$.

Then $(X_n)_{n\in\mathbb{N}_0}$ is asymptotically normally distributed with error $O(1/\sqrt{n})$ and

$$\mathbb{E}X_n = -\frac{\rho'(1)}{\rho}n + O(1), \qquad \mathrm{Var}(X_n) = \sigma^2 n + O(1).$$

The following example shall demonstrate the application of Theorem 1.4.9.

Example 1.4.10 (Average weight).
Once again, we return to the example with the binary words without consecutive ones. This time we are interested in the asymptotic distribution of the weight. In Example 1.2.10 we already constructed the bivariate generating function

$$F(z,u) = \frac{1+uz}{1-z-uz^2},$$

where $[u^k z^n]F(z,u) = a_{k,n}$ is the number of admissible binary words with length n and weight k. The random variable X_n then represents the weight of a NAF of length n. With the notation of Theorem 1.4.9, we obviously choose the decomposition $A(z,u) = 1 + uz$ and $C(z,u) = 1 - z - uz^2$. The functions $A(z,u)$ and $C(z,u)$ are analytic everywhere, and

[9] Analogous to ordinary generating functions, $[u^k z^n]F(z,u)$ denotes the coefficient of $u^k z^n$ in $F(z,u)$.

$C(z,1) = 1 - z - z^2$ has the two zeros $\frac{-1\pm\sqrt{5}}{2}$. For us, the zero closest to 0 is relevant, so we choose $\rho = \frac{-1+\sqrt{5}}{2} \approx 0.618$ and fix $r = 1$. Also, $A(\rho,1) = 1 + \rho \neq 0$.

Furthermore, $C_z(\rho,1) := \frac{\partial C}{\partial z}(\rho,1) = -1 - 2\rho \neq 0$. Thus, by the implicit function theorem, we have a unique analytic function $\rho(u)$ in a neighborhood of $u = 1$ such that $C(\rho(u),u) = 0$ and $\rho(1) = \rho$.

By implicit differentiation we may also compute the derivatives of this function ρ; we find

$$\rho'(1) = -\frac{C_u(\rho,1)}{C_z(\rho,1)}, \qquad \rho''(1) = -\frac{C_{uu}(\rho,1) + 2C_{uz}(\rho,1)\cdot\rho'(1) + C_{zz}(\rho,1)\cdot\rho'(1)^2}{C_z(\rho,1)}.$$

With the help of SageMath [45], we find that $\sigma^2 = \frac{\sqrt{5}}{25} \neq 0$.

Overall, all conditions of Theorem 1.4.9 are met and therefore we obtain that $(X_n)_{n\in\mathbb{N}_0}$ is asymptotically normal distributed with error $O(1/\sqrt{n})$ and

$$\mathbb{E}X_n = n\cdot\frac{1}{\varphi\sqrt{5}} + O(1), \qquad \mathrm{Var}(X_n) = n\cdot\frac{\sqrt{5}}{25} + O(1),$$

where $\varphi = \frac{1}{\rho} = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio. The expression for the expected weight coincides with the result of Example 1.2.10. $\triangle$

This concludes our short introduction to limiting distributions and particularly asymptotic normality.

1.4.2 Martingales and stopping times

Martingales are a very specific kind of stochastic process that can, in some sense, be interpreted as "fair games". In this section we present some important definitions and theorems with respect to martingale theory. As it turns out, we can use martingale theory in Section 2.4.1 in order to prove a relation between Chebyshev polynomials and a special type of random walks (Proposition 2.4.4) in a very elegant way.

The concept of martingales builds upon a generalization of the expected value. Consider the following observations based on a real-valued random variable $X : \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, \mu)$:

- By definition, X is $\mathcal{F}$-measurable. Furthermore, X contains all the information necessary to construct a minimal sub-σ-algebra $\mathcal{C}$ over which X is measurable ("induced σ-algebra", $\mathcal{C} = \{X^{-1}(B) \mid B \subseteq \mathbb{R},\ B \text{ Borel set}\}$).
- However, in general, X is not measurable over an arbitrary sub-σ-algebra $\mathcal{C}$ of $\mathcal{F}$.

For real-valued random variables, the expected value (as long as it exists) is a real number—which can again be interpreted as a (constant) real-valued random variable. Following this interpretation, the expected value yields a random variable which is measurable over *every* sub-σ-algebra of $\mathcal{F}$; in particular with respect to the trivial (and smallest) sub-σ-algebra $\{\emptyset, \Omega\}$.

In this context, the following theorem provides the theoretic foundation in order to generalize the expected value.

Theorem 1.4.11.
Let X be an μ-integrable real-valued random variable over the probability space $(\Omega, \mathcal{F}, \mu)$ and $\mathcal{C} \subseteq \mathcal{F}$ a sub-σ-algebra of $\mathcal{F}$. Then, up to almost sure equality, there is a unique $\mu|_{\mathcal{C}}$-integrable random variable Y over the probability space $(\Omega, \mathcal{C}, \mu|_{\mathcal{C}})$ such that for all $C \in \mathcal{C}$ the equation

$$\int_C Y \, d\mu|_{\mathcal{C}} = \int_C X \, d\mu \tag{1.2}$$

holds.

Proof. This is Theorem 15.1 in [4, §15], the proof is given there on p. 117 f. □

Definition 1.4.12 (Conditional expectation).
With the notation of Theorem 1.4.11, the random variable Y is called *conditional expectation* of X with respect to the sub-σ-algebra $\mathcal{C}$. We write $Y =: \mathbb{E}[X|\mathcal{C}]$.

The conditional expectation $\mathbb{E}[X|\mathcal{C}]$ can be interpreted as a "simplified" version of the random variable X in such a sense that measurability over some given sub-σ-algebra $\mathcal{C}$ is constructed.

Lemma 1.4.13.
Let X, Y be integrable real-valued random variables over the probability space $(\Omega, \mathcal{F}, \mu)$, and let $\mathcal{C}$ be a sub-σ-algebra of $\mathcal{F}$. Then the following properties hold:

(a) The (classical) expected value is the conditional expectation with respect to the trivial sub-σ-algebra, $\mathbb{E}X = \mathbb{E}[X|\{\emptyset, \Omega\}]$ almost surely.

(b) The conditional expectation is linear: for α, $\beta \in \mathbb{R}$ we have almost surely

$$\mathbb{E}[\alpha X + \beta Y|\mathcal{C}] = \alpha\mathbb{E}[X|\mathcal{C}] + \beta\mathbb{E}[Y|\mathcal{C}].$$

(c) If the random variable X is $\mathcal{C}$-measurable, then $\mathbb{E}[X|\mathcal{C}] \overset{\text{a.s.}}{=} X$ as well as $\mathbb{E}[X \cdot Y|\mathcal{C}] \overset{\text{a.s.}}{=} X \cdot \mathbb{E}[Y|\mathcal{C}]$.

Proof. Properties (a) and (b) both follow directly from the definition of the conditional expectation. For property (c), we refer to the statement from (15.21) in [4, §15]. □

Definition 1.4.14 (Martingale).
Let T be a totally ordered index set and let $(M_t)_{t \in T}$ be a stochastic process that is *adopted to*

the filtration $(\mathcal{F}_t)_{t\in T}$, meaning that $(\mathcal{F}_t)_{t\in T}$ is an increasing family of sub-σ-algebras of $\mathcal{F}$ (i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \le t$) such that M_t is a $\mathcal{F}_t$-measurable random variable over the probability space $(\Omega, \mathcal{F}, \mu)$ for all $t \in T$. Then $(M_t)_{t\ge 0}$ is called a *martingale* if for all $s < t$

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s \quad \text{almost surely.}$$

In the context of "fair games" this property may be interpreted such that the "best prognosis" for a future observation is the last available observation.

Remark (Discrete-time martingales).
In the case of discrete-time processes, $(M_n)_{n\in\mathbb{N}_0}$ is a martingale if

$$\mathbb{E}|M_n| < \infty \quad \text{and} \quad \mathbb{E}[M_{n+1}|M_0, M_1, \ldots, M_n] = M_n$$

holds for all $n \in \mathbb{N}_0$, where the random variables in the conditional part of the expected value represent the sigma algebra that they induce naturally.

Example 1.4.15 (Symmetric random walks).
We demonstrate that symmetric random walks $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ are martingales.

Without loss of generality assume $S_0 = 0$ almost surely. Then, obviously, $\mathbb{E}|S_n| \le n$ for all $n \in \mathbb{N}_0$ as after n steps, the random walk cannot be farther away from the origin than n steps. Furthermore, as the next state of the random walk only depends on the current state, we may write

$$\mathbb{E}[S_{n+1}|S_0, S_1, \ldots, S_n] = \mathbb{E}[S_{n+1}|S_n] = \frac{1}{2}\cdot(S_n + 1) + \frac{1}{2}\cdot(S_n - 1) = S_n,$$

which follows from the definition of the symmetric random walk (if the current state is m, then the choices for the next state are $m+1$ and $m-1$, both with probability $1/2$). This proves that symmetric random walks are martingales. △

The interpretation of martingales as "fair games" can be stressed even further: in principle, when playing a fair game, observations should not influence future predictions. In other words, we would expect $\mathbb{E}M_n = \mathbb{E}M_0$ to hold. And indeed, this follows directly from the characteristic property (1.2) of the conditional expectation. Moreover, in the context of *stopping times*, this is known as the *optional stopping theorem*.

Definition 1.4.16 (Stopping time).
Let T be a totally ordered index set and $(\mathcal{F}_t)_{t\in T}$ a filtration with respect to the σ-algebra $\mathcal{F}$. Then a function $\tau : \Omega \to T$ is said to be a *stopping time*, if for all $t \in T$ the property $\{\tau \le t\} = \{\omega \in \Omega \mid \tau(\omega) \le t\} \in \mathcal{F}_t$ holds.

Theorem 1.4.17 (Optional stopping theorem for martingales).
Let $(X_n)_{n\in\mathbb{N}_0}$ be a martingale adopted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$. Furthermore, let τ be a

stopping time with respect to the same filtration. Then the stopped process $(X_{\tau\wedge n})_{n\in\mathbb{N}_0}$ where $X_{\tau\wedge n}(\omega) := X_{\min\{\tau(\omega),n\}}(\omega)$ is also a martingale. In particular, we have $\mathbb{E}X_\tau = \mathbb{E}X_0$.

Proof. See statement 17.6 in [4, §17]. □

Example 1.4.18 (Hitting time).
Let $(X_n)_{n\in\mathbb{N}_0}$ be a real-valued stochastic process over $(\Omega, \mathcal{F}, \mu)$ adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ of $\mathcal{F}$. Then for a Borel set $A \subseteq \mathbb{R}$ we find that $\tau_A : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ with

$$\tau_A(\omega) := \inf\{n \in \mathbb{N}_0 \mid X_n(\omega) \in A\}$$

is a stopping time and is called *hitting time* of $(X_n)_{n\in\mathbb{N}_0}$ in A. △

At this point, we end our digression on martingale theory as we have all the tools necessary for the alternative proof of Proposition 2.4.3. For a rigorous treatment of martingale theory, we refer to [4, Chapter IV] and [5, Section 35].

Overall, this concludes our short introduction to some of the most central tools within the asymptotic analysis of discrete structures. In Chapter 2 and Chapter 3 these tools will be utilized in order to perform a precise asymptotic analysis of the behavior of several parameters related to special classes of lattice paths and trees, respectively.

2 Analysis of Lattice Paths

2.1 Introduction

Lattice paths as well as their stochastic incarnation—random walks—are interesting and classical objects of study. Several authors have investigated a variety of parameters related to lattice paths. For example, Banderier and Flajolet gave an asymptotic analysis of the number of special lattice paths (*walks/paths*, *bridges*, *meanders*, and *excursions*) in [3]. De Brujin, Knuth, and Rice [10] analyzed the expected height of certain lattice paths, and Panny and Prodinger [38] determined the asymptotic behavior of such paths with respect to several notions of height. However, lattice paths (and/or random walks) are more than just mathematically fascinating objects—they do have a vast variety of real-world applications in Biology (e.g. [8], [9]), Physics and Chemistry (e.g. [32], [41, Chapter 5]), language theory and complexity theory (e.g. [2], [20]), and many more.

Within this chapter, we analyze several special classes of lattice paths, i.e. lattice paths that fulfill certain restrictions (for instance the class of non-negative paths) asymptotically. These analyses are particularly interesting, as not only most of the concepts we introduced in Chapter 1 are used, but also some very special approaches are developed.

Formally, we define a lattice path as follows:

Definition 2.1.1 (Lattice path).
Let $\mathcal{S} \subseteq \mathbb{Z}$ be a set of integers and $n \in \mathbb{N}_0$. Then every integer sequence $(s_k)_{0\leq k\leq n}$ is called *lattice path of length n relative to* $\mathcal{S}$ if we have $s_{k+1} - s_k \in \mathcal{S}$ for all $k \in \{0, 1, \ldots, n-1\}$. The elements in $\mathcal{S}$ are called *allowed steps*. Furthermore, lattice paths with $\mathcal{S} = \{-1, 1\}$ are said to be *simple* lattice paths.

Remark.
As we have already mentioned in Section 1.4, lattice paths can be interpreted as realizations of special stochastic processes: *random walks*. We use this connection heavily in Section 2.4, where we provide an asymptotic analysis for a special class of lattice paths by investigating the respective probabilities.

Visually, we will represent lattice paths as so-called *directed lattice paths*, i.e. instead of considering the lattice path as a sequence of integers $(s_k)_{0\le k\le n}$, we consider the points $(k, s_k)_{0\le k\le n}$ in the plane. For example, see Figure 2.1 for such a representation. Thus, we

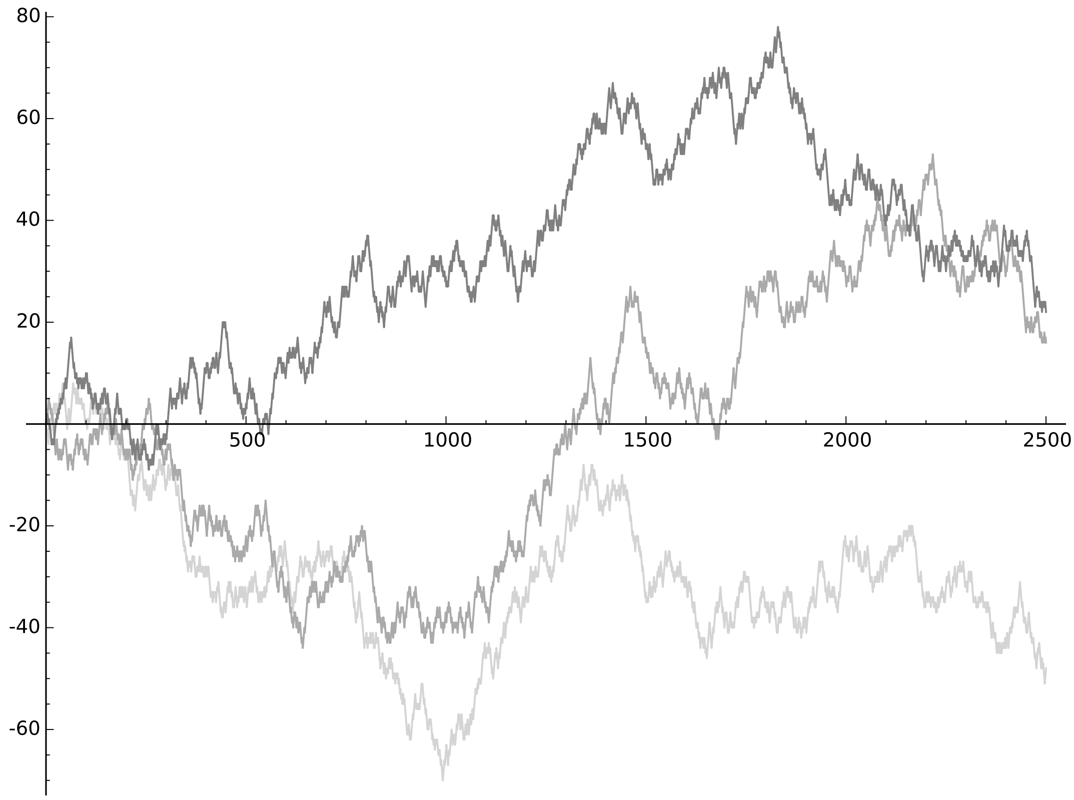

Figure 2.1: Directed (simple) lattice paths starting at 0

may also refer to positive and negative steps in $\mathcal{S}$ as *up-steps* and *down-steps*, respectively. Furthermore, notions like *height* or *altitude* seems quite natural for directed lattice paths.

The various classes of lattice paths we will investigate within this thesis are given in the following definition.

Definition 2.1.2 (Special lattice paths).
Let $(s_k)_{0\le k\le n}$ be a lattice path relative to $\mathcal{S} \subseteq \mathbb{Z}$ starting at 0 (so $s_0 = 0$).

(a) If the path ends in 0 (that is $s_n = 0$), then the lattice path is called *bridge*.

(b) If the lattice path visits only non-negative integers (i.e. $s_k \ge 0$ for all $k \in \{0, 1, \ldots, n\}$), then the lattice path is said to be a *meander*.

(c) If a lattice path is a *bridge* as well as a *meander*, then it is called an *excursion*.

(d) If a lattice path is never farther away from the start than at the end, we call it *extremal*. Formally, this means $\max_{0\le k\le n} |a_k| = |a_n|$.

(e) Finally, an extremal meander is said to be a *culminating* path.

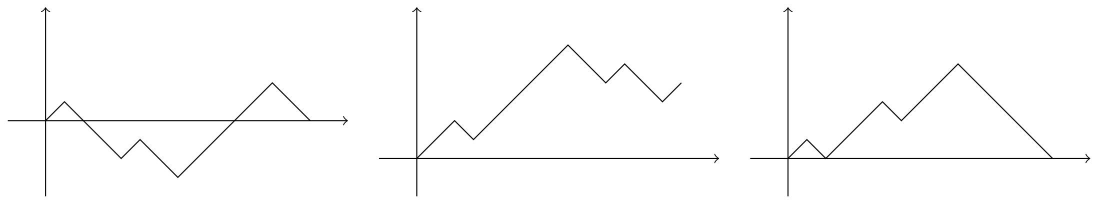

Figure 2.2: Illustration of a bridge, a meander, and an excursion (all simple)

Bridges, meanders and excursions are illustrated in Figure 2.2, culminating paths and extremal lattice paths are treated separately in Section 2.4.

In the following three sections we will discuss how these special classes of lattice paths can be analyzed asymptotically. While the basic analysis of unrestricted paths and bridges is rather straightforward, we need to develop special techniques for parameters of simple unrestricted paths as well as for meanders and excursions. Note that Section 2.2 and Section 2.3 are loosely based on [3] and [38], and in 2.4 we discuss some previously unknown results for culminating paths. This section is an adapted version of [19], which is a paper that was written in cooperation with Clemens Heuberger, Helmut Prodinger, and Stephan Wagner.

2.2 Unrestricted Paths and Bridges

First of all, note that the main difference between all of the special classes of lattice paths introduced in Definition 2.1.2 is the allowed height. Therefore, a bivariate generating function modeling the final height of a lattice path starting at 0 is desirable. Fortunately, a minor adaption of the symbolic method from Section 1.2 can be used to construct this object.

Let $\mathcal{S} := \{b_1, \ldots, b_m\} \subseteq \mathbb{Z}$ be the set of allowed steps. Furthermore, let $c := -\min_j b_j$ and $d := \max_j b_j$, and assume additionally that both c and d are positive. This means that there is both a positive as well as a negative allowed step, which ensures the existence of bridges and excursions.

When constructing combinatorial classes, we assumed that the size of the respective combinatorial objects is a non-negative integer. In this case, as we want to model the altitude of termination, it makes sense to drop this restriction. In particular, this means that we may interpret $\mathcal{S}$ as a combinatorial class with corresponding generating function $S(u) := \sum_{j=1}^{m} u^{b_j}$, which is said to be the *characteristic (Laurent) polynomial* of $\mathcal{S}$.

For Laurent polynomials (i.e. polynomials in u and u^{-1}), all the operations on combinatorial classes work in exactly the same way. In particular, we know that $\mathcal{S}^n$ contains all possible choices for lattice paths of length n, where the *size* of such an object exactly is its terminating height. The coefficient of u^k ($k \in \mathbb{Z}$) in the corresponding generating function $S(u)^n$ gives the number of lattice paths relative to $\mathcal{S}$ that terminate at height k after exactly n steps.

Example 2.2.1.
Consider the class of *simple lattice paths*, i.e. the class of paths relative to $\mathcal{S} = \{-1, 1\}$. The corresponding generating function for the height is $S(u) = u^{-1} + u$. By the argumentation above, for lattice paths of length n we find

$$S(u)^n = (u^{-1} + u)^n = \sum_{k=0}^{n} \binom{n}{j} u^{n-2j}.$$

Thus, for $k \equiv n \bmod 2$, we find that $[u^k]S(u)^n = \binom{n}{(n-k)/2}$. Because of the simple form of the generating function of the height of such paths, we are already able to derive the number of unrestricted lattice paths relative to $\mathcal{S}$ of length n as 2^n (which follows from summing over all possible heights—which can be done by setting $u = 1$), and $\binom{2n}{n}$ for the number of bridges of length $2n$ relative to $\mathcal{S}$. Combinatorially, this follows immediately from the fact that a bridge has to have equally many “down-steps” as “up-steps”. $\triangle$

However, if $\mathcal{S}$ is larger, then the analysis becomes more complicated. We are not quite finished with constructing the *bivariate* generating function: the variable u corresponds to the terminating height of the lattice path—but we also want to keep track of the length. Therefore, we consider the class $\{\rightarrow\} \times \mathcal{S}$, where “$\rightarrow$” represents a step of length 1 (the corresponding generating function is $z \mapsto z$).

From what we know regarding the symbolic method, the class of lattice paths can now be constructed as $(\{\rightarrow\} \times \mathcal{S})^*$, and the respective bivariate generating function is given by

$$W(z, u) = \frac{1}{1 - zS(u)},$$

where the exponent of z marks the length of the lattice path, and the exponent of u the terminating height. As usual, we are interested in singularities of $W(z, u)$. These are determined by the equation $1 - zS(u) = 0$, or equivalently, the polynomial equation

$$u^c - zu^c S(u) = 0. \tag{2.1}$$

The investigation of (2.1) will provide fruitful results for enumerating some of the special classes of lattice paths we want to study. Note that this equation is also called *kernel equation*, and the quantity $K(z, u) := u^c - z(u^c S(u))$ is said to be the *kernel* of the lattice paths determined by $\mathcal{S}$. Also, observe that the kernel equation defines a plane algebraic curve which we will refer to as the *characteristic curve* of $\mathcal{S}$.

The following lemma summarizes our findings up to here and provides the analysis for unrestricted paths.

Lemma 2.2.2 (Analysis of unrestricted paths, [3, Theorem 1]).
We follow [3, Proof of Theorem 1]. The bivariate generating function of lattice paths (with

z marking length and u marking final height) relative to a set $\mathcal{S} \subseteq \mathbb{Z}$ of allowed steps with characteristic polynomial $S(u)$ is a rational function of the form

$$W(z,u) = \frac{1}{1 - zS(u)}.$$

In particular, there are precisely $S(1)^n = |\mathcal{S}|^n$ (unrestricted) lattice paths relative to $\mathcal{S}$.

Proof. The form of the generating function has already been proved above. For the number of unrestricted lattice paths just note that the coefficient of z^n in $W(z,1)$ exactly counts the number of lattice paths of length n with arbitrary height. Then, because of

$$W(z,1) = \frac{1}{1 - S(1)z} = 1 + S(1)z + S(1)^2 z^2 + \dots$$

the statement follows. □

Apart from our proof via the generating function, it is trivial to see that the number of unrestricted paths of length n is given by $S(1)^n$—after all, we know $S(1) = |\mathcal{S}|$, and a lattice path can be thought of as a sequence of n allowed steps. However, this does not really help us for the asymptotic analysis of bridges: this actually requires that we have a close look at the kernel equation. And for doing so, it will prove convenient to represent the characteristic polynomial as $S(u) = \sum_{k=-c}^{d} s_k u^k$, where $s_k = 1$ if $k \in \mathcal{S}$ and $s_k = 0$ otherwise.

Observe that examining the kernel equation $1 - zS(u) = 0$ near $z = 0$ reveals that the equation can only be satisfied, if one of the relations

$$zu^{-c} \sim 1 \quad \text{or} \quad zu^{d} \sim 1 \qquad \text{for } z \to 0 \tag{2.2}$$

holds. We know that the characteristic equation $u^c - z(u^c S(u)) = 0$ is an equation of degree $c + d$. Such an equation is known to have $c + d$ roots, which (in dependence of z) are said to be the *branches* of the characteristic curve defined by the kernel equation.

As suggested by (2.2), we expect c "small branches" (which we will denote as $u_1, \dots, u_c$), and d "large branches" (denoted as $v_1, \dots, v_d$). "Small" and "large" relates to the behavior of the branch near 0, meaning that

$$u_j(z) \sim \exp\Big(\frac{2i(j-1)\pi}{c}\Big) z^{1/c}, \qquad v_k(z) \sim \exp\Big(\frac{2i(k-1)\pi}{d}\Big) z^{-1/d}$$

for $z \to 0$ and $j \in \{1, 2, \dots, c\}$, $k \in \{1, 2, \dots, d\}$. As stated in [3], this very informal discussion is backed up by the theory of Newton-Puiseux expansions, for which we refer to [42, Section 2.5]. In combination with some fundamental results from complex analysis, we are able to prove the following characterization of bridges.

Theorem 2.2.3 (Bridges and paths with fixed final height, [3, Theorem 1]). The generating function of bridges (i.e. lattice paths that end in 0) relative to a set of allowed steps $\mathcal{S}$ is an algebraic function given by

$$V(z) = z \cdot \sum_{j=1}^{c} \frac{u_j'(z)}{u_j(z)} = z \cdot \frac{d}{dz} \log(u_1(z)u_2(z) \cdots u_c(z)), \tag{2.3}$$

where $u_1(z), u_2(z), \ldots, u_c(z)$ denote all the small branches of the kernel equation. Generally, the generating function $W_k(z)$ corresponding to the paths that end at height k is, for $-\infty < k < c$

$$W_k(z) = z \cdot \sum_{j=1}^{c} \frac{u_j'(z)}{u_j(z)^{k+1}}, \tag{2.4}$$

and if $v_1(z)$, $v_2(z)$, $\ldots$, $v_d(z)$ denote the large branches of the kernel equation, then for $-d < k < \infty$ we have

$$W_k(z) = -z \cdot \sum_{j=1}^{d} \frac{v_j'(z)}{v_j(z)^{k+1}}. \tag{2.5}$$

Proof. We follow [3, Proof of Theorem 1]. In Lemma 2.2.2 we already established that the bivariate generating function $W(z,u)$ of lattice paths relative to $\mathcal{S}$ (where z marks the length of the lattice path and u marks the final height) has the form

$$W(z,u) = \frac{1}{1 - zS(u)}.$$

By definition, we have $W_k(z) = [u^k]W(z,u)$ and especially $V(z) = [u^0]W(z,u)$. The principal idea behind this proof is that we want to apply Cauchy's integral formula in order to obtain the desired generating functions.

For $|z| < 1/S(|u|)$, W is an analytic function in both arguments. Note that for fixed positive u, the radius of convergence of $z \mapsto W(z,u)$ is precisely $1/S(u)$. Furthermore, we know that the radius of convergence of $W_k(z)$ has to be at least $1/S(1)$. This is due to the fact that there are certainly more unrestricted lattice paths relative to $\mathcal{S}$ than those ending in k, meaning that the coefficients of $W_k(z)$ are dominated by the coefficients of $W(z,1)$—and as $W(z,1)$ has radius of convergence $1/S(1)$, $W_k(z)$ cannot have a smaller radius of convergence.

Let us concentrate on $|z| < r$, where $r := \frac{1}{2}S(1)^{-1}$. Observe that as $1/S(u)$ is a continuous function on $\mathbb{R}_{>0}$, there has to be an interval $(\alpha, \beta) \subseteq \mathbb{R}_{>0}$ such that $1/S(u) > r$ on this interval. Thus, the function $W(z,u)$ is analytic on the product domain $\{z \in \mathbb{C} \mid |z| < r\} \times \{u \in \mathbb{C} \mid \alpha < |u| < \beta\}$, as the restrictions imply $|zS(u)| < 1$.

Therefore, with $\gamma = \frac{\alpha+\beta}{2}$, Cauchy's integral formula can be applied, yielding

$$W_k(z) = [u^k]W(z,u) = \frac{1}{2\pi i} \oint_{|u|=\gamma} \frac{W(z,u)}{u^{k+1}}\, du.$$

Recall that for $z \to 0$, the large branches tend to ∞, while the small branches tend to 0. Thus, for sufficiently small z, we have $|v_j(z)| > \gamma$ for all $j \in \{1, 2, \ldots, d\}$ and distinct values for all small branches $u_j(z)$. Therefore, the integrand $\frac{W(z,u)}{u^{k+1}} = \frac{1}{u^{k+1}(1-zS(u))}$ has c simple poles at $u = u_j(z)$ for $j \in \{1, 2, \ldots, c\}$. Furthermore, in order to avoid the case where we have a singularity at 0, we assume $k < c$: as $\frac{1}{u^{k+1}(1-zS(u))} \stackrel{u\to 0}{=} O(u^{c-(k+1)})$ for fixed z the condition $k < c$ ensures boundedness for $u \to 0$.

Now, by the residue theorem (and with $u_j = u_j(z)$) we find

$$W_k(z) = \frac{1}{2\pi i} \oint_{|u|=\gamma} \frac{1}{u^{k+1}(1-zS(u))}\, du$$

$$= \sum_{j=1}^{c} \operatorname{Res}\left(\frac{1}{u^{k+1}(1-zS(u))},\ u = u_j\right) = \sum_{j=1}^{c} \frac{-1}{z u_j^{k+1} S'(u_j)}.$$

Observe that substituting any branch $u(z)$ in the kernel equation and differentiating with respect to z yields $-S(u(z)) - zS'(u(z))u'(z) = 0$ from which $\frac{1}{S'(u(z))} = -z^2 u'(z)$ can be obtained by using the kernel equation again. Substituting this into the expression for $W_k(z)$ from above proves (2.4) and because $k = 0 < c$ this also proves (2.3).

For (2.5) we note that $\frac{1}{u^{k+1}(1-zS(u))} \stackrel{|u|\to\infty}{=} O(u^{-d-(k+1)})$, which results in $\frac{1}{u^{k+1}(1-zS(u))} = o(u^{-1})$ for $|u| \to \infty$ as long as $k > -d$. In particular, this causes $\frac{1}{u^{k+1}(1-zS(u))}$ to decay sufficiently fast such that the contribution of the integral vanishes if the radius of the circle of integration tends towards ∞. Then, by subtracting the residues at the "large branches", and with $v_j = v_j(z)$ this results in

$$W_k(z) = -\sum_{j=1}^{d} \operatorname{Res}\left(\frac{1}{u^{k+1}(1-zS(u))}, u = v_j\right) = \sum_{j=1}^{d} \frac{1}{z v_j^{k+1} S'(v_j)}.$$

Then, following the same strategy as above proves (2.5). □

The following examples shall illustrate how the "small" and "large" branches of a kernel can be obtained, and how information on the generating function is then extracted.

Example 2.2.4 (Counting bridges).

(a) Consider the set of simple lattice paths induced by $\mathcal{S} = \{-1, 1\}$. The associated characteristic polynomial is given by $S(u) = u^{-1} + u$. Thus, the bivariate generating function for all paths (as of Lemma 2.2.2) is

$$W(z,u) = \frac{1}{1 - z(u^{-1} + u)},$$

and the kernel equation is given by $u - z(1 + u^2) = 0$. Solving this equation for u yields a small branch u_1 and a large branch v_1, namely

$$u_1(z) = \frac{1 - \sqrt{1-4z^2}}{2z}, \qquad v_1(z) = \frac{1 + \sqrt{1-4z^2}}{2z}.$$

As mentioned above, these branches come from solving the kernel equation. The decision which of the two solutions constitutes the small branch and the large branch is simple, because v_1 has a pole in 0 and thus tends towards ∞ for $z \to 0$. Therefore, v_1 indeed is the large branch, which leaves u_1 to be the small branch.

By Theorem 2.2.3, the generation function of simple bridges is therefore given by

$$V(z) = z \cdot \frac{u_1'(z)}{u_1(z)} = \frac{1}{\sqrt{1-4z^2}} = (1-4z^2)^{-1/2}.$$

From the considerations in the example above we know that $[z^{2n}]V(z) = \binom{2n}{n}$—and we may now use Singularity Analysis (Theorem 1.3.3, $\alpha = 1/2$) in order to obtain an asymptotic estimate for this central binomial coefficient:

$$\binom{2n}{n} = [z^{2n}](1-4z^2)^{-1/2} = [z^n](1-4z)^{-1/2} \sim \frac{4^n}{\sqrt{n\pi}}.$$

Note that this also solves the problem from the introduction of Chapter 1 (determining the number of "zero-sum game series") in a more satisfactory manner.

(b) Now consider the slightly more complicated case $\mathcal{S} = \{-1, 0, 1\}$. In this case, the bivariate generating function is given by $W(z,u) = \frac{1}{1-z(u^{-1}+1+u)}$, which yields the kernel equation $u - z(1+u+u^2) = 0$. Like before, this equation can be solved and we obtain

$$u_1(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z}, \qquad v_1(z) = \frac{1-z+\sqrt{1-2z-3z^2}}{2z}$$

for the small and large branch, respectively. Thus, the generating function of bridges is given by

$$V(z) = \frac{1}{\sqrt{1-2z-3z^2}} = \frac{1}{\sqrt{(1+z)(1-3z)}}.$$

As the singularity at $z = 1/3$ is the dominant one, with the help of Theorem 1.3.3 we find $[z^n]V(z) \sim \sqrt{\frac{3}{4}}\frac{3^n}{\sqrt{n\pi}}$. Alternatively, by following the same idea with the characteristic polynomial as above, we find $[z^n]V(z) = [u^0](u^{-1}+1+u)^n = [u^n](1+u+u^2)^n$, which are exactly the *trinomial numbers* enumerated by sequence A002426 in [34].

(c) Finally, let us have a look at $\mathcal{S} = \{-2, 1\}$. Determining the branches is slightly harder in this case as a cubic equation is involved: the bivariate generating function is $W(z,u) = \frac{1}{1-z(u^{-2}+u)}$, and the corresponding kernel equation is $u^2 - z(1+u^3) = 0$. The corresponding characteristic curve is illustrated in Figure 2.3. As can be seen in the figure, there are two branches that join into 0 for $z \to 0^+$ ("small branches"), and one branch that escapes towards ∞ for $z \to 0^-$ ("large branch").

With the help of SageMath [45], we are able to compute all those branches as well as the generating function for bridges relative to $\mathcal{S}$. However, as the resulting expressions

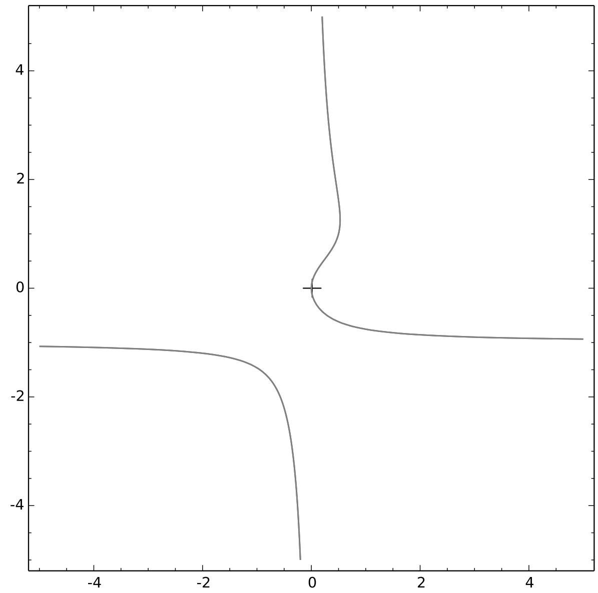

Figure 2.3: Characteristic curve $u^2 - z(1+u^3) = 0$ in the (z,u)-plane

are rather complicated we only give the Newton-Puiseux and power series expansions for the two small branches and the generating function, respectively:

$$\begin{aligned} u_1(z) &= z^{1/2} + \frac{1}{2}z^2 + \frac{5}{8}z^{7/2} + z^5 + \frac{231}{128}z^{13/2} + \frac{7}{2}z^8 + \dots \\ u_2(z) &= -z^{1/2} + \frac{1}{2}z^2 - \frac{5}{8}z^{7/2} + z^5 - \frac{231}{128}z^{13/2} + \frac{7}{2}z^8 \mp \dots \\ V(z) &= 1 + 3z^3 + 15z^6 + 84z^9 + 495z^{12} + \dots. \end{aligned}$$

Therefore, we cannot easily obtain an asymptotic estimate by means of Singularity Analysis: we cannot access the singularities of $V(z)$. However, we can revert to the approach that uses powers of the characteristic polynomial:

$$[z^{3n}]V(z) = [u^0](u^{-2} + u)^{3n} = [u^{6n}](1+u^3)^{3n} = \binom{3n}{n}. \qquad \triangle$$

Actually, this last example also brings us back to the central theme of this thesis: we would like to have asymptotic results for the number of bridges relative to a set of allowed steps $\mathcal{S}$ in all cases, and not only in those where we are actually able to determine the dominant singularities. However, before we discuss such a result for bridges, we have to define two lattice path parameters.

Definition 2.2.5 (Structural constant and lattice path period).
Let $\mathcal{S} \subseteq \mathbb{Z}$ be a set of allowed steps.

(a) Then the unique positive solution $\tau > 0$ of the equation $S'(u) = 0$, where $S(u)$ is the Laurent polynomial associated to $\mathcal{S}$, is said to be the *structural constant* of $\mathcal{S}$.

(b) Let $p \in \mathbb{Z}$ be the largest integer such that the characteristic Laurent polynomial $S(u)$ can be written as $S(u) = u^b \cdot T(u^p)$ for some $b \in \mathbb{Z}$ and another Laurent polynomial $T(u)$. Then the system of paths induced by $\mathcal{S}$ is said to have *period* p. Furthermore, we call such a system of paths *reduced*, if $\gcd(b, p) = 1$ holds.

Remark.
The period p of a system of paths can be interpreted combinatorially: note that $S(u) = u^b \cdot T(u^p)$ implies $S(u)^n = u^{bn} \cdot T(u^p)^n$. This means that the height of a path from this system that has length n is congruent to $nb \bmod p$. Therefore, it takes the height of a path a minimum of $\ell = \frac{p}{\gcd(b,p)}$ steps to return to the "original" congruence class.

Systems of non-reduced paths can be scaled in order to obtain an equivalent system of reduced paths.

Now, the following theorem provides an asymptotic result for bridges based on the so-called *saddle-point method*. Essentially, this approach approximates the integral obtained when applying Cauchy's integral formula to the generating function in a way such that the error can be controlled. Consult [17, Chapter VIII] for a thorough discussion of the saddle-point method.

Theorem 2.2.6 (Asymptotic analysis of bridges, [3, Theorem 3]).
Let $\tau > 0$ be the *structural constant* of a set of allowed steps $\mathcal{S} \subseteq \mathbb{Z}$ that induces a reduced system of paths with period p. Then the number of bridges admits the asymptotic behavior

$$[z^{np}]V(z) \sim \frac{p}{\tau}\sqrt{\frac{S(\tau)}{S''(\tau)}}\,\frac{S(\tau)^{np}}{\sqrt{2\pi np}}$$

and $[z^n]V(z) = 0$ for $n \nmid p$.

Proof. See [3, Proof of Theorem 3] as well as the remark on periodic paths in [3, Section 3.3]. □

Example 2.2.7.
Let us revisit (c) from Example 2.2.4; consider $\mathcal{S} = \{-2, 1\}$. As we are able to write the characteristic Laurent polynomial as $S(u) = u^{-2} + u = u^{-2}(1 + u^3)$, we can observe that the period of the walk is $p = 3$. Furthermore, the structural constant is the unique positive solution of the equation $S'(\tau) = 0$, which is equivalent to $-2\tau^{-3} + 1 = 0$, and thus $\tau = \sqrt[3]{2}$. Applying Theorem 2.2.6 then yields that for the number of bridges relative to $\mathcal{S}$, we have

$$\binom{3n}{n} = [z^{3n}]V(z) \sim \frac{3}{\sqrt[3]{2}}\sqrt{\frac{3\sqrt[3]{2}/2}{3\sqrt[3]{2}^2/2}}\,\frac{(27/4)^n}{\sqrt{6\pi n}} = \frac{3}{2}\frac{(27/4)^n}{\sqrt{3\pi n}}. \qquad \triangle$$

Before turning to the analysis of meanders and excursions, we want to get back to *simple* lattice paths, i.e. lattice paths relative to $\mathcal{S} = \{-1, 1\}$. Remember that the structure of these paths was so simple that we could basically read off their behavior—and exactly because of this simplicity, the class of simple paths is an excellent candidate for investigating more complex parameters.

In particular, following the deliberations of Prodinger and Panny in [38], we want to sketch the asymptotic analysis of the so-called *maximal deviation* of a simple lattice path.

Definition 2.2.8 (Maximal deviation).
Let $(s_k)_{0\leq k\leq n}$ be a simple lattice path of length n starting in 0. Then the parameter $\delta_{\max}^{(n)} := \max\{|s_k| \mid 0 \leq k \leq n\}$ is called *maximal deviation*. Simultaneously, $\delta_{\max}^{(n)}$ is considered as a random variable assigning each of the 2^n possible (and equally likely) lattice paths their maximal deviation.

Note that, for example, the maximal deviations of the lattice paths represented in Figure 2.2 are 3, 6, and 5, respectively.

We are interested in determining $\mathbb{E}\delta_{\max}^{(n)}$, the expected (or average) maximal deviation for simple lattice paths of length n. As usual, we will determine an exact formula for this expected value first, and then carry out an asymptotic analysis based on that. Let us start with an useful observation for paths with a given bound h for the deviation that end on a fixed altitude ℓ. For the sake of clarity, let $\psi_{h,\ell}(z)$ denote the generating function of this class of paths, that is $[z^n]\psi_{h,\ell}$ gives the number of simple lattice paths starting in 0 and ending in ℓ, for which $\delta_{\max}^{(n)} \leq h$ holds.

Proposition 2.2.9 ([38, Theorem 2.1]).
With the substitution $z = v/(1+v^2)$, we have

$$\psi_{h,\ell}(z) = v^{|\ell|}\frac{1+v^2}{1-v^2}\frac{1-v^{2(h-|\ell|+1)}}{1+v^{2h+2}}.$$

Proof. We follow [38, Proof of Theorem 2.1]. First of all, note that there are some relations between the generating functions $\psi_{h,\ell}$: Assume that for some lattice path we know $\delta_{\max}^{(n-1)} \leq h$, and assume that this path ends in $(n-1,\ell)$ (i.e. this path would be counted by the coefficient $[z^{n-1}]\psi_{h,\ell}(z)$). Such a path can be *continued* to a path that ends in $(n, \ell\pm 1)$—and also vice versa: every path with h-bounded maximal deviation that ends in (n,ℓ) can be constructed from a path that ends in $(n-1, \ell\pm 1)$ (as long as $|\ell \pm 1| \leq h$). Overall, this yields the relations

$$\psi_{h,-h}(z) = z\psi_{h,-h+1}(z), \qquad \psi_{h,h}(z) = z\psi_{h,h-1}(z),$$

as well as

$$\psi_{h,\ell}(z) = z\psi_{h,\ell-1}(z) + z\psi_{h,\ell+1}(z) \quad \text{for } |\ell| < h,\ \ell \neq 0$$

and $\psi_{h,0}(z) = 1 + z\psi_{h,-1}(z) + z\psi_{h,1}(z)$, because we have to account for the path of length 0. Rewriting this system of linear recurrences in a matrix yields

$$\begin{pmatrix} 1 & -z & & & & & \\ -z & 1 & -z & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -z & 1 & -z & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -z & 1 & -z \\ & & & & & -z & 1 \end{pmatrix} \begin{pmatrix} \psi_{h,-h}(z) \\ \psi_{h,-h+1}(z) \\ \vdots \\ \psi_{h,0}(z) \\ \vdots \\ \psi_{h,h-1}(z) \\ \psi_{h,h}(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

By Cramer's rule and the fact that the determinant of a block-triangular matrix is the product of the determinants of the matrices on the main diagonal, we find

$$\psi_{h,\ell}(z) = \frac{z^{|\ell|} a_h(z) a_{h-|\ell|}(z)}{a_{2h+1}(z)},$$

where $a_j(z)$ denotes the determinant of the matrix from the system above with j rows. Using Laplace's method for computing the determinants, it is easy to prove the recurrence $a_{j+1}(z) = a_j(z) - z^2 a_{j-1}(z)$ with $a_0(z) = a_1(z) = 1$.

In [27, Proof of Theorem 1] the very same sequence of determinants has been analyzed. In particular, it is easy to check that with $z = v/(1+v^2)$,

$$a_j(z) = \frac{1}{1-v^2} \frac{1-v^{2j+2}}{(1+v^2)^j}$$

solves the linear recurrence. Substituting this solution into the representation of $\psi_{h,\ell}(z)$ obtained above, we find

$$\psi_{h,\ell}(z) = v^{|\ell|} \frac{1+v^2}{1-v^2} \frac{1-v^{2(h-|\ell|+1)}}{1+v^{2h+2}}. \qquad \square$$

As a simple corollary, we find a representation of $\psi_h(z) := \sum_{|\ell| \le h} \psi_{h,\ell}(z)$, the generating function of all simple paths with maximal deviation less than or equal to h. This only uses the fact that due to symmetry we may write $\psi_h(z) = \psi_{h,0}(z) + 2\sum_{\ell=1}^{h} \psi_{h,\ell}(z)$.

Corollary 2.2.10 ([38, Theorem 2.2]).
The generating function $\psi_h(z)$ of all simple lattice paths with maximal deviation less than or equal to h can be represented as

$$\psi_h(z) = \frac{(1+v^2)(1-v^{h+1})^2}{(1-v)^2(1+v^{2h+2})},$$

where we used the substitution $z = v/(1+v^2)$.

For the rest of this section, let $c_n^{(h)}$ and $d_n^{(h)}$ denote the number of simple lattice paths with maximal deviation $\leq h$ and $> h$, respectively. By construction, we have $\psi_h(z) = \sum_{n\geq 0} c_n^{(h)} z^n$ and $d_n^{(h)} = 2^n - c_n^{(h)}$. Now, by exploiting Cauchy's integral formula, we can obtain an exact formula for these coefficients.

Proposition 2.2.11 ([38, Theorem 2.3, Corollary 2.7]).
The quantities $c_n^{(h)}$ and $d_n^{(h)}$ are given by

$$c_n^{(h)} = 2^n - 2\sum_{\lambda\geq 0}\sum_{0\leq \ell\leq h}\left[\binom{n}{\lceil \frac{n-(h+2)}{2}\rceil - 2\lambda(h+1)-\ell} + \binom{n}{\lfloor \frac{n-(h+2)}{2}\rfloor - 2\lambda(h+1)-\ell}\right] \tag{2.6}$$

and

$$d_n^{(h)} = 2\sum_{\lambda\geq 0}\sum_{0\leq \ell\leq h}\left[\binom{n}{\lceil \frac{n-(h+2)}{2}\rceil - 2\lambda(h+1)-\ell} + \binom{n}{\lfloor \frac{n-(h+2)}{2}\rfloor - 2\lambda(h+1)-\ell}\right]. \tag{2.7}$$

Proof. We follow [38, Proof of Theorem 2.3]. Let γ be a small contour that winds around the origin once. Then, by using Cauchy's integral formula and the substitution $z = v/(1+v^2)$, we find

$$\begin{aligned} c_n^{(h)} &= \frac{1}{2\pi i}\oint_\gamma \psi_h(z)\frac{dz}{z^{n+1}} = \frac{1}{2\pi i}\oint_{\tilde\gamma} \frac{(1+v^2)^n(1+v)(1-v^{h+1})^2}{(1-v)(1+v^{2h+2})}\frac{dv}{v^{n+1}} \\ &= [v^n]\frac{(1+v^2)^n(1+v)(1-v^{h+1})^2}{(1-v)(1+v^{2h+2})}. \end{aligned}$$

In order to extract the coefficient of v^n, we split the term into two summands.

$$\begin{aligned} c_n^{(h)} &= [v^n]\frac{(1+v^2)^n(1+v)(1-v^{h+1})^2}{(1-v)(1+v^{2h+2})} = [v^n]\frac{(1+v^2)^n(1+v)((1+v^{2h+2})-2v^{h+1})}{(1-v)(1+v^{2h+2})} \\ &= [v^n]\frac{(1+v^2)^n(1+v)}{1-v} - 2[v^{n-(h+1)}]\frac{(1+v^2)^n(1+v)}{(1-v)(1+v^{2h+2})}. \end{aligned} \tag{2.8}$$

Consider the first summand in (2.8). By expanding everything, we find

$$[v^n]\frac{(1+v^2)^n(1+v)}{1-v} = [v^n]\sum_{\lambda\geq 0}\sum_{j\geq 0}\binom{n}{j}v^{2j+\lambda}(1+v) = \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{j} + \sum_{j=0}^{\lfloor (n-1)/2\rfloor}\binom{n}{j} = 2^n.$$

This also implies that the second summand in (2.8) is equal to $d_n(h)$, as $d_n^{(h)} = 2^n - c_n^{(h)}$. In order to analyze this summand we need to expand the denominator $\frac{1}{(1-v)(1+v^{2h+2})}$:

$$\begin{aligned} \frac{1}{(1-v)(1+v^{2h+2})} &= \sum_{\lambda\geq 0}\sum_{\ell\geq 0}(-1)^\lambda v^{2\lambda(h+1)+\ell} = \sum_{\lambda\geq 0}\sum_{\ell\geq 0} v^{4\lambda(h+1)+\ell} - \sum_{\lambda\geq 0}\sum_{\ell\geq 0} v^{(4\lambda+2)(h+1)+\ell} \\ &= \sum_{\lambda\geq 0}\sum_{\ell=0}^{2h+1} v^{4\lambda(h+1)+\ell}. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} d_n^{(h)} &= 2[v^{n-(h+1)}]\Big(\sum_{j\geq 0}\binom{n}{j}v^{2j}\Big)\Big(\sum_{\lambda\geq 0}\sum_{\ell=0}^{2h+1} v^{4\lambda(h+1)+\ell}\Big)(1+v) \\ &= 2[v^{n-(h+1)}]\Big(\sum_{j\geq 0}\binom{n}{j}v^{2j}\Big)\Big(\sum_{\lambda\geq 0}\sum_{\ell=0}^{2h+1} v^{4\lambda(h+1)+\ell}\Big) \\ &\qquad + 2[v^{n-(h+2)}]\Big(\sum_{j\geq 0}\binom{n}{j}v^{2j}\Big)\Big(\sum_{\lambda\geq 0}\sum_{\ell=0}^{2h+1} v^{4\lambda(h+1)+\ell}\Big). \end{aligned}$$

The statement of the proposition then follows from investigating the cases $n \equiv h \bmod 2$ and $n \not\equiv h \bmod 2$ and then using the floor and ceiling notation to obtain one result for both cases. $\square$

Remember that the coefficients $d_n^{(h)}$ give the number of simple lattice paths of length n starting at 0 that leave the interval $[-h, h]$ at least once. Therefore, the difference $d_n^{(h-1)} - d_n^{(h)}$ gives the number of lattice paths with maximal deviation exactly h. This idea enables us to compute the expected maximal deviation.

Theorem 2.2.12 (Expected maximal deviation, exact; [38, Theorem 2.8]). Assuming that all simple lattice paths of length n are equally likely, the expected maximal deviation for simple lattice paths is given by

$$\begin{aligned} \mathbb{E}\delta_{\max}^{(n)} &= 2^{-n}\sum_{h\geq 0} d_n^{(h)} \\ &= 2^{-n+1}\sum_{\substack{h,\lambda\geq 0\\ 0\leq \ell\leq h}}\left[\binom{n}{\lceil \frac{n-(h+2)}{2}\rceil - 2\lambda(h+1)-\ell} + \binom{n}{\lfloor \frac{n-(h+2)}{2}\rfloor - 2\lambda(h+1)-\ell}\right]. \end{aligned}$$

Proof. As explained above, the number of simple lattice paths with maximal deviation h is given by $d_n^{(h-1)} - d_n^{(h)}$. Therefore, the expected value is given by the sum

$$\mathbb{E}\delta_{\max}^{(n)} = \sum_{h=0}^{n} h\cdot \frac{d_n^{(h-1)} - d_n^{(h)}}{2^n} = 2^{-n}\sum_{h=0}^{n} h(d_n^{(h-1)} - d_n^{(h)}).$$

By summation by parts this can be rewritten as

$$2^{-n}\sum_{h=0}^{n} h(d_n^{(h-1)} - d_n^{(h)}) = 2^{-n}\left(\sum_{h=0}^{n} d_n^{(h)} - ((n+1)d_n^{(n)} - 0\cdot d_n^{(-1)})\right) = 2^{-n}\sum_{h\geq 0} d_n^{(h)}. \qquad \square$$

Determining the asymptotic behavior of $\mathbb{E}\delta_{\max}^{(n)}$ is rather technical. We follow the approach presented in [38]. First of all, like the original authors we only consider the case of even path lengths. Now, note that the problem in approximating $\mathbb{E}\delta_{\max}^{(2n)}$ is that binomial sums of the form

$$2^{-2n}\sum_{a\leq k\leq b}\binom{2n}{n+k}$$

appear. As it turns out, the complement of the error function,

$$\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

can be used in order to obtain the main term of the asymptotic contribution of these sums.

Proposition 2.2.13 ([38, Theorem 2.9]).
Let $\varepsilon > 0$. Assume $0 \le a \le b = O(n^{1/2+\varepsilon})$ and $k = O(n^{1/2+\varepsilon})$. Then the following holds:

(a) For the scaled sum of shifted central binomial coefficients we have

$$2^{-2n} \sum_{a \le k \le b} \binom{2n}{n+k} = \frac{1}{2}\left(\operatorname{erfc}\Big(\frac{a-1/2}{\sqrt{n}}\Big) - \operatorname{erfc}\Big(\frac{b+1/2}{\sqrt{n}}\Big)\right)(1 + O(n^{-1+\varepsilon})).$$

(b) The shifted central binomial coefficient itself can be estimated by

$$2^{-2n}\binom{2n}{n+k} = \left(\operatorname{erfc}\Big(\frac{k}{\sqrt{n}}\Big) - \operatorname{erfc}\Big(\frac{k+1/2}{\sqrt{n}}\Big) + \frac{1}{2}\frac{ke^{-k^2/n}}{\sqrt{\pi n^3}}\right)(1 + O(n^{-1+\varepsilon})),$$

as well as

$$2^{-2n}\binom{2n}{n+k} = \left(\operatorname{erfc}\Big(\frac{k-1/2}{\sqrt{n}}\Big) - \operatorname{erfc}\Big(\frac{k}{\sqrt{n}}\Big) - \frac{1}{2}\frac{ke^{-k^2/n}}{\sqrt{\pi n^3}}\right)(1 + O(n^{-1+\varepsilon})).$$

Sketch of the proof. There are three central concepts to this proof: first of all, the asymptotic equality

$$2^{-2n}\binom{2n}{n+k} \sim 2^{-2n}\binom{2n}{n} e^{-k^2/n}$$

holds if $k = O(n^{1/2+\varepsilon})$. Furthermore, from Example 2.2.4 we already know the second central concept, namely the asymptotic equality

$$2^{-2n}\binom{2n}{n} \sim \frac{1}{\sqrt{n\pi}}.$$

It can be shown that these two results can be combined in the sense that we have

$$2^{-2n}\binom{2n}{n+k} = \frac{e^{-k^2/n}}{\sqrt{n\pi}}(1 + O(n^{-1+\varepsilon}))$$

for $k = O(n^{1/2+\varepsilon})$. Finally, the third central idea of this proof is to replace the summation over $a \le k \le b$ by integration over $(a - 1/2, b + 1/2)$ (i.e. with continuity correction). Carefully controlling the error made by this approximation then proves the statement of the proposition above. □

Note that in Section 2.4 we will present another approach for estimating the shifted central binomial coefficients that does not require the complement of the error function.

In any case, with these estimates we are able to give a first approximation of $\mathbb{E}\delta^{(2n)}_{\max}$. In order to do so, we need to define the following two arithmetic functions:

$$\sigma(m) := \sum_{\substack{m=(4\lambda+1)(h+1),\\ \lambda,h\geq 0}} 1, \qquad \tau(m) := \sum_{\substack{m=(4\lambda+3)(h+1),\\ \lambda,h\geq 0}} 1.$$

Proposition 2.2.14 ([38, Theorem 2.10]).
The expected maximal deviation of simple lattice path with length $2n$ can be expressed as

$$\mathbb{E}\delta^{(2n)}_{\max} = \Bigg(2\sum_{m\geq 1}\Big[(\sigma(m)-\tau(m))\operatorname{erfc}\Big(\frac{m}{2\sqrt{n}}\Big)\Big] \\ + \frac{1}{\sqrt{n^3\pi}}\sum_{m\geq 1}\Big[(\sigma(m)-\tau(m))me^{-m^2/n}\Big]\Bigg)(1+O(n^{-1+\varepsilon})).$$

Proof. We follow the approach from [38, Proof of Theorem 2.10]. Starting from the representation proved in Theorem 2.2.12, splitting the sums according to the parity of h yields

$$2^{-2n+2}\sum_{\substack{h,\lambda\geq 0\\ h \text{ even}}}\sum_{0\leq \ell\leq h}\binom{2n}{n+\frac{h+2+4\lambda(h+1)+2\ell}{2}}$$

for even h and

$$2^{-2n+1}\sum_{\substack{h,\lambda\geq 0\\ h\text{ odd}}}\Bigg(\sum_{0\leq\ell\leq h-1}2\binom{2n}{n+\frac{h+3+4\lambda(h+1)+2\ell}{2}}+\binom{2n}{n+\frac{h+1+4\lambda(h+1)}{2}}+\binom{2n}{n+\frac{3h+3+4\lambda(h+1)}{2}}\Bigg)$$

for odd h. The binomial coefficients have already been rewritten such that the estimates from Proposition 2.2.13 can be applied. Doing so and estimating the sums over the ranges $0\leq\ell\leq h$ and $0\leq\ell\leq h-1$ by the result from (a), and the two separate binomial coefficients by the two estimates from (b), respectively, gives

$$\mathbb{E}\delta^{(n)}_{\max} = 2\sum_{h,\lambda\geq 0}\Big[\operatorname{erfc}\Big(\frac{(4\lambda+1)(h+1)}{2\sqrt{n}}\Big)-\operatorname{erfc}\Big(\frac{(4\lambda+3)(h+1)}{2\sqrt{n}}\Big)\Big](1+O(n^{-1+\varepsilon})) \\ + \frac{1}{\sqrt{n^3\pi}}\sum_{k,\lambda\geq 0}\Big[(4\lambda+1)(k+1)\exp\Big(\frac{-[(4\lambda+1)(k+1)]^2}{n}\Big) \\ -(4\lambda+3)(k+1)\exp\Big(\frac{-[(4\lambda+3)(k+1)]^2}{n}\Big)\Big](1+O(n^{-1+\varepsilon})).$$

Note that the second sum comes from the estimates from (b), and as this sum is only over all odd h, we may substitute and sum over $k=2h+1$ instead. Furthermore, by tails pruning and tails completion, the error made when expanding the range of summation to infinity is exponentially small. Finally, the statement of the theorem follows by definition of $\sigma(m)$ and $\tau(m)$. $\square$

From the asymptotic expansion in Proposition 2.2.14, we can obtain the main contribution by computing the Mellin transform of $\operatorname{erfc}(x)$ and applying *Mellin inversion*, we find (like described in [38, (2.21)])

$$\operatorname{erfc}(x) = \frac{1}{2\pi i}\frac{1}{\sqrt{\pi}}\int_{c-i\infty}^{c+i\infty}\frac{1}{z}\Gamma\Big(\frac{z+1}{2}\Big)x^{-z}\,dz, \qquad \text{for } c,x>0.$$

Using this substitution in the expressions above, bringing everything under the integral, and then shifting the line of integration to the left (while adding back the residues of the poles we pass) then gives a nice asymptotic expansion.

Proposition 2.2.15 ([38, Theorem 2.12]).
The asymptotic contribution of the first sum in the representation from Proposition 2.2.13 is

$$2\sum_{m\ge1}\Big[(\sigma(m)-\tau(m))\operatorname{erfc}\Big(\frac{m}{2\sqrt{n}}\Big)\Big] = \sqrt{n\pi}-\frac{1}{2}+O(n^{-r}) \tag{2.9}$$

for all $r>0$.

Before we sketch the proof for this statement, we observe the following: by definition of $\sigma(m)$ and $\tau(m)$ and some simple rewriting, we find

$$\sum_{m\ge1}\frac{\sigma(m)-\tau(m)}{m^s} = \zeta(s)\beta(s),$$

where $\beta(s)$ is the Dirichlet beta function.

Remark.
The Dirichlet beta function is also often called Catalan beta function, and it is defined by

$$\beta(s) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^s}.$$

It can be expressed in terms of the Hurwitz zeta function as $\beta(s) = 4^{-s}(\zeta(s,1/4)-\zeta(s,3/4))$. Amongst many other interesting properties, it satisfies the zeta-like functional equation (cf. [36, Table 3.7.1])

$$\beta(1-s) = (\pi/2)^{-s}\sin(\pi s/2)\Gamma(s)\beta(s),$$

which also implies that $\beta(s)$ has zeros at all negative odd integers.

Proof of Proposition 2.2.15. We follow the approach from [38, Proof of Theorem 2.12]. The left-hand side of (2.9) can be rewritten as

$$\sum_{m\ge1}(\sigma(m)-\tau(m))\frac{1}{2\pi i}\frac{2}{\sqrt{\pi}}\int_{c-i\infty}^{c+i\infty}\frac{1}{z}\Gamma\Big(\frac{z+1}{2}\Big)\Big(\frac{m}{2\sqrt{n}}\Big)^{-z}\,dz.$$

As sketched in [38, Proof of Theorem 2.12], summation and integration may be interchanged (assuming $c > 1$) such that we obtain

$$\frac{1}{2\pi i}\frac{2}{\sqrt{\pi}}\int_{c-i\infty}^{c+i\infty}\frac{1}{z}(2\sqrt{n})^{z}\zeta(z)\beta(z)\Gamma\Big(\frac{z+1}{2}\Big)\,dz.$$

The integrand has only two simple poles for $z = 1$ and $z = 0$ as the poles of the gamma function cancel against the zeros of the beta function. Shifting the line of integration to the left thus produces a contribution of $\sqrt{n\pi}$ (from the pole at $z = 1$) and $-1/2$ (from the pole at 0). Computing these residues requires the values $\beta(1) = \pi/4$ and $\beta(0) = 1/2$, which can be taken from, for example, [35, Table 3.7.1]. As the integrand has no other poles and as all of the occurring functions behave sufficiently well, the line of integration may be shifted arbitrarily far to the left (which also yields the error term in (2.9)). Overall, the statement follows. □

Remark.
By using a similar technique, we can prove that the other sum contributes the following:

$$\sum_{m\geq 1}(\sigma(m)-\tau(m))me^{-m^2/n} = \frac{n\pi}{8} + O(n^{-r})$$

for all $r > 0$. For more details, see [38, Theorem 2.13].

Combining the results from this remark and Proposition 2.2.15, we find the following expansion:

Theorem 2.2.16 (Expected maximal deviation, asymptotics; [38, Theorem 2.14]).
Let $\varepsilon > 0$ and $n \in \mathbb{N}_0$. Assuming that all simple lattice paths of length $2n$ are equally likely, the expected maximal deviation for simple lattice paths is given asymptotically by

$$\mathbb{E}\delta_{\max}^{(2n)} = \sqrt{n\pi} - \frac{1}{2} + O(n^{-1/2+\varepsilon}).$$

This concludes our section on unrestricted lattice paths and bridges. Nevertheless, note that there is a plethora of other interesting parameters of lattice paths (like returns to 0, number of peaks, other notions of height like the *maximal span*, or even the area between a lattice path/bridge and the positive axis, ...).

2.3 Meanders and Excursions

Now, let us turn to the investigation of special restricted lattice paths: meanders and excursions. Recall that a meander is a non-negative walk and an excursion is a non-negative bridge. In particular, this means that depending on the current position of a meander, not all steps in the set of allowed steps $\mathcal{S}$ might be admissible—which, in turn, complicates the process of finding a suitable representation of the generating function that can be used for analyzing meanders and excursions.

Fortunately, the problems introduced by this restriction can be solved by means of the *kernel method*, which basically makes use of the small branches $u_j(z)$ of the characteristic curve that we encountered before in order to express the generating function of meanders in a similar way as the generating function of unrestricted paths from Lemma 2.2.2.

Let us denote the bivariate generating function of meanders as

$$M(z,u) := \sum_{n,k\geq 0} m_{n,k} z^n u^k = \sum_{n\geq 0} \mu_n(u) z^n = \sum_{k\geq 0} M_k(z) u^k,$$

where z and u correspond to the length and the final height of a meander, respectively. In particular, $m_{n,k}$ denotes the number of meanders of length n that end in k. The function $\mu_n(u)$ holds information on the possible ending heights after n steps, and $M_k(z)$ encodes the number of lattice paths that end in k. As the number of meanders and excursions is dominated by the number of unrestricted paths and bridges, $M(z,u)$ has to be analytic for $|u| < 1$ and $|z| < 1/S(1)$, where $S(u)$ is the characteristic Laurent polynomial of the set of allowed steps $\mathcal{S}$.

For the given problem, a recursive approach arises naturally: meanders of length n can be extended to meanders of length $n+1$ as long as only those steps are chosen that keep the walk above the x-axis. With the quantities we just defined, this can be expressed as[1]

$$\mu_0(u) = 1, \qquad \mu_{n+1}(u) = S(u)\mu_n(u) - \{u^{<0}\}S(u)\mu_n(u).$$

This is because multiplying $\mu_n(u)$ with $S(u)$ yields the Laurent polynomial encoding all possible heights that can be reached by taking an arbitrary step in $\mathcal{S}$ from a meander of length n. In order to ensure that we obtain only meanders, the part of $\mu_n(u)S(u)$ that corresponds to “non-meanders” has to be subtracted.

Multiplying this equation with z^n and summing over n then gives

$$\frac{M(z,u)-1}{z} = S(u)M(z,u) - \{u^{<0}\}S(u)M(z,u).$$

By simple manipulation we obtain the fundamental functional equation for meanders. It is given by

$$M(z,u) = 1 + zS(u)M(z,u) - z\{u^{<0}\}S(u)M(z,u). \tag{2.10}$$

Note that because $\mathcal{S}$ is a finite set, $S(u)$ also only involves finitely many negative powers. Especially, we have

$$\{u^{<0}\}S(u)M(z,u) = \{u^{<0}\}S(u)\sum_{k=0}^{c-1} M_k(z)u^k = \sum_{k=0}^{c-1} M_k(z)\cdot\{u^{<0}\}S(u)u^k.$$

[1] For a Laurent polynomial $f(u)$, the notation $\{u^{<0}\}f(u)$ denotes the principal part of u, i.e. the summands of $f(u)$ that contain a negative power of u.

By defining $r_k(u) := \{u^{<0}\}S(u)u^k$, we can rewrite (2.10) in the following way:

$$M(z,u)(1-zS(u)) = 1 - z\sum_{k=0}^{c-1} M_k(z)r_k(u). \tag{2.11}$$

This is the point where we apply the kernel method: observe that (2.11) currently contains $c+1$ unknown functions, which makes it impossible to tackle the problem directly. However, remember that the small branches $u_j(z)$ are solutions of the kernel equation $1-zS(u)=0$ near $z=0$. Therefore, substituting any of the small branches $u_1(z)$, …, $u_c(z)$ into (2.11) lets the left-hand side vanish and produces an equation in the c unknown functions $M_0(z)$, …, $M_{c-1}(z)$.

However, the kernel method is even more powerful than that: we investigate the equations that result from substituting the small branches into (2.11) in a small neighborhood of the origin, such that the following conditions hold:

(a) The neighborhood is sufficiently small such that $|z| < 1/S(1)$ holds,

(b) All the small branches are distinct and satisfy $|u_j(z)| < 1$.

Observe that as the large branches tend towards infinity for $z \to 0$, (b) can only hold for small branches in a suitable neighborhood of the origin. Then, by multiplying with $u_j(z)^c$ in order to clear the denominators of the $r_k(u)$, we obtain the following system of c equations:

$$u_j(z)^c - z\sum_{k=0}^{c-1} u_j(z)^c r_k(u_j(z))M_k(z) = 0, \qquad j \in \{1,2,\dots,c\}.$$

Following the argumentation in [3], the determinant of this system is a Vandermonde-type determinant—and thus not 0. Nevertheless, in order to compute the unknown functions $M_k(z)$, we follow another strategy. In [7, Remark after the proof of Theorem 13], the authors remark that for z in the small neighborhood of the origin from before, the quantity

$$N(z,u) := u^c - z\sum_{k=0}^{c-1} u^c r_k(u)M_k(z)$$

is a monic polynomial in u with $u_1(z)$, …, $u_c(z)$ as roots. Therefore, it factors as

$$N(z,u) = \prod_{j=1}^{c}(u-u_j(z)). \tag{2.12}$$

Altogether, these considerations are the core of the proof of the following theorem.

Theorem 2.3.1 (Meanders and excursions, [3, Theorem 2]).
For a finite allowed set of steps $\mathcal{S} \subseteq \mathbb{Z}$, the bivariate generating function of meanders (where z marks the length of the path and u marks the final height) is algebraic and has the shape

$$M(z,u) = \frac{\prod_{j=1}^{c}(u-u_j(z))}{u^c(1-zS(u))} = -\frac{1}{z}\prod_{j=1}^{d}\frac{1}{u-v_j(z)}, \tag{2.13}$$

where $u_j(z)$ and $v_j(z)$ denote the small and large branches of the characteristic curve of $\mathcal{S}$, respectively. In particular, the generating function of excursions, $E(z) = M(z,0)$, is given by

$$E(z) = \frac{(-1)^{c-1}}{z} \prod_{j=1}^{c} u_j(z) = \frac{(-1)^{d-1}}{z} \prod_{j=1}^{d} \frac{1}{v_j(z)}. \tag{2.14}$$

Proof. By (2.11) and the factorization (2.12), we directly obtain

$$M(z,u)u^c(1-zS(u)) = u^c - z\sum_{k=0}^{c-1} u^c r_k(u) M_k(z) = \prod_{j=1}^{c}(u-u_j(z)).$$

With a similar argument for $1-zS(u)$ as for $N(z,u)$ above, it is easy to see that the kernel factors in the following way:

$$u^c(1-zS(u)) = -z \cdot \prod_{j=1}^{c}(u-u_j(z)) \cdot \prod_{j=1}^{d}(u-v_j(z)).$$

Combining these results yields (2.13). Finally, (2.14) can be obtained simply by using $E(z) = M(z,0)$. □

Before discussing an example for generating functions of meanders and excursions, we remark how the asymptotic behavior of the number of excursions looks like. Similarly to Theorem 2.2.6, the following result is based on the *saddle-point method*—and again, we do not discuss the proof in detail. Recall from Definition 2.2.5 that the *structural constant* of a set of allowed steps $\mathcal{S} \subseteq \mathbb{Z}$ is the unique positive solution to the equation $S'(\tau) = 0$, and that the *period* of the class of paths induced by $\mathcal{S}$ is the largest integer p such that the Laurent polynomial $S(u)$ can be written as $S(u) = u^b T(u^p)$ for some $b \in \mathbb{Z}$ and some other Laurent polynomial T.

Furthermore, note that in [3, Section 3] it is shown that among the small branches, there is exactly one analytic positive branch (u_1, for example) that dominates all other branches in modulus in the sense of

$$|u_j(z)| < u_1(|z|) \quad \text{for } j \in \{2,3,\dots,c\} \text{ and } 0 < |z| \leq 1/S(\tau).$$

We call this special dominating small branch the *principal branch*. As we will see in the following theorem, the asymptotic behavior of excursions depends on the non-principal branches.

Theorem 2.3.2 (Asymptotic analysis of excursions, [3, Theorem 3]).
Let $\mathcal{S} \subseteq \mathbb{Z}$ be a finite set of allowed steps with structural constant $\tau > 0$ that induces a reduced system of paths with period p. Then the number of excursions admits the following asymptotic behavior:

$$[z^{np}]E(z) \sim p(-1)^{c-1}\sqrt{\frac{2S(\tau)^3}{S''(\tau)}}\, Y_1\Big(\frac{1}{S(\tau)}\Big)\frac{S(\tau)^{np}}{2\sqrt{\pi n^3 p^3}},$$

where the function $Y_1(z)$ is defined as the product of all non-principal branches, $Y_1(z) := u_2(z)\cdots u_c(z)$, and $[z^n]E(z) = 0$ for $n \nmid p$.

In other words, the number of excursions of length np is of order $\Theta(S(\tau)^{np}/\sqrt{n^3})$.

To conclude this section, let us illustrate the results from Theorem 2.3.1 and Theorem 2.3.2 in an example.

Example 2.3.3 (Excursions and Catalan paths).
For the sake of simplicity, let us revisit the example from the introduction of this thesis: simple zero-sum games. We already know from Example 2.2.4 that simple zero-sum game series can be interpreted as bridges relative to $\mathcal{S} = \{-1, 1\}$. In the context of these games, meanders and excursions also have interesting interpretations: meanders correspond to a series of games where after each new round we still have won at least as many games as we have lost. And naturally, excursions correspond to zero-sum game series of this kind. In fact, non-negative simple lattice paths that end in 0 are called *Catalan paths*, or *Dyck paths*.

Recall that $S(u) = u^{-1} + u = u^{-1}(1 + u^2)$. Thus, the period of $\mathcal{S}$ is 2, and it is easy to show that the structural constant is $\tau = 1$.

Solving the kernel equation $1 - zS(u) = 0$ yields the small and large branches

$$u_1(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z}, \quad \text{and} \quad v_1(z) = \frac{1 + \sqrt{1 - 4z^2}}{2z},$$

respectively. Thus, by (2.13), the generating function of meanders relative to $\mathcal{S} = \{-1, 1\}$ is given by

$$M(z, u) = \frac{2zu - 1 + \sqrt{1 - 4z^2}}{2zu - 2z^2 - 2z^2u^2},$$

and by (2.14), the corresponding generating function of excursions is

$$E(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2}.$$

Because of the simple form of $E(z)$, we can analyze the asymptotic behavior of the corresponding counting sequence both by means of singularity analysis as well as through Theorem 2.3.2.

When studying $E(z)$ in the context of singularity analysis ($\alpha = -1/2$), we obtain

$$\begin{aligned}
[z^{2n}]E(z) = [z^{2n}]\frac{1 - \sqrt{1 - 4z^2}}{2z^2} &= \frac{1}{2}[z^{2n+2}](1 - \sqrt{1 - 4z^2}) = \frac{-1}{2}[z^{2n+2}](1 - 4z^2)^{1/2} \\
&= \frac{-1}{2}[t^{n+1}](1 - 4t)^{1/2} = \frac{-1}{2}4^{n+1}[t^{n+1}](1 - t)^{1/2} \\
&\sim 4^{n+1}\frac{n^{-3/2}}{-2\Gamma(-1/2)} = \frac{4^n}{\sqrt{\pi n^3}}.
\end{aligned}$$

On the other hand, consider the statement of Theorem 2.3.2. As we only have one small branch ($c = 1$), the function $Y_1(u)$ is constant with value 1. Furthermore, we already know that $p = 2$ and $\tau = 1$. And due to $S''(u) = 2u^{-3}$, we find

$$[z^{2n}]E(z) \sim 2\sqrt{\frac{2S(1)^3}{S''(1)}}\frac{S(1)^{2n}}{2\sqrt{\pi n^3 2^3}} = \frac{4^n}{\sqrt{\pi n^3}},$$

which is the same result as from singular analysis.

Alternatively, we can expand the function $E(z)$ by means of the binomial series, such that we obtain

$$\begin{aligned}
[z^{2n}]E(z) &= [z^{2n}]\frac{1-\sqrt{1-4z^2}}{2z^2} = \frac{1}{2}[z^{2n+2}](1-(1-4z^2)^{1/2}) \\
&= \frac{1}{2}[z^{2n+2}]\left(1-\sum_{k\geq 0}(-1)^k\binom{1/2}{k}(4z^2)^k\right) = \frac{4^{n+1}}{2}(-1)^n\binom{1/2}{n+1} \\
&= \frac{4^{n+1}}{2}(-1)^n\frac{(1/2)(-1/2)(-3/2)\cdots(1/2-n)}{(n+1)!} = \frac{(2n)!}{(n+1)!\,n!} \\
&= \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

The integers generated by this sequence appear very often within enumerative combinatorics: the *Catalan numbers*, $\frac{1}{n+1}\binom{2n}{n} = C_n$. They arise in a vast number of different problems like, for example, triangulation of convex polygons, counting the number of "balanced parenthesis"-expressions, counting special classes of trees[2], and many more. Catalan numbers are enumerated by sequence A000108 in [34], and by the analysis above, asymptotically we have $C_n \sim \frac{4^n}{\sqrt{\pi n^3}}$. For an extensive collection of problems involving Catalan numbers see [43]. △

2.4 Culminating paths

In this section[3] we will concentrate on the asymptotic analysis of the culminating paths and extremal lattice paths from Definition 2.1.2. Note that culminating paths have also been studied in [8]—however, for the sake of consistency, we stick to the terminology introduced in [19] and refer to culminating paths as *admissible lattice paths*.

All investigations within this chapter strongly depend on the interpretation of lattice paths as realizations of random walks. In particular, we will see that extremal lattice paths can be analyzed as a special case of admissible lattice paths by a simple reflection argument. With this in mind, the following definition of the stochastic pendant of admissible lattice paths will prove very useful.

[2] This is discussed in detail in Section 3.2.

[3] Note that this is an adapted version of a paper written in cooperation with Clemens Heuberger, Helmut Prodinger, and Stephan Wagner, see [19].

Definition 2.4.1 (Admissible random walks).
Let $(S_k)_{0\le k\le n}$ be a simple symmetric random walk on $\mathbb{N}_0$ or $\mathbb{Z}$ of length n starting at 0. That is, we have $\mathbb{P}(S_0 = 0) = 1$ as well as

$$\mathbb{P}(S_k = j-1 \mid S_{k-1} = j) = \mathbb{P}(S_k = j+1 \mid S_{k-1} = j) = \frac{1}{2} \quad \text{for } j \ge 1,$$
$$\mathbb{P}(S_k = 1 \mid S_{k-1} = 0) = 1,$$

for random walks defined on $\mathbb{N}_0$, and

$$\mathbb{P}(S_k = j-1 \mid S_{k-1} = j) = \mathbb{P}(S_k = j+1 \mid S_{k-1} = j) = \frac{1}{2} \quad \text{for } j \in \mathbb{Z}$$

for random walks on $\mathbb{Z}$. Then $(S_k)_{0\le k\le n}$ is said to be *admissible* of height h if the random walk stays within the interval $[0,h]$ and ends in h, i.e. $S_k \in [0,h]$ for all k with $0 \le k \le n$ and $\mathbb{P}(S_n = h) = 1$. It is called *admissible* if the random walk is admissible of any height $h \in \mathbb{N}$.

Note that based on this definition, the admissible lattice paths encountered above are exactly the realizations of admissible random walks.

In order to simplify our investigations, we let $p_n^{(h)}$ and $q_n^{(h)}$ denote the probability that a simple symmetric random walk on $\mathbb{N}_0$ and $\mathbb{Z}$ is admissible of height h, respectively. Furthermore, we denote the probability that a random walk over $\mathbb{N}_0$ and $\mathbb{Z}$ is admissible at all with $p_n := \sum_{h\ge 0} p_n^{(h)}$ and $q_n := \sum_{h\ge 0} q_n^{(h)}$.

Remember that an admissible lattice path is a non-negative lattice path ending in its maximum. This special class is also visualized in Figure 2.4, where all admissible lattice paths of length 5 are depicted. There are three admissible lattice paths of height 3, and one of height 1 and 5, respectively. Note that when considering simple random walks on $\mathbb{Z}$, every possible lattice path has the same probability 2^{-n}. Admissible random walks on $\mathbb{Z}$ are enumerated by sequence A167510 in [34].

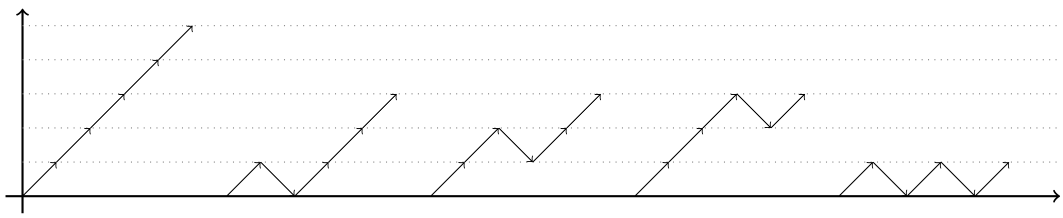

Figure 2.4: Admissible lattice paths of length 5

However, in the case of random walks on $\mathbb{N}_0$, the probability depends on the number of visits to 0: if there are v such visits (including the initial state), then the path occurs with probability 2^{-n+v}. Note that by "folding down" (i.e., reflecting about the x-axis) some sections between consecutive visits to 0, or the section between the last visit and the end, 2^ν lattice paths on $\mathbb{Z}$ can be formed, where the random walk is never farther away from the start than at the end. These are exactly *extremal lattice paths*—and by construction, the number of extremal lattice paths of length n is given by $p_n 2^n$. To illustrate this idea of extremal lattice paths, all paths of this form of length 3 are given in Figure 2.5.

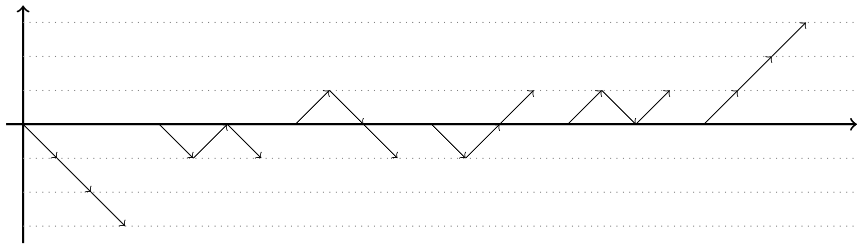

Figure 2.5: Extremal lattice paths of length 3

One of our motivations for investigating admissible random walks originates from a conjecture in [47]. There, Zhao introduced the notion of a *bidirectional ballot sequence*:

Definition 2.4.2 ([47, Definition 3.1]).
A 0-1 sequence is called a *bidirectional ballot sequence* if every prefix and suffix contains strictly more 1's than 0's. The number of bidirectional ballot sequences of length n is denoted by B_n.

Bidirectional ballot sequences are strongly related to admissible random walks on $\mathbb{Z}$. In fact, every bidirectional ballot sequence of length $n+2$ bijectively corresponds to an admissible lattice path of length n on $\mathbb{Z}$: given an admissible lattice path, every up-step corresponds to a 1, and down-steps correspond to 0. Adding a 1 both at the beginning and at the end of the constructed string gives a bidirectional ballot sequence of length $n+2$.

Therefore, bidirectional ballot walks may also be seen as lattice paths with unique minimum and maximum.

In [47], Zhao also shows that $B_n = \Theta(2^n/n)$, states (without detailed proof) that $B_n \sim 2^n/(4n)$ and conjectures that

$$\frac{B_n}{2^n} = \frac{1}{4n} + \frac{1}{6n^2} + O\Big(\frac{1}{n^3}\Big).$$

In this section, we want to give a detailed analysis of the asymptotic behavior of admissible random walks. By exploiting the bijection between admissible random walks and bidirectional ballot sequences, we also prove a stronger version of Zhao's conjecture.

In order to do so, we use a connection between Chebyshev polynomials and the probabilities $p_n^{(h)}$ and $q_n^{(h)}$ (cf. Proposition 2.4.3 and Proposition 2.4.4, respectively), which we explore in detail in Section 2.4.1. This allows us to determine explicit representations of the probabilities p_n and q_n, which are given in Theorem 2.4.5. The analysis of the asymptotic behavior of admissible random walks of given length shall focus in particular on the height of these random walks. In this context, we define random variables H_n and $\widetilde{H}_n$ by

$$\mathbb{P}(H_n = h) := \frac{p_n^{(h)}}{p_n}, \qquad \mathbb{P}(\widetilde{H}_n = h) := \frac{q_n^{(h)}}{q_n}.$$

These random variables model the height of admissible random walks on $\mathbb{N}_0$ and $\mathbb{Z}$, respectively. Besides an asymptotic expansion for p_n and q_n, we are also interested in the behavior of the expected height and its variance. The asymptotic analysis of these expressions, which is based on an approach featuring the Mellin transform, is carried out in Section 2.4.2 and Section 2.4.3, and the results are given in Theorem 2.4.8 and Theorem 2.4.10, respectively. Finally, Zhao's conjecture is proved in Corollary 2.4.11.

2.4.1 Chebyshev Polynomials and Random Walks

We denote the Chebyshev polynomials of the first and second kind by T_h and U_h, respectively, i.e.,

$$\begin{aligned}
T_{h+1}(x) &= 2xT_h(x) - T_{h-1}(x) &&\text{for } h \geq 1, && T_0(x) = 1, && T_1(x) = x,\\
U_{h+1}(x) &= 2xU_h(x) - U_{h-1}(x) &&\text{for } h \geq 1, && U_0(x) = 1, && U_1(x) = 2x.
\end{aligned}$$

In the following propositions, we show that these polynomials occur when analyzing admissible random walks.

Proposition 2.4.3 ([23]).
The probability that a simple symmetric random walk $(S_k)_{0\leq k\leq n}$ of length n on $\mathbb{Z}$ is admissible of height h is

$$q_n^{(h)} = \mathbb{P}(0 \leq S_0, S_1, \dots, S_n \leq h \text{ and } S_n = h) = 2[z^{n+1}]\frac{1}{U_{h+1}(1/z)} \tag{2.15}$$

for $h \geq 0$ and $n \geq 0$.

Proof. We follow [23] and consider the $(h+1)\times(h+1)$ transfer matrix

$$M_h = \begin{pmatrix}
0 & \frac{1}{2} & 0 & \cdots & \cdots & 0\\
\frac{1}{2} & 0 & \frac{1}{2} & \cdots & \cdots & 0\\
0 & \frac{1}{2} & 0 & \ddots & & 0\\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & & \ddots & \ddots & \frac{1}{2}\\
0 & 0 & 0 & \dots & \frac{1}{2} & 0
\end{pmatrix},$$

which has the following simple yet useful property: if $w_{n,k}^{(h)}$ is the probability that $0 \le S_0, S_1, \dots, S_n \le h$ and $S_n = k$, then the recursion

$$w_n^{(h)} \cdot M_h = w_{n+1}^{(h)},$$

holds for the vectors $w_n^{(h)} = (w_{n,0}^{(h)}, w_{n,1}^{(h)}, \dots, w_{n,h}^{(h)})$. In particular, we have $w_n^{(h)} = w_0^{(h)} \cdot M_h^n$. The initial vector is $w_0^{(h)} = e_0 = (1, 0, \dots, 0)$. Since we also want that $S_n = h$, we multiply by the vector $e_h = (0, \dots, 0, 1)^\top$ at the end to extract only the last entry $w_{n,h}^{(h)}$. This yields the generating function

$$\sum_{n \ge 0} q_n^{(h)} z^n = \sum_{n \ge 0} e_0 M_h^n e_h z^n = e_0 (I - zM_h)^{-1} e_h.$$

Cramer's rule yields

$$\sum_{n \ge 0} q_n^{(h)} z^n = \frac{z^h 2^{-h}}{\det(I - zM_h)}.$$

The determinant of $I - zM_h$ can be computed recursively in h by means of row expansion, see (for instance) [1, p.97]:

$$\det(I - zM_{h+2}) = \det(I - zM_{h+1}) - \frac{z^2}{4} \det(I - zM_h).$$

Comparing this with the recursion for the Chebyshev polynomials and checking the initial values, we find that $\frac{2^{h+1} \det(I - zM_h)}{z^{h+1}} = U_{h+1}(1/z)$. Therefore, we obtain

$$\sum_{n \ge 0} q_n^{(h)} z^n = \frac{2}{zU_{h+1}(1/z)}$$

from which (2.15) follows by extracting the coefficient of z^n. □

An analogous statement holds for admissible random walks on $\mathbb{N}_0$ with the sole difference that in this case, the Chebyshev polynomials of the first kind occur.

Proposition 2.4.4 ([23]).
The probability that a random walk $(S_k)_{0 \le k \le n}$ of length n on $\mathbb{N}_0$ is admissible of height h is given by

$$p_n^{(h)} = \mathbb{P}(0 \le S_0, S_1, \dots, S_n \le h \text{ and } S_n = h) = 2[z^{n+1}] \frac{1}{T_{h+1}(1/z)} \tag{2.16}$$

for $h \ge 0$ and $n \ge 1$.

Proof. Following the approach discussed in [23], we observe that for random walks with a reflective barrier at 0, the $(h+1)\times(h+1)$ transfer matrix has the form

$$\widetilde{M}_h = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \frac{1}{2} \\ 0 & 0 & 0 & \dots & \frac{1}{2} & 0 \end{pmatrix}.$$

By the same approach involving Cramer's rule as in the proof of Proposition 2.4.3, we find the generating function

$$\sum_{n\geq 0} p_n^{(h)} z^n = e_0(I - z\widetilde{M}_h)^{-1} e_h = \frac{z^h 2^{1-h}}{\det(I - z\widetilde{M}_h)},$$

where we have the recursion

$$\det(I - z\widetilde{M}_{h+2}) = \det(I - z\widetilde{M}_{h+1}) - \frac{z^2}{4}\det(I - z\widetilde{M}_h)$$

for the determinant of $I - z\widetilde{M}_h$. Finally, (2.16) follows from $\frac{2^{h-1}\det(I - z\widetilde{M}_{h-1})}{z^h} = T_h(1/z)$, which can be proved again by verifying that the same recursion holds for the Chebyshev-T polynomials and that the initial values agree. □

A rather interesting fact is that this result can also be obtained by a completely different stochastic approach, namely by means of martingale theory:

Alternate martingale proof [46]. Note that all the probability-theoretic concepts necessary for this approach are given in Section 1.4.2.

Let $(S_k)_{k\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$. We have already seen in Example 1.4.15 that every symmetric random walk on $\mathbb{Z}$ is a martingale itself. Based on this random walk we define the stochastic process $(M_k)_{k\geq 0}$ with $M_k := e^{itS_k}(\cos t)^{-k}$, where t with $|t| < \frac{\pi}{2}$ is a parameter. We want to show that $(M_k)_{k\geq 0}$ is a martingale as well.

Recall that in order to do so, we have to show

$$\mathbb{E}|M_k| < \infty \quad \text{as well as} \quad \mathbb{E}[M_{k+1}|M_0, M_1, \dots, M_k] = M_k$$

for all $k \geq 0$. The boundedness of the expected value follows immediately from the definition of M_k and because S_k is a real-valued random variable. In particular, we have

$$\mathbb{E}|M_k| = \underbrace{\mathbb{E}(|e^{itS_k}|)}_{=1} \cdot |\cos t|^{-k} = |\cos t|^{-k} < \infty,$$

as $|t| < \frac{\pi}{2}$. For the second martingale property we make use of the fact that we can write the random variable S_k as a sum of identical and independently distributed random variables $(X_k)_{k\geq 0}$ with $\mathbb{P}(X_k = -1) = \mathbb{P}(X_k = 1) = \frac{1}{2}$, that is $S_k = \sum_{j=1}^{k} X_j$. Then, we may write

$$\begin{aligned}\mathbb{E}[M_{k+1}|M_0, M_1, \ldots, M_k] &= \mathbb{E}[e^{itS_{k+1}}(\cos t)^{-(k+1)}|M_0, \ldots M_k] \\ &= (\cos t)^{-1}\mathbb{E}[e^{itX_{k+1}} \cdot \underbrace{e^{itS_k}(\cos t)^{-k}}_{=M_k}|M_0, \ldots, M_k].\end{aligned}$$

As M_k is certainly measurable with respect to the σ-algebra induced by $M_0, \ldots, M_k$, we may use statement (c) of Lemma 1.4.13.

$$\mathbb{E}[M_{k+1}|M_0, M_1, \ldots, M_k] = M_k \cdot (\cos t)^{-1}\mathbb{E}[e^{itX_{k+1}}|M_0, \ldots, M_k].$$

Furthermore, as $e^{itX_{k+1}}$ is independent of $M_0, \ldots, M_k$ by construction, the conditional expectation degenerates to the “standard” expected value, finally resulting in

$$\mathbb{E}[M_{k+1}|M_0, M_1, \ldots, M_k] = M_k \cdot (\cos t)^{-1}\mathbb{E}e^{itX_{k+1}} = M_k(\cos t)^{-1}\underbrace{\left(\frac{1}{2}e^{it} + \frac{1}{2}e^{-it}\right)}_{=\cos t} = M_k.$$

This proves that $(M_k)_{k\geq 0}$ actually is a martingale—and thus, we may apply the optional stopping theorem, Theorem 1.4.17 with the hitting time $\tau_h := \inf\{k \in \mathbb{N}_0 \mid |S_k| = h\}$. This yields

$$1 = \mathbb{E}M_0 = \mathbb{E}M_{\tau_h} = \frac{1}{2}\Big(e^{iht} + e^{-iht}\Big)\mathbb{E}((\cos t)^{-\tau_h}) = \cos(ht)\mathbb{E}((\cos t)^{-\tau_h}).$$

After substituting $t = \arccos z$, we find

$$1 = \cos(h \arccos z) \cdot \mathbb{E}z^{-\tau_h} \quad\iff\quad \mathbb{E}z^{-\tau_h} = \frac{1}{\cos(h \arccos z)} = \frac{1}{T_h(z)},$$

where the last equality holds due to [11, 3.11.6]. We can use this in order to get the probability generating function

$$\sum_{n\geq 0}\mathbb{P}(\tau_h = n)z^n = \mathbb{E}z^{\tau_h} = \frac{1}{T_h(1/z)}.$$

Note that due to the folding argument from the introduction of this section, $\mathbb{P}(\tau_h = n)$ is also the probability that a simple symmetric random walk on $\mathbb{N}_0$ reaches h for the first time after n steps. Finally, $p_n^{(h)} = [z^{n+1}]\frac{2}{T_{h+1}(1/z)}$ follows from the fact that every admissible random walk of length n and height h hits $h+1$ for the first time with probability $1/2$ after one more step. □

Remark.
The coefficients of $\frac{1}{T_h(1/z)}$ have also been studied in [25]. There, the case of fixed h is investigated, whereas we mostly focus on the asymptotic behavior of $\sum_{h\geq 0} p_n^{(h)}$ for $n \to \infty$.

Using the results from Proposition 2.4.3 and Proposition 2.4.4, we may give explicit representations of the probabilities $p_n^{(h)}$ and $q_n^{(h)}$ by investigating the Chebyshev polynomials thoroughly.

Remark (Iverson's notation).
We use the Iversonian notation

$$[\![expr]\!] = \begin{cases} 1 & \text{if } expr \text{ is true,} \\ 0 & \text{otherwise,} \end{cases}$$

popularized in [18, Chapter 2].

In the following theorem and throughout the rest of the section, m will denote a half-integer, i.e., $m \in \frac{1}{2}\mathbb{N} = \{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$. While this convention may seem unusual, it simplifies many of our formulas and is therefore convenient for calculations.

Theorem 2.4.5 ([39]).
With $\tau_{h,k} := (h+1)(2k+1)/2$ and $\upsilon_{h,k} := (h+2)(2k+1)/2$, we have

$$p_{2m-1}^{(h)} = \frac{4}{4^m} \sum_{k \geq 0} (-1)^k \frac{\tau_{h,k}}{m} \binom{2m}{m - \tau_{h,k}} \cdot [\![h + 1 \equiv 2m \bmod 2]\!], \tag{2.17}$$

$$q_{2m-2}^{(h)} = \frac{4}{4^m} \sum_{k \geq 0} \frac{2\upsilon_{h,k}^2 - m}{(2m-1)m} \binom{2m}{m - \upsilon_{h,k}} \cdot [\![h \equiv 2m \bmod 2]\!] \tag{2.18}$$

for $h \geq 0$ and half-integers $m \in \frac{1}{2}\mathbb{N}$ with $m \geq 1$.

Proof. The following approach is discussed in [23] and [39]. We begin with the analysis of $p_n^{(h)}$. The probabilities are related to the Chebyshev-T polynomials by Proposition 2.4.4. It is a well-known fact (cf. [36, 22:3:3]) that these polynomials have the explicit representation

$$T_h(x) = \frac{(x - \sqrt{x^2 - 1}\,)^h + (x + \sqrt{x^2 - 1}\,)^h}{2},$$

which immediately yields

$$\frac{1}{T_h(1/z)} = z^h \frac{2}{(1 - \sqrt{1 - z^2}\,)^h + (1 + \sqrt{1 - z^2}\,)^h} =: z^h Y(z^2). \tag{2.19}$$

By applying Cauchy's integral formula, we obtain the coefficients of the factor $Y(t)$ encountered in (2.19). We choose a sufficiently small circle around 0 as the integration contour γ. Thus, we get

$$\begin{aligned} [t^n]Y(t) &= [t^n] \frac{2}{(1 - \sqrt{1 - t}\,)^h + (1 + \sqrt{1 - t}\,)^h} \\ &= \frac{1}{2\pi i} \oint_\gamma \frac{2}{(1 - \sqrt{1 - t}\,)^h + (1 + \sqrt{1 - t}\,)^h} \cdot \frac{1}{t^{n+1}}\, dt. \end{aligned}$$

We want to simplify the expression $\sqrt{1-t}$ in this integral. This can be achieved by the substitution $t = \frac{4u}{(1+u)^2}$, which gives us $dt = (1-u)\cdot\frac{4}{(1+u)^3}\,du$ and $\sqrt{1-t} = \frac{1-u}{1+u}$. Also, the new integration contour is $\tilde{\gamma}$, which is still a contour that winds around the origin once. Then, again by Cauchy's integral formula, we obtain

$$\begin{aligned}
[t^n]Y(t) &= \frac{1}{2\pi i}\oint_{\tilde{\gamma}}(1-u)\frac{(1+u)^{2n+h-1}}{2^{2n+h-1}(1+u^h)}\cdot\frac{1}{u^{n+1}}\,du \\
&= [u^n](1-u)\frac{(1+u)^{2n+h-1}}{2^{2n+h-1}(1+u^h)}.
\end{aligned}$$

Expanding the factor $\frac{(1+u)^{2n+h-1}}{1+u^h}$ into a series with the help of the geometric series and the binomial theorem yields

$$\frac{(1+u)^{2n+h-1}}{1+u^h} = \sum_{k\ge 0}(-1)^k u^{kh}(1+u)^{2n+h-1} = \sum_{k\ge 0}(-1)^k u^{kh}\sum_{j=0}^{2n+h-1}\binom{2n+h-1}{j}u^j,$$

and therefore

$$[u^\ell]\frac{(1+u)^{2n+h-1}}{1+u^h} = \sum_{k\ge 0}(-1)^k\binom{2n+h-1}{\ell-hk}.$$

This allows us to expand the expression encountered before, that is

$$\begin{aligned}
[t^n]Y(t) &= [u^n](1-u)\frac{(1+u)^{2n+h-1}}{2^{2n+h-1}(1+u^h)} \\
&= \frac{1}{2^{2n+h-1}}\sum_{k\ge 0}(-1)^k\left[\binom{2n+h-1}{n-hk}-\binom{2n+h-1}{n-hk-1}\right].
\end{aligned}$$

Using the binomial identity

$$\binom{N-1}{\alpha}-\binom{N-1}{\alpha-1} = \frac{N-2\alpha}{N}\binom{N}{\alpha},$$

the expression above can be simplified so that, together with (2.19), we find

$$\frac{1}{T_h(1/z)} = 2\sum_{n\ge 0}\left(\frac{z}{2}\right)^{2n+h}\sum_{k\ge 0}(-1)^k\frac{2hk+h}{2n+h}\binom{2n+h}{n-hk}.$$

By plugging this into (2.16), we obtain

$$\begin{aligned}
p_n^{(h)} &= 2[z^{n+1}]\frac{1}{T_{h+1}(1/z)} \\
&= 4[z^{n+1}]\sum_{\ell\ge 0}\left(\frac{z}{2}\right)^{2\ell+h+1}\sum_{k\ge 0}(-1)^k\frac{2(h+1)k+h+1}{2\ell+h+1}\binom{2\ell+h+1}{\ell-(h+1)k} \\
&= \frac{1}{2^{h-1}}[z^{n-h}]\sum_{\ell\ge 0}\left(\frac{z}{2}\right)^{2\ell}\sum_{k\ge 0}(-1)^k\frac{2(h+1)k+h+1}{2\ell+h+1}\binom{2\ell+h+1}{\ell-(h+1)k}.
\end{aligned}$$

Combinatorially, it is clear that $p_n^{(h)} = 0$ for n and h of different parity, as only heights of the same parity as the length can be reached by a random walk starting at the origin. This can also be observed in the representation above. Assuming $n \equiv h \bmod 2$, we can write $n-h = 2\ell$ or equivalently $\frac{n-h}{2} = \ell$. This gives us

$$\begin{aligned} p_n^{(h)} &= \frac{1}{2^{n-1}} \sum_{k\geq 0} (-1)^k \frac{2(h+1)k+h+1}{n+1} \binom{n+1}{\frac{n-h}{2} - (h+1)k} \\ &= \frac{1}{2^{n-1}} \sum_{k\geq 0} (-1)^k \frac{(h+1)(2k+1)}{n+1} \binom{n+1}{\frac{n+1}{2} - \frac{1}{2}(h+1)(2k+1)}. \end{aligned}$$

Substituting $n = 2m-1$ with a half-integer $m \in \frac{1}{2}\mathbb{N}$ such that $h+1 \equiv 2m \bmod 2$, and recalling that $\tau_{h,k} = (h+1)(2k+1)/2$, the representation in (2.17) is proved.

For the second part, we consider the explicit representation

$$U_h(x) = \frac{(x+\sqrt{x^2-1}\,)^{h+1} - (x-\sqrt{x^2-1}\,)^{h+1}}{2\sqrt{x^2-1}}$$

of the Chebyshev-U polynomials, which is equivalent to

$$\frac{1}{U_h(1/z)} = z^h \frac{2\sqrt{1-z^2}}{(1+\sqrt{1-z^2}\,)^{h+1} - (1-\sqrt{1-z^2}\,)^{h+1}}. =: z^h \tilde{Y}(z^2).$$

Again, we investigate the factor $\tilde{Y}(t)$ based on Cauchy's integration formula and the substitution $t = \frac{4u}{(1+u)^2}$:

$$\begin{aligned} [t^n]\tilde{Y}(t) &= \frac{1}{2\pi i} \oint_\gamma \frac{2\sqrt{1-t}}{(1+\sqrt{1-t}\,)^{h+1} - (1-\sqrt{1-t}\,)^{h+1}} \cdot \frac{1}{t^{n+1}}\, dt \\ &= \frac{1}{2\pi i} \oint_{\tilde{\gamma}} (1-u)^2 \frac{2^{-(2n+h)}(1+u)^{2n+h-1}}{1-u^{h+1}} \frac{1}{u^{n+1}}\, du \\ &= [u^n](1-u)^2 \frac{2^{-(2n+h)}(1+u)^{2n+h-1}}{1-u^{h+1}}. \end{aligned}$$

By expanding $(1+u)^{2n+h-1}$ and $\frac{1}{1-u^{h+1}}$, we find

$$\begin{aligned} &[u^n](1-u)^2 \frac{2^{-(2n+h)}(1+u)^{2n+h-1}}{1-u^{h+1}} \\ &\quad = \frac{1}{2^{2n+h}} \sum_{k\geq 0} \left[\binom{2n+h-1}{n-k(h+1)} - 2\binom{2n+h-1}{n-k(h+1)-1} + \binom{2n+h-1}{n-k(h+1)-2} \right], \end{aligned}$$

and thus, because of $\frac{1}{U_h(1/z)} = z^h \tilde{Y}(z^2)$, we find

$$\begin{aligned} \frac{1}{U_h(1/z)} &= \sum_{n\geq 0} \left(\frac{z}{2}\right)^{2n+h} \\ &\quad \times \sum_{k\geq 0} \left[\binom{2n+h-1}{n-k(h+1)} - 2\binom{2n+h-1}{n-k(h+1)-1} + \binom{2n+h-1}{n-k(h+1)-2} \right]. \end{aligned} \tag{2.20}$$

When investigating $\frac{1}{U_{h+1}(1/z)}$, the binomial coefficients on the right-hand side of (2.20) have the form

$$\binom{2n+h}{n-k(h+2)} - 2\binom{2n+h}{n-k(h+2)-1} + \binom{2n+h}{n-k(h+2)-2}.$$

They can be simplified by using the binomial identity

$$\binom{2N}{N-\alpha+1} - 2\binom{2N}{N-\alpha} + \binom{2N}{N-\alpha-1} = \frac{4\alpha^2-(2N+2)}{(2N+1)(2N+2)}\binom{2N+2}{N+1-\alpha}$$

where $2N \in \mathbb{N}_0$ and $N-\alpha \in \mathbb{N}_0$. Thus, the expression above can be written as

$$\frac{(h+2)^2(2k+1)^2-(2n+h+2)}{(2n+h+1)(2n+h+2)}\binom{2n+h+2}{n+\frac{h}{2}+1-\frac{1}{2}(h+2)(2k+1)}$$

so that we obtain

$$\frac{1}{U_{h+1}(1/z)} = \sum_{n\geq 0}\Big(\frac{z}{2}\Big)^{2n+h+1}\sum_{k\geq 0}\frac{(h+2)^2(2k+1)^2-(2n+h+2)}{(2n+h+1)(2n+h+2)} \times \binom{2n+h+2}{n+\frac{h}{2}+1-\frac{1}{2}(h+2)(2k+1)}.$$

By the same combinatorial argument as before, $q_n^{(h)} = 0$ holds for n and h of different parities. The representation above is in accordance with this observation. Plugging this representation of $\frac{1}{U_{h+1}(1/z)}$ into (2.15) yields

$$q_n^{(h)} = \frac{1}{2^n}\sum_{k\geq 0}\frac{(h+2)^2(2k+1)^2-(n+2)}{(n+1)(n+2)}\binom{n+2}{\frac{n+2}{2}-\frac{1}{2}(h+2)(2k+1)}$$

for $n \equiv h \bmod 2$. Finally, (2.18) follows from substituting $n = 2m-2$ for a suitable half-integer $m \in \frac{1}{2}\mathbb{N}$ (such that $2m-2 \geq 0$) with $h \equiv 2m \bmod 2$, and because $\upsilon_{h,k} := (h+2)(2k+1)/2$. $\square$

With explicit formulae for the probabilities $p_n^{(h)}$ and $q_n^{(h)}$, we can start to work towards the analysis of the asymptotic behavior of admissible random walks.

2.4.2 Admissible Random Walks on $\mathbb{N}_0$

In this section, we begin to develop the tools required for a precise analysis of the asymptotic behavior of admissible random walks on $\mathbb{N}_0$.

Recalling the result of Theorem 2.4.5, we find that in the half-integer representation $p_{2m-1}^{(h)}$, the shifted central binomial coefficient $\binom{2m}{m-\tau_{h,k}}$ appears. Hence, for the purpose of obtaining an expansion for $p_{2m-1} = \sum_{h\geq 0} p_{2m-1}^{(h)}$, analyzing the asymptotic behavior of binomial coefficients in the central region is necessary.

Lemma 2.4.6.
For $n \in \frac{1}{2}\mathbb{N}$ and $|\alpha| \leq n^{2/3}$ such that $n-\alpha \in \mathbb{N}$, we have

$$\binom{2n}{n-\alpha} \sim \frac{4^n}{\sqrt{n\pi}} \exp\left(-\frac{\alpha^2}{n}\right) \cdot S(\alpha, n)$$

with $S(\alpha, n) := \sum_{\ell, j \geq 0} c_{\ell j} \frac{\alpha^{2j}}{n^\ell}$ and

$$\begin{aligned} c_{\ell j} = [\alpha^{2j} n^{-\ell}] &\Bigg(\sum_{r\geq 0} \frac{d_r}{(2n)^r}\Bigg)\Bigg(\sum_{r\geq 0} \frac{(-1)^r d_r}{(n+\alpha)^r}\Bigg)\Bigg(\sum_{r\geq 0} \frac{(-1)^r d_j}{(n-\alpha)^r}\Bigg) \\ &\times \Bigg(\sum_{r\geq 0} (-1)^r \binom{-1/2}{r} \frac{\alpha^{2r}}{n^{2r}}\Bigg)\Bigg(\sum_{r\geq 0} \frac{1}{r!} \frac{\alpha^{4r}}{n^{3r}} \Bigg[\sum_{t\geq 0} \frac{-1}{(t+2)(2t+3)} \frac{\alpha^{2t}}{n^{2t}}\Bigg]^r\Bigg), \end{aligned} \tag{2.21}$$

where the coefficients d_r come from the higher-order Stirling approximation of the factorial, cf. (2.22). Additionally, the estimate

$$S(\alpha, n) = 1 + O\left(\frac{1+|\alpha|}{n}\right)$$

holds for $|\alpha| \leq n^{2/3}$ and we know that $c_{00} = 1$ as well as $c_{\ell j} = 0$ if $j > \frac{2}{3}\ell$.

If $|\alpha| > n^{2/3}$, the term

$$\binom{2n}{n-\alpha}/4^n = O(\exp(-n^{1/3}))$$

decays faster than any power of n.

Proof. We begin by recalling the higher-order Stirling approximation (cf. [17, p. 760])

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \Bigg(\sum_{j\geq 0} \frac{d_j}{n^j}\Bigg). \tag{2.22}$$

An explicit representation of the coefficients d_j can be found in [33]. From the logarithmic representation of the factorial (see [17, p. 766]), the expansion

$$\frac{1}{n!} \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{e}{n}\right)^n \Bigg(\sum_{j\geq 0} \frac{(-1)^j d_j}{n^j}\Bigg) \tag{2.23}$$

for the reciprocal factorial follows.

Let us assume $|\alpha| \leq n^{2/3}$. Then, by applying (2.22) and (2.23) to the shifted central binomial coefficient, we obtain

$$\begin{aligned} \binom{2n}{n-\alpha} &= \frac{(2n)!}{(n-\alpha)!\,(n+\alpha)!} \\ &= \frac{1}{\sqrt{n\pi}} \left(1 - \frac{\alpha^2}{n^2}\right)^{-1/2} \frac{(2n)^{2n}}{(n+\alpha)^{n+\alpha}(n-\alpha)^{n-\alpha}} \\ &\qquad \times \Bigg(\sum_{r\geq 0} \frac{d_r}{(2n)^r}\Bigg)\Bigg(\sum_{r\geq 0} \frac{(-1)^r d_r}{(n+\alpha)^r}\Bigg)\Bigg(\sum_{r\geq 0} \frac{(-1)^r d_r}{(n-\alpha)^r}\Bigg). \end{aligned}$$

The factor $\left(1-\frac{\alpha^2}{n^2}\right)^{-1/2}$ can be expanded as a binomial series, resulting in

$$\left(1-\frac{\alpha^2}{n^2}\right)^{-1/2} = \sum_{r\geq 0}(-1)^r\binom{-1/2}{r}\frac{\alpha^{2r}}{n^{2r}}.$$

The remaining factor is handled by means of the identity $n^n = \exp(n\log n)$, which leads to

$$\begin{aligned}\frac{(2n)^{2n}}{(n+\alpha)^{n+\alpha}(n-\alpha)^{n-\alpha}} &= \exp(2n\log(2n)-(n+\alpha)\log(n+\alpha)-(n-\alpha)\log(n-\alpha))\\ &= \exp(2n\log 2+2n\log n-(n+\alpha)(\log n+\log(1+\alpha/n))\\ &\qquad -(n-\alpha)(\log n+\log(1-\alpha/n)))\\ &= 4^n\exp(\alpha\log(1-\alpha/n)-\alpha\log(1+\alpha/n)\\ &\qquad -n\log(1-\alpha/n)-n\log(1+\alpha/n)).\end{aligned}$$

By expanding the logarithm into a power series, we can simplify this expression to

$$\begin{aligned}\frac{(2n)^{2n}}{(n+\alpha)^{n+\alpha}(n-\alpha)^{n-\alpha}} &= 4^n\exp\left(2\left[-\sum_{t\geq 0}\frac{1}{2t+1}\frac{\alpha^{2t+2}}{n^{2t+1}}+\sum_{t\geq 0}\frac{1}{2t+2}\frac{\alpha^{2t+2}}{n^{2t+1}}\right]\right)\\ &= 4^n\exp\left(-\frac{\alpha^2}{n}\right)\exp\left(-\frac{\alpha^4}{n^3}\sum_{t\geq 0}\frac{1}{(t+2)(2t+3)}\frac{\alpha^{2t}}{n^{2t}}\right)\\ &= 4^n\exp\left(-\frac{\alpha^2}{n}\right)\left(\sum_{r\geq 0}\frac{1}{r!}\frac{\alpha^{4r}}{n^{3r}}\left[\sum_{t\geq 0}\frac{-1}{(t+2)(2t+3)}\frac{\alpha^{2t}}{n^{2t}}\right]^r\right).\end{aligned}$$

We also use

$$\frac{1}{n\pm\alpha} = \frac{1}{n}\frac{1}{1\pm\frac{\alpha}{n}} = \frac{1}{n}\sum_{r\geq 0}\left(\mp\frac{\alpha}{n}\right)^r.$$

By the symmetry of the binomial coefficient, the resulting asymptotic expansion has to be symmetric in α. Assembling all these expansions yields the asymptotic formula

$$\binom{2n}{n-\alpha} \sim \frac{4^n}{\sqrt{n\pi}}\exp\left(-\frac{\alpha^2}{n}\right)\cdot S(\alpha,n),$$

where $S(\alpha,n)$ is defined as in the statement of the lemma.

Note that $d_0 = 1$, and thus the first summand of the series in (2.21) is 1—which gives $c_{00} = 1$. Only in the last series, the exponent of α is greater than the exponent of $1/n$, with the maximal difference being induced by α^{4r}/n^{3r}. Thus, if $j > \frac{2}{3}\ell$, we have $c_{\ell j} = 0$. Together with $|\alpha| \leq n^{2/3}$, this implies the estimate for $S(\alpha,n)$.

For $|\alpha| > n^{2/3}$, we can use the monotonicity of the binomial coefficient to obtain

$$\binom{2n}{n-\alpha} \leq \binom{2n}{n-\lceil n^{2/3}\rceil},$$

for which the exponential factor ensures fast decay,

$$\exp\Big(-\frac{\lceil n^{2/3}\rceil^2}{n}\Big) = O\big(\exp\big(-n^{1/3}\big)\big),$$

and as everything else is of polynomial growth, the statement of the lemma follows. □

Now that we have an asymptotic expansion for the shifted central binomial coefficient, let us look at our explicit formula in (2.17) again: we have

$$p_{2m-1}^{(h)} = \frac{4}{4^m}\sum_{k\ge 0}(-1)^k \frac{\tau_{h,k}}{m}\binom{2m}{m-\tau_{h,k}} \cdot [\![h+1 \equiv 2m \bmod 2]\!],$$

where $\tau_{h,k} = (h+1)(2k+1)/2$. Therefore, the total probability for a random walk of length $2m-1$ on $\mathbb{N}_0$ to be admissible is given by

$$p_{2m-1} = \sum_{h\ge 0} p_{2m-1}^{(h)} = \frac{4}{4^m}\sum_{\substack{h,k\ge 0\\ h+1\equiv 2m \bmod 2}}(-1)^k \frac{\tau_{h,k}}{m}\binom{2m}{m-\tau_{h,k}}.$$

The terms where $\tau_{h,k} > m^{2/3}$ can be neglected in view of the last statement in Lemma 2.4.6, as their total contribution decays faster than any power of m: note that there are only $O(m^2)$ such terms (trivially, h, $k \le m$), each of which contributes $O(m\exp(-m^{1/3}))$ to the sum. For all other values of h and k, we can replace the binomial coefficient by its asymptotic expansion. This gives us, for any $L > 0$,

$$\begin{aligned} p_{2m-1} = {} & \frac{4}{\sqrt{m\pi}}\sum_{\substack{h,k\ge 0,\tau_{h,k}\le m^{2/3}\\ h+1\equiv 2m \bmod 2}}(-1)^k\frac{\tau_{h,k}}{m}\exp\Big(-\frac{\tau_{h,k}^2}{m}\Big)\sum_{\ell=0}^{L-1}\sum_{j\ge 0}c_{\ell j}\frac{\tau_{h,k}^{2j}}{m^\ell} \\ & + O\Big(\frac{1}{\sqrt{m}}\sum_{\substack{h,k\ge 0,\tau_{h,k}\le m^{2/3}\\ h+1\equiv 2m \bmod 2}}\frac{\tau_{h,k}^{2J(L)+1}}{m^{L+1}}\Big), \end{aligned}$$

where $J(L) \le \frac{2}{3}L$ since $c_{\ell j} = 0$ for $j > \frac{2}{3}\ell$. Since the sum clearly contains $O(m^{4/3})$ terms, the error is at most $O(m^{-1/2+4/3+2/3(2J(L)+1)-(L+1)}) = O(m^{1/2-L/9})$. The exponent can be made arbitrarily small by choosing L accordingly. Finally, if we extend the sum to the full range (all integers h, $k \ge 0$ such that $h+1 \equiv 2m \bmod 2$) again, we only get another error term of order $O(\exp(-m^{1/3}))$, which can be neglected. In summary, we have

$$p_{2m-1} \sim \frac{4}{\sqrt{m\pi}}\sum_{\substack{h,k\ge 0\\ h+1\equiv 2m \bmod 2}}(-1)^k\frac{\tau_{h,k}}{m}\exp\Big(-\frac{\tau_{h,k}^2}{m}\Big)\sum_{\ell,j\ge 0}c_{\ell j}\frac{\tau_{h,k}^{2j}}{m^\ell}. \tag{2.24}$$

This sum can be analyzed with the help of the Mellin transform and the converse mapping theorem (cf. Theorem 1.3.13). In order to follow this approach, we will investigate those terms in (2.24) whose growth is not obvious more precisely. That is, we will focus on the contribution of terms of the form

$$\sum_{\substack{h,k\geq 0\\ h+1\equiv 2m \bmod 2}} (-1)^k \tau_{h,k}^{2j+1} \exp\Big(-\frac{\tau_{h,k}^2}{m}\Big).$$

We are also interested in the expected height and the corresponding variance and higher moments of admissible random walks. Asymptotic expansions for these can be obtained by analyzing moments of the random variable H_n with $\mathbb{P}(H_n = h) := \frac{p_n^{(h)}}{p_n}$, as stated in the introduction. For the sake of convenience, let us consider the r-th shifted moment $\mathbb{E}(H_{2m-1}+1)^r$. We know

$$\mathbb{E}(H_{2m-1}+1)^r = \sum_{h\geq 0}(h+1)^r \mathbb{P}(H_{2m-1} = h) = \frac{\sum_{h\geq 0}(h+1)^r p_{2m-1}^{(h)}}{p_{2m-1}}.$$

The asymptotic behavior of the denominator is related to the behavior of the sum from above—and fortunately, the behavior of the numerator is related to the behavior of the very similar sum

$$\sum_{\substack{h,k\geq 0\\ h+1\equiv 2m \bmod 2}} (-1)^k \tau_{h,k}^{2j+1}(h+1)^r \exp\Big(-\frac{\tau_{h,k}^2}{m}\Big).$$

The following lemma analyzes sums of this structure asymptotically.

Lemma 2.4.7.
Let j, $r\in\mathbb{N}_0$. Then we have

$$\sum_{\substack{h,k\geq 0\\ h+1\equiv 2m \bmod 2}} (-1)^k \tau_{h,k}^{2j+1}(h+1)^r \exp\Big(-\frac{\tau_{h,k}^2}{m}\Big) = 2^{r-1}\Gamma\Big(j+1+\frac{r}{2}\Big)\beta(r+1)m^{j+1+r/2} + O(m^{-K}) \quad (2.25)$$

for any fixed $K > 0$, where $\beta(\,\cdot\,)$ denotes the Dirichlet beta function.

Proof of Lemma 2.4.7. If we substitute $m = x^{-2}$, the left-hand side of (2.25) becomes

$$f(x) := \sum_{\substack{h,k\geq 0\\ h+1\equiv 2m \bmod 2}} (-1)^k \tau_{h,k}^{2j+1}(h+1)^r \exp(-\tau_{h,k}^2 x^2).$$

This is a typical example of a harmonic sum, cf. [15, §3], and the Mellin transform can be applied to obtain its asymptotic behavior. First of all, from Lemma 1.3.9 we know that the Mellin transform of a harmonic sum of the form $f(x) = \sum_{k\geq 1} a_k g(b_k x)$ can be factored as $\sum_{k\geq 1} a_k b_k^{-s} g^*(s)$, provided that the half-plane of absolute convergence of the Dirichlet series

$\Lambda(s) = \sum_{k\geq 1} a_k b_k^{-s}$ has non-empty intersection with the fundamental strip of the Mellin transform g^* of the base function g. In this particular case, the Dirichlet series is

$$\Lambda(s) := \sum_{\substack{h,k\geq 0 \\ h+1\equiv 2m \bmod 2}} (-1)^k \tau_{h,k}^{2j+1-s}(h+1)^r,$$

and the base function is $g(x) = \exp(-x^2)$, with Mellin transform $g^*(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)$ and fundamental strip $\langle 0, \infty\rangle$.

Now we simplify the Dirichlet series. For $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 2j+2+r$, the sum

$$\Lambda(s) = 2^{s-(2j+1)} \sum_{\substack{h,k\geq 0 \\ h+1\equiv 2m \bmod 2}} (-1)^k (h+1)^{2j+1+r-s}(2k+1)^{2j+1-s}$$

converges absolutely because it is dominated by the zeta function. In view of the definition of the β function, this simplifies to

$$\Lambda(s) = 2^{s-(2j+1)}\beta(s-(2j+1))\kappa_{2m}(s-(2j+1+r)),$$

where $\kappa_{2m}(s)$ depends on the parity of $2m$. We find

$$\kappa_{2m}(s) = \sum_{\substack{h\geq 0 \\ h+1\equiv 2m \bmod 2}} (h+1)^{-s} = \begin{cases} 2^{-s}\zeta(s) & \text{for } m \in \mathbb{N}, \\ (1-2^{-s})\zeta(s) & \text{for } m \notin \mathbb{N}. \end{cases}$$

Thus, the Mellin transform of f is

$$f^*(s) = \Lambda(s)g^*(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)2^{s-(2j+1)}\beta(s-(2j+1))\kappa_{2m}(s-(2j+1+r)).$$

By the converse mapping theorem (see Theorem 1.3.13), the asymptotic growth of $f(x)$ for $x \to 0$ can be found by considering the analytic continuation of $f^*(s)$ further to the left of the complex plane and investigating its poles. The theorem may be applied because $\Lambda(s)$ has polynomial growth and $\Gamma(s/2)$ decays exponentially along vertical lines of the complex plane.

We find that $f^*(s)$ has a simple pole at $s = 2j+2+r$, which comes from the zeta function in the definition of κ_{2m}. There are no other poles: β is an entire function, and the poles of Γ cancel against the zeros of β (at all odd negative integers, see the earlier remark).

The asymptotic contribution from the pole of f^* is

$$\begin{aligned} \operatorname{Res}(f^*, s = 2j+2+r)\cdot x^{-(2j+2+r)} &= \frac{1}{2}\Gamma\left(j+1+\frac{r}{2}\right)2^{r+1}\beta(r+1)\frac{1}{2}x^{-(2j+2+r)} \\ &= 2^{r-1}\Gamma\left(j+1+\frac{r}{2}\right)\beta(r+1)m^{j+1+r/2}, \end{aligned}$$

which does not depend on the parity of $2m$, as the respective residue of κ_{2m} is $\frac{1}{2}$ in either case. Finally, the O-term in (2.25) comes from the fact that f^* may be continued analytically arbitrarily far to the left in the complex plane without encountering any additional poles. □

Remark.
In Lemma 2.4.7, particular values of the Dirichlet beta function are required. To compute the asymptotic expansions for the first moments, we need $\beta(1) = \pi/4$, $\beta(2) = G \approx 0.91597$, as well as $\beta(3) = \pi^3/32$, where G is the Catalan constant. These values are taken from [36, Table 3.7.1].

At this point, all that remains to obtain asymptotic expansions is to multiply the contributions resulting from Lemma 2.4.7 with the correct coefficients and contributions from (2.24).

Theorem 2.4.8 (Asymptotic analysis of admissible random walks on $\mathbb{N}_0$).
The probability that a random walk on $\mathbb{N}_0$ is admissible can be expressed asymptotically as

$$p_n = \sqrt{\frac{\pi}{2n}} - \frac{5\sqrt{2\pi}}{24\sqrt{n^3}} + \frac{127\sqrt{2\pi}}{960\sqrt{n^5}} - \frac{1571\sqrt{2\pi}}{16128\sqrt{n^7}} - \frac{1896913\sqrt{2\pi}}{184320\sqrt{n^9}} + O\Big(\frac{1}{\sqrt{n^{11}}}\Big), \tag{2.26}$$

where $\sqrt{\pi/2} \approx 1.25331$. The expected height of admissible random walks is given by

$$\mathbb{E}H_n = 2G\sqrt{\frac{2n}{\pi}} - 1 + \frac{5\sqrt{2}G}{6\sqrt{\pi n}} - \frac{131\sqrt{2}G}{720\sqrt{\pi n^3}} + \frac{1129\sqrt{2}G}{12096\sqrt{\pi n^5}} + O\Big(\frac{1}{\sqrt{n^7}}\Big), \tag{2.27}$$

where $2G\sqrt{2/\pi} \approx 1.46167$, and the variance of H_n can be expressed as

$$\mathbb{V}H_n = \frac{\pi^3 - 32G^2}{4\pi}n + \frac{\pi^3 - 40G^2}{6\pi} - \frac{\pi^3 - 12G^2}{180\pi n} + \frac{11\pi^3 - 265G^2}{1890\pi n^2} + O\Big(\frac{1}{n^3}\Big), \tag{2.28}$$

where $(\pi^3 - 32G^2)/(4\pi) \approx 0.33092$. Generally, the r-th moment is asymptotically given by

$$\mathbb{E}H_n^r \sim \frac{2^{r/2+2}}{\pi}\Gamma\Big(\frac{r}{2}+1\Big)\beta(r+1)n^{r/2}. \tag{2.29}$$

Moreover, if $\eta = h/\sqrt{n}$ satisfies $3/\sqrt{\log n} < \eta < \sqrt{\log n}/2$ and $h \equiv n \bmod 2$, we have the local limit theorem [46]

$$\mathbb{P}(H_n = h) = \frac{p_n^{(h)}}{p_n} \sim \frac{2\phi(\eta)}{\sqrt{n}} = \frac{8\eta}{\pi\sqrt{n}} \sum_{k\geq 0}(-1)^k(2k+1)\exp\Big(-\frac{(2k+1)^2\eta^2}{2}\Big) \tag{2.30}$$

$$= \frac{2\sqrt{2\pi}}{\eta^2\sqrt{n}} \sum_{k\geq 0}(-1)^k(2k+1)\exp\Big(-\frac{\pi^2(2k+1)^2}{8\eta^2}\Big). \tag{2.31}$$

Remark.
Note that the asymptotic behavior of the moments of H_n readily implies that the normalized random variable $H_n/\sqrt{n}$ converges weakly to the distribution whose density is given by $\phi(\eta)$ (see [17, Theorem C.2]). The local limit law (2.30) is somewhat stronger.

Proof. With (2.24) and the result of Lemma 2.4.7, obtaining an asymptotic expansion of p_{2m-1} is only a question of developing the shifted central binomial coefficient and multiplying with the correct growth contributions from (2.25). By doing so (with the help of SageMath [45]: corresponding SageMath code can be found in Appendix A or online at `http://arxiv.org/src/1503.08790/anc/random-walk_NN.ipynb`), an asymptotic expansion in the half-integer m is obtained. Substituting $m = (n+1)/2$ then gives (2.26).

The results in (2.27) and (2.28) are obtained by considering

$$\mathbb{E}(H_n+1)^r = \frac{\sum_{h\geq 0}(h+1)^r p_n^{(h)}}{p_n},$$

making use of (2.24) and Lemma 2.4.7 again. Note that we have $\mathbb{E}H_n = \mathbb{E}(H_n+1)-1$, as well as $\mathbb{V}H_n = \mathbb{E}(H_n+1)^2 - [\mathbb{E}(H_n+1)]^2$. For higher moments, we only give the principal term of the asymptotics, which corresponds to the coefficient c_{00} in (2.24), but in principle it would be possible to calculate further terms as well.

The fact that the two series in (2.30) and (2.31) that represent the density $\phi(\eta)$ are equal is a simple consequence of the Poisson sum formula, Theorem 1.3.2. In fact, as the sum in (2.30) takes the same values for k and $-(k+1)$, we can write

$$\sum_{k\geq 0}(-1)^k(2k+1)\exp\Bigl(-\frac{(2k+1)^2\eta^2}{2}\Bigr) = \frac{1}{2}\sum_{k\in\mathbb{Z}}(-1)^k(2k+1)\exp\Bigl(-\frac{(2k+1)^2\eta^2}{2}\Bigr).$$

This sum can be represented as the sum of the function

$$f(x) := (2x+1)\exp\Bigl(-\frac{(2x+1)^2\eta^2}{2} + i\pi x\Bigr)$$

over all integers. The Fourier transform of f can be computed easily, we find

$$\hat{f}(t) = \frac{\sqrt{2\pi^3}}{4\eta^3}(1-2t)\exp\Bigl(-\frac{\pi^2(1-2t)^2}{8\eta^2} + i\pi t\Bigr).$$

In particular, for an integer $k\in\mathbb{Z}$ we find $\hat{f}(k+1) = \frac{\sqrt{2\pi^3}}{4\eta^3}(-1)^k(2k+1)\exp\Bigl(-\frac{\pi^2(2k+1)^2}{8\eta^2}\Bigr)$. Therefore, Theorem 1.3.2 yields

$$\frac{1}{2}\sum_{k\in\mathbb{Z}}(-1)^k(2k+1)\exp\Bigl(-\frac{(2k+1)^2\eta^2}{2}\Bigr) = \frac{\sqrt{2\pi^3}}{8\eta^3}\sum_{k\in\mathbb{Z}}(-1)^k(2k+1)\exp\Bigl(-\frac{-\pi^2(2k+1)^2}{8\eta^2}\Bigr),$$

which, considering the fact that a similar symmetry argument like above can be used for this other sum, proves the equality of (2.30) and (2.31).

It remains to prove (2.30). To this end, we follow [46] and revisit the explicit expression (recall that we set $n = 2m-1$)

$$p_{2m-1}^{(h)} = \frac{4}{4^m}\sum_{k\geq 0}(-1)^k\frac{\tau_{h,k}}{m}\binom{2m}{m-\tau_{h,k}}.$$

First of all, we can eliminate all k with $\tau_{h,k} > m^{2/3}$, since their total contribution is at most $O(m\exp(-m^{1/3}))$ as before. For all other values of k, we replace the binomial coefficient according to Lemma 2.4.6 by

$$\binom{2m}{m-\tau_{h,k}} = \frac{4^m}{\sqrt{\pi m}}\exp\Big(-\frac{\tau_{h,k}^2}{m}\Big)\Big(1+O\Big(\frac{1+\tau_{h,k}}{m}\Big)\Big).$$

Note here that

$$\tau_{h,k} = \frac{(h+1)(2k+1)}{2} = \frac{h}{2}(2k+1)\Big(1+O\Big(\frac{1}{h}\Big)\Big),$$

and likewise

$$\frac{\tau_{h,k}^2}{m} = \frac{h^2(2k+1)^2}{2n}\Big(1+O\Big(\frac{1}{h}+\frac{1}{n}\Big)\Big).$$

It follows that

$$\frac{\tau_{h,k}}{m}\exp\Big(-\frac{\tau_{h,k}^2}{m}\Big) = \frac{h(2k+1)}{n}\exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big)\Big(1+O\Big(\frac{1}{h}+\frac{hk^2+1}{n}\Big)\Big).$$

We are assuming that $\tau_{h,k} \le m^{2/3} = ((n+1)/2)^{2/3}$, which implies $hk^2/n = O(n^{1/3}/h)$. In view of our assumptions on h, this means that the error term is $O(n^{-1/6}\sqrt{\log n})$. Thus we have

$$\begin{aligned} p_n^{(h)} = p_{2m-1}^{(h)} &= \frac{4\sqrt{2}h}{\sqrt{\pi n^3}} \\ &\times \sum_{\substack{k\ge 0 \\ \tau_{h,k}\le((n+1)/2)^{2/3}}} (-1)^k(2k+1)\exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big)\Big(1+O\Big(\frac{\sqrt{\log n}}{n^{1/6}}\Big)\Big) \\ &\qquad + O\big(n\exp(-(n/2)^{1/3})\big). \end{aligned}$$

Adding all terms $\tau_{h,k} > m^{2/3} = ((n+1)/2)^{2/3}$ back only results in a negligible contribution that decays faster than any power of n again, but we need to be careful with the O-term inside the sum, as we have to bound the accumulated error by the sum of the absolute values. We have

$$\sum_{k\ge 0}(2k+1)\exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big) = O(n/h^2),$$

which can be seen e.g. by approximating the sum by an integral (or by means of the Mellin transform again), so

$$\begin{aligned} p_n^{(h)} &= \frac{4\sqrt{2}h}{\sqrt{\pi n^3}}\sum_{k\ge 0}(-1)^k(2k+1)\exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big) + O\Big(\frac{\sqrt{\log n}}{hn^{2/3}}\Big) \\ &= \frac{4\sqrt{2}\eta}{\sqrt{\pi}n}\sum_{k\ge 0}(-1)^k(2k+1)\exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big) + O\Big(\frac{\log n}{n^{7/6}}\Big). \end{aligned}$$

Since $p_n = \sqrt{\frac{\pi}{2n}}(1 + O(n^{-1}))$, this yields

$$\begin{aligned}\frac{p_n^{(h)}}{p_n} &= \frac{8\eta}{\pi\sqrt{n}}\sum_{k\geq 0}(-1)^k(2k+1)\exp\Big(-\frac{\eta^2(2k+1)^2}{2}\Big) + O\Big(\frac{\log n}{n^{2/3}}\Big)\\ &= \frac{2\phi(\eta)}{\sqrt{n}} + O\Big(\frac{\log n}{n^{2/3}}\Big).\end{aligned}$$

We still have to show that this is a valid representation, i.e. $\frac{2\phi(\eta)}{\sqrt{n}}$ has to be the dominant contribution and may not be absorbed by the O-term. First, let us assume $\eta \geq 1$.

Note that $(2k+1)\exp\Big(-\frac{\eta^2(2k+1)^2}{2}\Big)$ is a monotonously decreasing sequence in k: we have

$$\begin{aligned}(2k+1)\exp\Big(-\frac{\eta^2(2k+1)^2}{2}\Big) &\geq (2k+3)\exp\Big(-\frac{\eta^2(2k+3)^2}{2}\Big)\\ \iff \qquad \frac{2k+1}{2k+3} &\geq \exp(-\eta^2(4k+4)),\end{aligned}$$

which is valid because the sequence on the left-hand side is increasing, while the sequence on the right-hand side is decreasing, as well as because the inequality holds for $k = 0$. Thus, the sum $\sum_{k\geq 0}(-1)^k(2k+1)\exp\Big(-\frac{\eta^2(2k+1)^2}{2}\Big)$ is an alternating sum where the Leibniz criterion can be applied. Therefore, the estimate

$$\begin{aligned}\sum_{k\geq 0}(-1)^k(2k+1)\exp\Big(-\frac{\eta^2(2k+1)^2}{2}\Big) &\geq \exp\Big(-\frac{\eta^2}{2}\Big) - 3\exp\Big(-\frac{9\eta^2}{2}\Big)\\ &= \exp\Big(-\frac{\eta^2}{2}\Big)\Big(1 - 3\exp(-4\eta^2)\Big) = \Omega\Big(\exp\Big(-\frac{\eta^2}{2}\Big)\Big)\end{aligned}$$

holds. By our assumptions on η we find that the sum can be bounded from below by $\Omega(\exp(-(\log n)/8)) = \Omega(n^{-1/8})$, which causes a lower bound of $\Omega\Big(\frac{1}{n^{5/8}\sqrt{\log n}}\Big)$ for $\frac{2\phi(\eta)}{\sqrt{n}}$. This means that the first term indeed dominates the error term in this case.

For $\eta < 1$, we use the alternative representation (2.31). By applying the same estimate from the argument with the alternating series above, we find that the sum can be bound below by $\Omega(n^{-\pi^2/72})$. This gives an overall bound of

$$\frac{2\phi(\eta)}{\sqrt{n}} = \Omega\left(\frac{n^{-\pi^2/72}}{\eta^2\sqrt{n}}\right) = \Omega\left(\frac{1}{n^{\frac{\pi^2+36}{72}}\log n}\right),$$

which lets us draw the same conclusion as above since $(\pi^2 + 36)/72 < 2/3$. □

Remark.
As stated in the introduction, the number $2^n p_n$ gives the number of extremal lattice paths on $\mathbb{Z}$—and thus, with the asymptotic expansion of p_n, we also have an asymptotic expansion for the number of extremal lattice paths on $\mathbb{Z}$ of given length.

This concludes our analysis of admissible random walks on $\mathbb{N}_0$. In the next section, we investigate admissible random walks on $\mathbb{Z}$.

2.4.3 Ballot Sequences and Admissible Random Walks on $\mathbb{Z}$

In principle, the approach we follow for the analysis of the asymptotic behavior of admissible random walks on $\mathbb{Z}$ is the same as in the previous section. However, due to the different structure of (2.18), some steps will need to be adapted.

With the notation of Lemma 2.4.6, we are able to express q_{2m-2} for a half-integer $m \in \frac{1}{2}\mathbb{N}$ with $m \geq 1$ as

$$q_{2m-2} \sim \frac{4}{\sqrt{m\pi}} \frac{1}{2m-1} \sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} \frac{2\upsilon_{h,k}^2 - m}{m} \exp\Big(-\frac{\upsilon_{h,k}^2}{m}\Big) \sum_{\ell,j\geq 0} c_{\ell,j} \frac{\upsilon_{h,k}^{2j}}{m^{\ell}}. \tag{2.32}$$

In analogy to our investigation of admissible random walks on $\mathbb{N}_0$, we also want to determine the expected height and variance of admissible random walks. These are related to the random variable $\widetilde{H}_n$, which we defined by

$$\mathbb{P}(\widetilde{H}_n = h) = \frac{q_n^{(h)}}{q_n}.$$

To make things easier, we will investigate moments of the form $\mathbb{E}(\widetilde{H}_n + 2)^r$. They can be computed by

$$\mathbb{E}(\widetilde{H}_n + 2)^r = \sum_{h\geq 0} (h+2)^r \mathbb{P}(\widetilde{H}_n = h) = \frac{\sum_{h\geq 0} (h+2)^r q_n^{(h)}}{q_n}.$$

Therefore, we are interested in the asymptotic contribution of

$$\sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} \frac{2\upsilon_{h,k}^2 - m}{m} \upsilon_{h,k}^{2j} (h+2)^r \exp\Big(-\frac{\upsilon_{h,k}^2}{m}\Big),$$

which is discussed in the following lemma.

Lemma 2.4.9.
Let $K > 0$ be fixed. Then we have the asymptotic expansion

$$\sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} \frac{2\upsilon_{h,k}^2 - m}{m} \exp\Big(-\frac{\upsilon_{h,k}^2}{m}\Big) = \frac{\sqrt{m\pi}}{4} + O(m^{-K}). \tag{2.33}$$

For $j \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} &\frac{2\upsilon_{h,k}^2 - m}{m} \upsilon_{h,k}^{2j} \exp\Big(-\frac{\upsilon_{h,k}^2}{m}\Big) \\ &= \Big(\frac{\log m}{2} + 2\gamma + \log 2 + \frac{1}{2}\psi\Big(j+\frac{1}{2}\Big) + \frac{1}{2j} + [\![m \notin \mathbb{N}]\!] \cdot (2\log 2 - 2)\Big) \\ &\qquad \times \frac{j}{2}\Gamma\Big(j+\frac{1}{2}\Big) m^{j+1/2} + O(m^{-K})\end{aligned} \tag{2.34}$$

where $\psi(s)$ is the digamma function. Finally, for $j \in \mathbb{N}_0$, $r \in \mathbb{N}$ we find

$$\sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} \frac{2\upsilon_{h,k}^2 - m}{m}\upsilon_{h,k}^{2j}(h+2)^r \exp\Big(-\frac{\upsilon_{h,k}^2}{m}\Big)$$
$$= j\Gamma\Big(j+\frac{1}{2}\Big)\kappa_{2m}(1-r)m^{j+1/2}$$
$$+\frac{1}{2}\Big(j+\frac{r}{2}\Big)\Gamma\Big(j+\frac{r+1}{2}\Big)(2^{r+1}-1)\zeta(r+1)m^{j+(r+1)/2} + O(m^{-K}), \quad (2.35)$$

where $\kappa_{2m}(s) = 2^{-s}\zeta(s)$ for $m \in \mathbb{N}$ and $\kappa_{2m}(s) = (1-2^{-s})\zeta(s) - 1$ otherwise.

Proof. Let j, $r \in \mathbb{N}_0$. We want to analyze the sum

$$\sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} \frac{2\upsilon_{h,k}^2 - m}{m}\upsilon_{h,k}^{2j}(h+2)^r \exp\Big(-\frac{\upsilon_{h,k}^2}{m}\Big)$$

asymptotically, where m is a half-integer in $\frac{1}{2}\mathbb{N}$ with $m \geq 1$.

In analogy to the proof of Lemma 2.4.7, we substitute $x^{-2} = m$, so that the sum becomes

$$\begin{aligned} f(x) :=& \sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} (2x^2\upsilon_{h,k}^2 - 1)\upsilon_{h,k}^{2j}(h+2)^r \exp(-\upsilon_{h,k}^2 x^2) \\ =& 2x^2 \sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} \upsilon_{h,k}^{2j+2}(h+2)^r \exp(-\upsilon_{h,k}^2 x^2) - \sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} \upsilon_{h,k}^{2j}(h+2)^r \exp(-\upsilon_{h,k}^2 x^2) \\ =:& 2x^2 f_1(x) - f_2(x). \end{aligned}$$

Both f_1 and f_2 are harmonic sums, and we determine their Mellin transforms as we did earlier in the proof of Lemma 2.4.7. By elementary properties of the Mellin transform, we know that $f^*(s) = 2f_1^*(s+2) - f_2^*(s)$. Let Λ_1 and Λ_2 be the Dirichlet series associated with the harmonic sums $f_1(x)$ and $f_2(x)$, respectively. We find

$$\begin{aligned} \Lambda_1(s) &= \sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} \upsilon_{h,k}^{2j+2-s}(h+2)^r = 2^{s-(2j+2)} \sum_{\substack{h,k\geq 0\\ h\equiv 2m \bmod 2}} (h+2)^{2j+2+r-s}(2k+1)^{2j+2-s} \\ &= (2^{s-(2j+2)} - 1)\zeta(s-(2j+2)) \sum_{\substack{h\geq 0\\ h\equiv 2m \bmod 2}} (h+2)^{2j+2+r-s}. \end{aligned}$$

We investigate the sum over h separately, and obtain

$$\kappa_{2m}(s) := \sum_{\substack{h\geq 0\\ h\equiv 2m \bmod 2}} (h+2)^{-s} = \begin{cases} 2^{-s}\zeta(s) & \text{for } m \in \mathbb{N}, \\ (1-2^{-s})\zeta(s) - 1 & \text{for } m \notin \mathbb{N}. \end{cases}$$

Therefore, we find the Mellin transform of the first harmonic sum to be

$$f_1^*(s) = \frac{1}{2}\Gamma\Big(\frac{s}{2}\Big)\Lambda_1(s) = \frac{1}{2}\Gamma\Big(\frac{s}{2}\Big)(2^{s-2j-2}-1)\zeta(s-2j-2)\kappa_{2m}(s-(2j+r)-2).$$

The Mellin transform of the second sum can be found in a completely analogous way: we have

$$f_2^*(s) = \frac{1}{2}\Gamma\Big(\frac{s}{2}\Big)(2^{s-2j}-1)\zeta(s-2j)\kappa_{2m}(s-(2j+r)).$$

Altogether, this yields the Mellin transform

$$\begin{aligned} f^*(s) &= 2f_1^*(s+2) - f_2^*(s) \\ &= \frac{s-1}{2}\Gamma\Big(\frac{s}{2}\Big)(2^{s-2j}-1)\zeta(s-2j)\kappa_{2m}(s-(2j+r)). \end{aligned}$$

As in the proof of Lemma 2.4.7, the growth conditions necessary for application of the converse mapping theorem (Theorem 1.3.13) hold.

In order to analyze the poles of $f^*(s)$, we need to distinguish three cases, as $\zeta(s-2j)$ has a simple pole at $s = 2j+1$ and $\kappa_{2m}(s-(2j+r))$ has a simple pole at $s = 2j+r+1$. The poles of $\Gamma(s/2)$ at even $s \leq 0$ are canceled by the zeros of $\zeta(s-2j)$, unless $s = j = 0$. In that case, the pole is canceled by the factor $(2^{s-2j}-1)$.

First, let $r = j = 0$. Then, $f^*(s)$ has a simple pole at $s = 1$, because one of the poles of $\zeta(s)$ or $\kappa_{2m}(s)$ cancels against the zero of $(s-1)$. Here, the residue of $f^*(s)$ is given by $\sqrt{\pi}/4$, which translates to a contribution of $\sqrt{m\pi}/4$. This proves (2.33).

Second, for $r = 0$ and $j > 0$, the function $f^*(s)$ has a pole of order 2 at $s = 2j+1$. By expanding all the occurring functions, we find the Laurent expansion

$$f^*(s) \asymp \begin{cases} \frac{j}{2}\Gamma\big(j+\frac{1}{2}\big)\Big[\frac{1}{(s-(2j+1))^2} + \frac{\frac{1}{2}\psi(j+\frac{1}{2})+2\gamma+\log 2+\frac{1}{2j}}{s-(2j+1)}\Big] + O(1) & \text{for } m \in \mathbb{N}, \\ \frac{j}{2}\Gamma\big(j+\frac{1}{2}\big)\Big[\frac{1}{(s-(2j+1))^2} + \frac{\frac{1}{2}\psi(j+\frac{1}{2})+2\gamma+3\log 2-2+\frac{1}{2j}}{s-(2j+1)}\Big] + O(1) & \text{for } m \notin \mathbb{N}, \end{cases}$$

where $\psi(s)$ is the digamma function (cf. [11, 5.2.2], see [11, §5.4(ii)] for special values). As the pole of order 2 contributes the factor $\frac{1}{2}m^{j+1/2}\log m$, and the pole of order 1 gives $m^{j+1/2}$, (2.34) is proved.

Finally, consider $r > 0$. In this case we have two separate single poles at $s = 2j+1$ and $s = 2j+r+1$. Computing the residues gives the growth contribution

$$j\Gamma\Big(j+\frac{1}{2}\Big)\kappa_{2m}(1-r)m^{j+1/2} + \Big(j+\frac{r}{2}\Big)\Gamma\Big(j+\frac{r+1}{2}\Big)\Big(2^r-\frac{1}{2}\Big)\zeta(r+1)m^{j+(r+1)/2},$$

which proves (2.35). □

Fortunately, when explicitly computing the expansion all the logarithmic terms cancel out and we obtain the same behavior for admissible paths of even and odd length. The following theorem summarizes our findings.

Theorem 2.4.10 (Asymptotic analysis of admissible random walks on $\mathbb{Z}$).
The probability that a random walk on $\mathbb{Z}$ is admissible has the asymptotic expansion

$$q_n = \frac{1}{n} - \frac{4}{3n^2} + \frac{88}{45n^3} - \frac{976}{315n^4} + \frac{3488}{675n^5} - \frac{276928}{31185n^6} + O\Big(\frac{1}{n^7}\Big). \tag{2.36}$$

The expected height of admissible random walks on $\mathbb{Z}$ is given by

$$\mathbb{E}\widetilde{H}_n = \frac{\sqrt{2\pi^3}}{4}\sqrt{n} - 2 + \frac{3\sqrt{2\pi^3}}{16\sqrt{n}} - \frac{539\sqrt{2\pi^3}}{5760\sqrt{n^3}} + \frac{50713\sqrt{2\pi^3}}{483840\sqrt{n^5}} + O\Big(\frac{1}{\sqrt{n^7}}\Big), \tag{2.37}$$

where $\sqrt{2\pi^3}/4 \approx 1.96870$, and the variance of $\widetilde{H}_n$ can be expressed as

$$\mathbb{V}\widetilde{H}_n = \frac{28\zeta(3) - \pi^3}{8}n + \frac{224\zeta(3) - 9\pi^3}{48} - \frac{1792\zeta(3) - 67\pi^3}{2880n} + \frac{107520\zeta(3) - 4189\pi^3}{120960n^2} + O\Big(\frac{1}{n^3}\Big), \tag{2.38}$$

where $(28\zeta(3) - \pi^3)/8 \approx 0.33141$. Generally, the r-th moment is asymptotically given by

$$\mathbb{E}\widetilde{H}_n^r \sim \frac{r}{\sqrt{\pi}}\Gamma\Big(\frac{r+1}{2}\Big)(2^{r+1} - 1)2^{-r/2}\zeta(r+1)n^{r/2}. \tag{2.39}$$

Moreover, if $\eta = h/\sqrt{n}$ satisfies $6/\sqrt{\log n} < \eta < \sqrt{\log n}/2$, we have the local limit theorem [46]

$$\mathbb{P}(\widetilde{H}_n = h) = \frac{q_n^{(h)}}{q_n} \sim \frac{2\chi(\eta)}{\sqrt{n}} = \frac{4\sqrt{2}}{\sqrt{\pi n}} \sum_{k\geq 0}((2k+1)^2\eta^2 - 1)\exp\Big(-\frac{(2k+1)^2\eta^2}{2}\Big) \tag{2.40}$$

$$= \frac{4\pi^2}{\eta^3\sqrt{n}} \sum_{k\geq 1}(-1)^{k-1}k^2 \exp\Big(-\frac{\pi^2k^2}{2\eta^2}\Big). \tag{2.41}$$

Proof. Overall, the strategy of the proof is the same as in Theorem 2.4.8. The asymptotic expansions were again computed with the help of SageMath [45], and corresponding SageMath code can be found in Appendix A.

The equality of (2.40) and (2.41) can again be seen after applying the Poisson summation formula, Theorem 1.3.2. Note that like before, due to symmetry we can write

$$\sum_{k\geq 0}((2k+1)^2\eta^2 - 1)\exp\Big(-\frac{(2k+1)^2\eta^2}{2}\Big) = \frac{1}{2}\sum_{k\in\mathbb{Z}}((2k+1)^2\eta^2 - 1)\exp\Big(-\frac{(2k+1)^2\eta^2}{2}\Big).$$

Thus, the function of interest for the Poisson summation formula is $f(x) = ((2x+1)^2\eta^2 - 1)\exp\Big(-\frac{(2x+1)^2\eta^2}{2}\Big)$, which has Fourier transform

$$\hat{f}(t) = \frac{\sqrt{2\pi^5}}{2\eta^3}t^2 \exp(i\pi(t-1))\exp\Big(-\frac{\pi^2t^2}{2\eta^2}\Big).$$

For integers $k \in \mathbb{Z}$, this evaluates to $\hat{f}(k) = \frac{\sqrt{2\pi^5}}{2\eta^3}(-1)^{k-1}k^2 \exp\left(-\frac{\pi^2 k^2}{2\eta^2}\right)$ which (taking into account the symmetry $\hat{f}(k) = \hat{f}(-k)$ and $\hat{f}(0) = 0$) proves the equality of (2.40) and (2.41).

The proof of the local limit law is similar to before—however, a few details differ. Note that because $\tau_{h,k} = (h+1)(2k+1)/2$ and $\upsilon_{h,k} = (h+2)(2k+1)/2$ differ only marginally, we may follow the corresponding deliberations from the proof of Theorem 2.4.8 up to the point where we have

$$\begin{aligned} q_n^{(h)} = q_{2m-2}^{(h)} = {} & \frac{4\sqrt{2}}{\sqrt{\pi(n+2)}} \\ & \times \sum_{\substack{k \geq 0 \\ \upsilon_{h,k} \leq ((n+2)/2)^{2/3}}} \frac{(h+2)^2(2k+1)^2 - (n+2)}{(n+1)(n+2)} \exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big)\Big(1 + O\Big(\frac{\sqrt{\log n}}{n^{1/6}}\Big)\Big) \\ & + O\big(n \exp(-(n/2)^{1/3})\big). \end{aligned}$$

Again, adding back all terms for which we have $\upsilon_{h,k} > ((n+2)/2)^{2/3}$ results in a negligible contribution that decays faster than any power of n. The error within the sum can be handled by investigating the contribution of the dominating sum

$$\begin{aligned} & \sum_{k \geq 0} \frac{(h+2)^2(2k+1)^2 + (n+2)}{(n+1)(n+2)} \exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big) \\ & = \frac{(h+2)^2}{(n+1)(n+2)} \sum_{k \geq 0} (2k+1)^2 \exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big) + \frac{1}{n+1} \sum_{k \geq 0} \exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big). \end{aligned}$$

By using the Mellin transform it is easy to see that

$$\begin{aligned} \sum_{k \geq 0} (2k+1)^2 \exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big) &= O\Big(\frac{n^{3/2}}{h^3}\Big), \\ \sum_{k \geq 0} \exp\Big(-\frac{h^2(2k+1)^2}{2n}\Big) &= O\Big(\frac{\sqrt{n}}{h}\Big). \end{aligned}$$

Overall, this means that we obtain

$$q_n^{(h)} = \frac{4\sqrt{2}}{\sqrt{\pi n^3}} \sum_{k \geq 0} ((2k+1)^2\eta^2 - 1) \exp\Big(-\frac{(2k+1)^2\eta^2}{2}\Big) + O\Big(\frac{\log n}{n^{5/3}}\Big),$$

and therefore

$$\begin{aligned} \frac{q_n^{(h)}}{q_n} &= \frac{4\sqrt{2}}{\sqrt{\pi n}} \sum_{k \geq 0} ((2k+1)^2\eta^2 - 1) \exp\Big(-\frac{(2k+1)^2\eta^2}{2}\Big) + O\Big(\frac{\log n}{n^{2/3}}\Big) \\ &= \frac{2\chi(\eta)}{\sqrt{n}} + O\Big(\frac{\log n}{n^{2/3}}\Big). \end{aligned}$$

Now all that remains to show is that $\frac{2\chi(\eta)}{\sqrt{n}}$ in fact is the dominant term and is not absorbed by the O-term.

First, assume $\eta \geq \frac{3}{2}$. As all the summands in

$$\sum_{k\geq 0}((2k+1)^2\eta^2-1)\exp\Big(-\frac{(2k+1)^2\eta^2}{2}\Big)$$

are obviously positive, the sum can be bound from below by $\Omega(\exp(-\eta^2/2))$. Because we assume $\eta < \sqrt{\log n}/2$, this bound can be simplified to $\Omega(n^{-1/8})$. Overall, we obtain a bound of $\Omega(\frac{1}{n^{5/8}})$, which shows that the first term actually is the dominant term.

For $\eta < \frac{3}{2}$, we use the representation from (2.41). It is easy to see, that the series is once again an alternating series where the Leibniz criterion can be applied, and thus the estimate

$$\begin{aligned}\sum_{k\geq 1}(-1)^{k-1}k^2\exp\Big(-\frac{\pi^2k^2}{2\eta^2}\Big) &\geq \exp\Big(-\frac{\pi^2}{2\eta^2}\Big)-4\exp\Big(-\frac{4\pi^2}{2\eta^2}\Big)\\ &= \exp\Big(-\frac{\pi^2}{2\eta^2}\Big)\Big(1-4\exp\Big(-\frac{3\pi^2}{2\eta^2}\Big)\Big)=\Omega\Big(\exp\Big(-\frac{\pi^2}{2\eta^2}\Big)\Big)\end{aligned}$$

holds. By taking into account that we assume $6/\sqrt{\log n} < \eta$, we find that

$$\frac{2\chi(\eta)}{\sqrt{n}} = \Omega\left(\frac{n^{-\pi^2/72}}{\eta^3\sqrt{n}}\right) = \Omega\left(\frac{\sqrt{(\log n)^3}}{n^{\frac{36+\pi^2}{72}}}\right),$$

which dominates the error term because of $(36+\pi^2)/72 < 2/3$. This proves the local limit law. $\square$

Remark.
As every simple symmetric random walk of length n on $\mathbb{Z}$ occurs with probability 2^{-n}, we know that the number of admissible random walks on $\mathbb{Z}$ is $2^n q_n$. Thus, an asymptotic expansion for the number of admissible random walks follows directly from (2.36) upon multiplication by 2^n. This is sequence A167510 in [34].

Furthermore, in the introduction we illustrated that admissible random walks are strongly related to bidirectional ballot sequences. Since every bidirectional ballot sequence of length $n+2$ corresponds to an admissible random walk of length n on $\mathbb{Z}$ (i.e., $B_n = 2^{n-2}q_{n-2}$), we are able to prove Zhao's conjecture that was mentioned in the introduction.

Corollary 2.4.11 (Bidirectional ballot walks).
The number of bidirectional ballot walks B_n of length n can be expressed asymptotically as

$$B_n = 2^n\Big(\frac{1}{4n}+\frac{1}{6n^2}+\frac{7}{45n^3}+\frac{10}{63n^4}+\frac{764}{4725n^5}+\frac{4952}{31185n^6}\Big)+O\Big(\frac{2^n}{n^7}\Big).$$

3 Analysis of Trees

3.1 Introduction

Trees—acyclic connected graphs—are one of the most important discrete structures. Like lattice paths, their applications span across many areas of research; for example Biology and Life Sciences (where *Galton-Watson trees*[1] are used to model branching processes, cf. [31]), and especially computer science: within computer science, trees are a fundamental concept for data structures as well as for algorithms (see [30] for some examples).

The aim of this chapter is to introduce some basic classes of trees, as well as to analyze these classes based on the tools discussed in Chapter 1. The results of these discussions are important for the real-world applications: for instance, precise information on some data structure allows to study the performance of algorithms that act on such a structure.

We begin by introducing some fundamentals that are used within this chapter.

Definition 3.1.1 (Trees and related concepts).
An undirected graph $F = (V, E)$ with vertices V and edges E is said to be a *forest* if F contains no cycles.

The components of a forest are called *trees*. In particular, this means that trees are *acyclic*, *connected* graphs. Two trees $T = (V, E)$ and $T' = (V', E')$ are isomorphic, if there is a bijective map $f : V \to V'$ such that if $e = \{v_1, v_2\} \in E$, we also have $f(e) := \{f(v_1), f(v_2)\} \in E'$ (i.e. the bijection is consistent with respect to the edges). The function f is also called an *isomorphism*. When counting trees, we usually count them *up to isomorphism*, meaning that isomorphic trees are only counted once.

A *labeling* of a tree $T = (V, E)$ is a bijective map $\ell\colon V \to \{1, 2, \ldots, |V|\}$. A graph with a labeling ℓ is called *labeled graph*. Two labeled graphs are isomorphic, if their underlying unlabeled graphs are isomorphic and their labelings coincide.

And finally, if two vertices in a tree T are joined by an edge, then the vertices are called *adjacent*. The *degree* of a vertex v, $\deg v$, counts the number of adjacent vertices. Vertices

[1] A lot of (mathematical) research is currently happening around Galton-Watson trees. See [12, Section 4.4.5] for an introduction.

with degree 0 or 1 are called *external nodes* or *leaves*, all other nodes are called *internal nodes*.

This definition is illustrated by Figure 3.1: F is a forest consisting of three trees (T_1, T_2, T_3). While T_1 and T_3 represent the same tree, T_2 is different. Furthermore, T_4, T_5, and T_6 are labeled trees: T_4 and T_5 are equal, but the labeling of T_6 is different (observe that the degree for the node with label 4 is 2, instead of 3 as for T_4 and T_5).

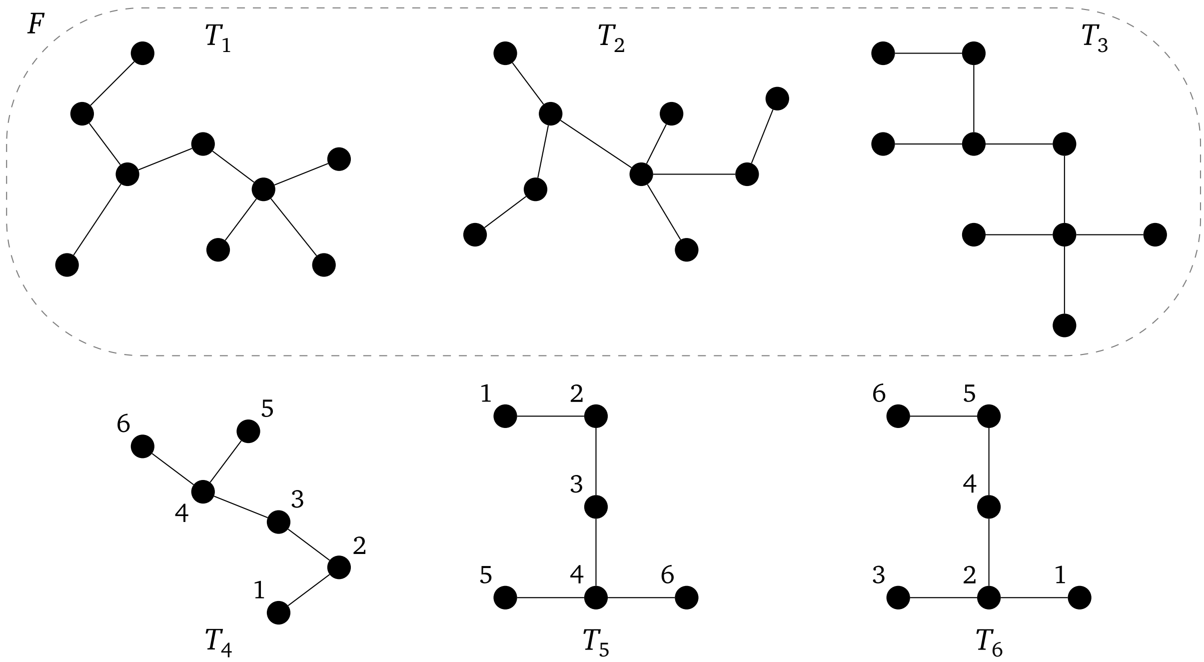

Figure 3.1: Illustration of Definition 3.1.1

Also, observe that not every labeling of a tree induces a distinct labeled tree: swapping the labels 5 and 6 in T_5 yields two different embeddings of the same labeled tree!

Essentially, the structure of directories on any computer can be interpreted as a tree. However, this tree possesses additional structure because all folders are seen relative to the uppermost entry, the root. In fact, this structure corresponds to so-called *rooted trees*.

Definition 3.1.2 (Rooted trees).
A rooted tree is a structure (V, E, r) where $T = (V, E)$ is a tree and r is some vertex in V. This particular vertex r is said to be the *root* of the tree.

The introduction of a root node simultaneously induces a direction on the tree: as every two vertices in a tree are connected by a unique path, every vertex has a fixed distance to the root. This distance is also called the *height*, the *depth* or the *level* of a node, which we denote as $d(v)$. Note that obviously, we have $d(r) = 0$. Then, for some vertex $v \in V$ with $d(v) = k$, we call an neighbor on level $k+1$ *successor*, and a neighbor on level $k-1$

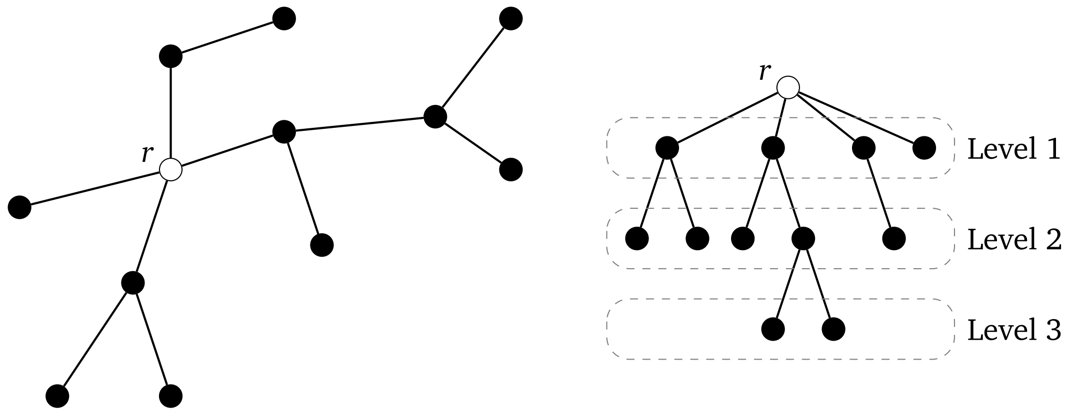

Figure 3.2: Structure of rooted trees

predecessor. Obviously, a vertex can have multiple successors, but only one predecessor. The structure induced by selecting a root node is illustrated in Figure 3.2.

Sometimes it also makes sense to distinguish between the various embeddings of a tree in the plane. This leads to the concept of *(rooted) plane trees*, where every vertex in a rooted tree is equipped with an additional "left-to-right" order for its successors. For example, the rooted trees depicted in Figure 3.3 are equal (as they only differ in their embedding), but when interpreting them as rooted plane trees, they are distinct. Note that in Section 3.2 we will explore a profound connection between simple excursions (i.e. Catalan or Dyck paths) and rooted plane trees.

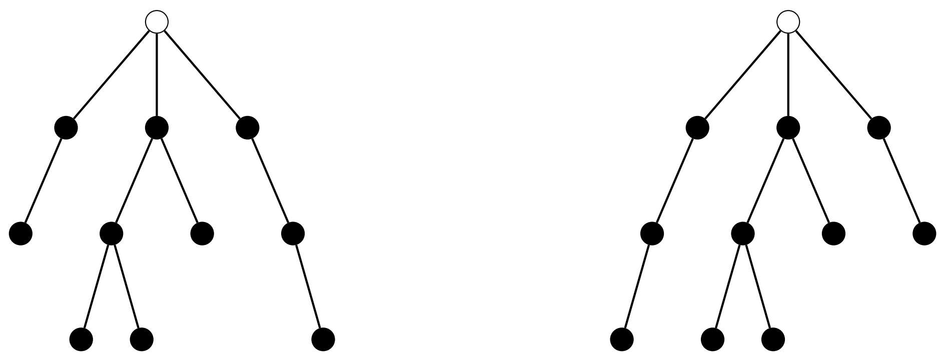

Figure 3.3: Two distinct rooted plane trees

Essentially, this is all the terminology we need in order to construct the special classes of trees that we will investigate within this chapter.

In Section 3.2 we will introduce a technique ("*Lagrange inversion*") as well as an extension to *Singularity Analysis* from Section 1.3.1; two methods that are especially useful when it comes to the analysis of trees. Afterwards, with the help of these results, the growth of several special classes of trees is discussed both from an exact and an asymptotic point of view. This section is mainly based on [12] and [17, Section I.5].

And finally, Section 3.3 revisits the results of [28], where the *average number of deepest nodes* in rooted plane trees is investigated.

3.2 Basic Tree Enumeration

Virtually all important algorithms that operate on trees (like traversing a tree, searching something in a tree, deleting a vertex from a tree etc.) are formulated and implemented recursively. This is because the structure of trees is predestined for a recursive approach: for instance, when considering rooted trees, then every successor of the root can be considered as the root of a sub-tree.

This nice property of trees also has consequences when it comes to generating functions: consider the generating function $T(z)$ modeling the growth of some special class of trees (with respect to the number of vertices of the trees). Then, either $T(z)$ or a very simple transformation of $T(z)$ fulfills a functional equation of the form $T(z) = z\Phi(T(z))$, where Φ is a suitable power series.

We want to demonstrate this fact for the the so-called *binary trees*, as well as for *Motzkin trees*.

Example 3.2.1 (Generating function of binary trees and Motzkin trees).
Binary trees are rooted plane trees where every vertex has either no, or exactly two successors. In other words, a binary tree is either just a leaf, or an internal node with exactly two binary trees attached. We are interested in the number of binary trees with respect to the number of *internal nodes*.

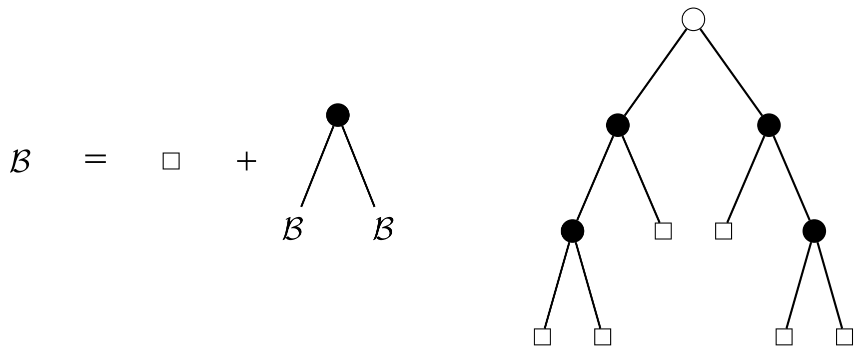

Figure 3.4: Binary trees: symbolic equation and example

If $B(z)$ denotes the generating function for binary trees, then by the recursive definition from above the equation $B(z) = 1 + zB(z)^2$ can be inferred by means of the symbolic method from Section 1.2: in fact, the combinatorial class of binary trees can be constructed as $\mathcal{B} = \{\square\} + \{\bullet\} \times \mathcal{B}^2$ (where $\square$ and $\bullet$ represent a leaf and an inner node, respectively). This is also illustrated in Figure 3.4.

Instead of investigating $B(z)$, we consider $\tilde{B}(z) := B(z) - 1$ and find

$$\tilde{B}(z) = z(1 + \tilde{B}(z))^2 = z\Phi(\tilde{B}(z))$$

with $\Phi(u) = (1+u)^2$.

Of course, we can alternatively also investigate the number of binary trees with respect to the number of all nodes: In this case, we have the construction $\mathcal{B}_\bullet = \{\bullet\} + \{\bullet\} \times \mathcal{B}_\bullet^2$ such that we obtain the functional equation $B_\bullet(z) = z + zB_\bullet(z)^2 = z\Phi(B_\bullet(z))$ for the generating function where $\Phi(u) = 1 + u^2$.

A very similar situation occurs when investigating *Motzkin trees*. These are rooted plane trees where every node has at most two successors. In this case, we are interested in the number of Motzkin trees with respect to the number of overall nodes. Similarly to the number of binary trees, the combinatorial class of Motzkin trees where the size is the number of nodes in the tree can be constructed as $\mathcal{M}_\bullet = \{\bullet\} + \{\bullet\} \times \mathcal{M}_\bullet + \{\bullet\} \times \mathcal{M}_\bullet^2$ (because, like above, a Motzkin tree is either a leaf, or an inner vertex with one or two Motzkin trees attached). Thus, the generating function fulfills

$$M_\bullet(z) = z + zM_\bullet(z) + zM_\bullet(z)^2 = z(1 + M_\bullet(z) + M_\bullet(z)^2) = z\Phi(M_\bullet(z)),$$

where $\Phi(u) = 1 + u + u^2$. $\triangle$

The generating functions of all of the classes we want to discuss admit a functional equation of the form $T(z) = z\Phi(T(z))$. This motivates the following investigations.

3.2.1 Exact results: Lagrange inversion

Basically, the technique of *Lagrange inversion* corresponds to integration by substitution—which is also used to prove the result. It provides an explicit representation of the coefficients of an implicitly defined generating function. However, before we explain Lagrange inversion in detail, we first prove an auxiliary result.

Lemma 3.2.2 (Analytic inversion, [17, Lemma IV.2]).
Let $F\colon \mathbb{C} \to \mathbb{C}$ be a function that is analytic in a neighborhood of $y_0 \in \mathbb{C}$, and define $z_0 := F(y_0)$. Then the following holds:

(a) If $F'(y_0) \neq 0$, then there is a function $g(z)$ that is analytic in some neighborhood V of z_0 such that $F \circ g \equiv \mathrm{id}_V$ holds on V.

(b) Otherwise, if $F'(y_0) = \cdots = F^{(m-1)}(y_0) = 0$ and $F^{(m)}(y_0) \neq 0$, then for all $\vartheta \in [0, 2\pi)$ there are distinct functions $g_1(z), \ldots, g_m(z)$ that are analytic in

$$V = \{z \in \mathbb{C} \mid |z - z_0| < r \text{ and } \operatorname{Arg}(z - z_0) \neq \vartheta\}$$

for some $r > 0$, such that $F \circ g_j \equiv \mathrm{id}_V$ holds on V for all $j \in \{1, 2, \ldots, m\}$.

Proof. We follow the sketch given in [17, Proof of Lemma IV.2]. Because of $F(y_0) = z_0$ and as zeros of analytic functions are isolated, there is a neighborhood U of y_0 such that $F(y) \neq z_0$ for all $y \in U \setminus \{y_0\}$. In particular, we can choose a $r > 0$ such that $F(y) \neq z_0$ for y satisfying $|y - y_0| = r$. Additionally, as z_0 is not an element of the closed set $\{F(y) \mid |y - y_0| = r\}$, there is a neighborhood V of z_0 such that $F(y) \neq z$ for $z \in V$ and $|y - y_0| = r$.

This allows us to define the functions $\sigma_k(z) := \frac{1}{2\pi i} \oint_{|y-y_0|=r} \frac{F'(y)}{F(y)-z} y^k \, dy$ for $k \in \{0, 1\}$. Furthermore, because the functions σ_k are defined as integrals of continuous functions over a compact set, they are continuous as well—and by fundamental theorems from (complex) analysis, it can be shown that the σ_k are even analytic: with Morera's theorem in mind, we investigate the integral of $\sigma_k(z)$ over some triangle in the complex plane. By Fubini's theorem, we are allowed to interchange the integrals—and by Cauchy's theorem, the inner integral is 0. Thus, by Morera's theorem, the functions σ_k are analytic.

Now, using the argument principle, we find

$$\sigma_0(z) = \frac{1}{2\pi i} \oint_{|y-y_0|=r} \frac{F'(y)}{F(y) - z} \, dy = m(y, z) \in \mathbb{Z},$$

where $m(y, z)$ denotes the order of y as a zero of $F(y) - z$. Because we already proved that σ_0 is continuous, it has to be constant on V as it additionally only assumes values in $\mathbb{Z}$.

The value of σ_1 can be found with the help of the residue theorem: we find

$$\sigma_1(z) = \sum_{\substack{y \in \mathbb{C} \\ |y-y_0|<r \\ F(y)=z}} m(y, z) y.$$

In (a), we assumed that y_0 is a simple zero of $F(y) = z_0$, which yields $\sigma_0(z_0) = 1$, and thus also $\sigma_0 \equiv 1$ on V. By setting $g(z) := \sigma_1(z)$, we find $g(z) = y$ for $F(y) = z$ in V—which proves (a).

For (b), we use a similar trick: assume that the neighborhood of y_0 is chosen sufficiently small such that $F'(y) \neq 0$ for all $y \in U \setminus \{y_0\}$. Now, as y_0 is a zero of order m, we find $\sigma_0 \equiv m$, meaning there have to be m distinct zeros in the disk with radius r. By using the statement from (a) for each of those zeros, we obtain m (locally defined) functions $g_1, \ldots,$ g_m. Locally, these functions act as inverse functions of F—and by analytic continuation onto the simply connected domain $V = \{z \in \mathbb{C} \mid |z - z_0| < r \text{ and } \operatorname{Arg}(z - z_0) \neq \vartheta\}$, we obtain the statement from (b). □

This is everything we need to prove the following theorem.

Theorem 3.2.3 (Lagrange Inversion Theorem, [17, Theorem A.2]).
Let $\Phi, H \in \mathbb{C}[\![u]\!]$ be formal power series with $\varphi(0) \neq 0$ and H not a constant. Then there is a

unique formal power series $y \in \mathbb{C}[\![z]\!]$ that fulfills the functional equation $y(z) = z\Phi(y(z))$. For $n > 0$, we find

$$[z^n]H(y(z)) = \frac{1}{n}[u^{n-1}]H'(u)\Phi(u)^n,$$

and in particular

$$[z^n]y(z) = \frac{1}{n}[u^{n-1}]\Phi(u)^n. \tag{3.1}$$

Proof. We follow the idea in [17, Proof of Theorem A.2], meaning that at first we interpret the occurring power series as formal power series in order to show existence and uniqueness of a power series $y(z)$. Afterwards, they are considered as analytic objects again.

In order to show that $y(z)$ exists and is uniquely determined, let $\Phi(u) = \sum_{j\geq 0} \varphi_j u^j$ with $\varphi_j \in \mathbb{C}$ for $j \geq 0$ and $\varphi_0 \neq 0$. By making the ansatz $y(z) = \sum_{j\geq 1} y_j z^j$, we find $y(z) = z(\varphi_0 + O(z)) = \varphi_0 z + O(z^2)$, which yields $y_1 = \varphi_0$. Iterating this approach yields

$$y(z) = z(\varphi_0 + \varphi_1(\varphi_0 z + O(z^2)) + O(z^2)) = \varphi_0 z + \varphi_0\varphi_1 z^2 + O(z^3),$$

that is $y_2 = \varphi_0\varphi_1$. Iteratively, the coefficients of $y(z)$ can be determined uniquely—which means that $y(z)$ itself exists and is uniquely determined.

From now on, the power series are interpreted as analytical objects again. Without loss of generality, for determining $[z^n]y(z)$ we may assume that the functions $\Phi(u)$ and $H(u)$ are polynomials of degree n (higher powers are not used anyhow); in particular we may assume that $\Phi(u)$ and $H(u)$ are entire functions.

In order to show analyticity of $y(z)$, consider the inverse function $\psi(u) := \frac{u}{\Phi(u)}$. Because of $\psi'(u) = \frac{\Phi(u)-u\Phi'(u)}{\Phi(u)^2}$, we find $\psi'(0) = \frac{1}{\Phi(0)} = \frac{1}{\varphi_0} \neq 0$. By the analytic inversion lemma (Lemma 3.2.2), this implies that $y(z)$ is analytic in some neighborhood of $z_0 = 0$. With $C_r := \{z \in \mathbb{C} \mid |z| = r\}$ and Cauchy's integral formula, we find

$$[z^n]H(y(z)) = \frac{1}{n}[z^{n-1}](H(y(z)))' = \frac{1}{n}\frac{1}{2\pi i}\oint_{C_r} \frac{(H(y(z)))'}{z^n}\,dz = \frac{1}{n}\frac{1}{2\pi i}\oint_{C_r} \frac{H'(y(z))}{\left(\frac{y(z)}{\Phi(y(z))}\right)^n} y'(z)\,dz,$$

which, by substituting $y(z) = u$ and $y'(z)\,dz = du$, yields

$$\frac{1}{n}\frac{1}{2\pi i}\oint_{y(C_r)} \frac{H'(u)\Phi(u)^n}{u^n}\,du = \frac{1}{n}[u^{n-1}]H'(u)\Phi(u)^n$$

by Cauchy's integral formula and the fact that the contour $y(C_r)$ still winds around the origin once (which is an implication of the inequality $|y(z) - \varphi_0 z| < |\varphi_0 z|$ for $|z| = r$ and r sufficiently small). Finally, (3.1) can be obtained by choosing $H(u) = u$. □

Examples for applications of Theorem 3.2.3 are given after discussing a method for obtaining the asymptotic behavior.

3.2.2 Asymptotic results: Singularity Analysis

Recall that Singularity Analysis essentially extracts information on the growth of a counting sequence by investigating the nature of the dominant singularities of its generating function, i.e. the singularities that are closest to the origin.

Ideally, we would like to apply these methods from Section 1.3.1 to a function that is determined by a functional equation of the form $y(z) = z\Phi(y(z))$. As it turns out, this can be done by explicitly constructing a Puiseux expansion of $y(z)$ and then applying Singularity Analysis to this expansion.

However, first of all we need to learn more about the location of the dominant singularity, i.e. we need to investigate the radius of convergence of $y(z)$.

Lemma 3.2.4 ([17, Proposition IV.5]).
Let $\Phi \in \mathbb{C}[\![u]\!]$ be analytic around 0 with radius of convergence $R > 0$ such that Φ is not of the form $\Phi(u) = a_0 + a_1 u$. Furthermore, assume that all coefficients of Φ are non-negative with $\Phi(0) \neq 0$. Then, if there is a $\tau \in (0, R)$ such that $\frac{\tau\Phi'(\tau)}{\Phi(\tau)} = 1$ holds, the function defined implicitly by $y(z) := z\Phi(y(z))$ is determined uniquely and has radius of convergence $r = \frac{\tau}{\Phi(\tau)}$.

Proof. Analogously to the proof of Theorem 3.2.3 we can show that $y(z)$ is analytic in a neighborhood of $z = 0$. Assume that $r > 0$ is the corresponding radius of convergence. Furthermore, by the same procedure as in the proof of Theorem 3.2.3, we can show inductively that all coefficients in the power series expansion of y are non-negative, which causes y to be increasing on the interval $(0, r)$. Now we want to show that $y(r) = \tau$.

First, assume $y(r) < \tau$, and let $x \in (0, r)$. Note that by using the non-negativity of the variance of the random variable S defined via

$$\mathbb{P}(S = k) = \frac{\varphi_k x^k}{\Phi(x)},$$

we can show that the function $x \mapsto \frac{x\Phi'(x)}{\Phi(x)}$ is strictly increasing on $(0, r)$. Together with the assumption, this yields

$$\frac{y(r)\Phi'(y(r))}{\Phi(y(r))} < \frac{\tau\Phi'(\tau)}{\Phi(\tau)} = 1.$$

This shows that $\psi'(y(r)) \neq 0$, which (due to Lemma 3.2.2) results in y being analytic in a neighborhood of r. This contradicts Pringsheim's theorem, which states that a power series with non-negative coefficients and radius of convergence r has a singularity at r.

Alternatively, assume $y(r) > \tau$. In this case, as we know that because y is increasing and continuous on $(0, r)$ with $y(0) = 0$, there has to be a $z_0 \in (0, r)$ such that $y(z_0) = \tau$. However, this implies $\psi'(y(z_0)) = \psi'(\tau) = 0$. Hence, y is not analytic near z_0—which is a contradiction because $z_0 < r$ is within the disc of absolute convergence.

Therefore, $y(r) = \tau$ has to hold, which means $r = \psi(\tau) = \frac{\tau}{\Phi(\tau)}$. This proves the lemma. $\square$

The first step is already achieved: we know that the singularities we are interested in have modulus $r = \frac{\tau}{\Phi(\tau)}$. All that remains now is finding a suitable expansion and applying Singularity Analysis. This is done in the following theorem.

Theorem 3.2.5 (Singular Inversion, [17, Theorem VI.6]).
With the assumptions of Lemma 3.2.4, the singular expansion of $y(z)$ near r has the form

$$y(z) = \tau - \sqrt{\frac{2\Phi(\tau)}{\Phi''(\tau)}}\sqrt{1-z/r} + \sum_{j\geq 2}(-1)^j d_j (1-z/r)^{j/2} \tag{3.2}$$

where the d_j are computable constants. Under the assumption that Φ is p-periodic (i.e. p is the largest integer such that Φ can be written as $\Phi(z) = z^b g(z^p)$ for some $b \in \mathbb{Z}$, $g \in \mathbb{C}[\![z]\!]$), the coefficients of $y(z)$ are asymptotically given by

$$[z^n]y(z) \sim p\sqrt{\frac{\Phi(\tau)}{2\Phi''(\tau)}}\frac{r^{-n}}{\sqrt{\pi n^3}}. \tag{3.3}$$

Sketch of the proof. We sketch the central ideas of this proof for the aperiodic case, i.e. for $p = 1$. The periodic case can then be handled by substitution. Details can be found in Theorem VI.6 as well as Remark VI.17 in [17].

First of all, it can be shown that the aperiodicity of Φ carries over to y. Based on this aperiodicity, we can show that r is the only dominant singularity of y.

Basically, as $\psi(u) = \frac{u}{\Phi(u)}$ is the inverse of $y(z)$, we consider an expansion of ψ around τ. We already know that $\psi'(\tau) = 0$, thus we need $\psi''(\tau)$. The second derivative is given by

$$\psi''(u) = -\frac{u\Phi''(u)}{\Phi(u)^2} - 2\frac{\Phi'(u)}{\Phi(u)^2} + 2\frac{u\Phi'(u)^2}{\Phi(u)^3} = 2\frac{\Phi'(u)}{\Phi(u)^2}\left(\frac{u\Phi'(u)}{\Phi(u)} - 1\right) - \frac{u\Phi''(u)}{\Phi(u)^2},$$

which yields $\psi''(\tau) = -\frac{\tau\Phi''(\tau)}{\Phi(\tau)^2} = -r\frac{\Phi''(\tau)}{\Phi(\tau)} < 0$. By Lemma 3.2.2, this means that there are 2 analytic solutions to $y(z) = z\Phi(y(z))$ in a slitted neighborhood of r, one of which corresponds to the $y(z)$ we have constructed for $|z| < r$.

Then, the singular expansion of ψ around $z = r$ gives

$$\begin{aligned} r - z = \psi(\tau) - \psi(y(z)) &= \psi(\tau) - \left(\psi(\tau) + \frac{\psi''(\tau)}{2}(y(z)-\tau)^2 + O((y(z)-\tau)^3)\right) \\ &= r\frac{\Phi''(\tau)}{2\Phi(\tau)}(y(z)-\tau)^2\left(1 + \tilde{d}_2(y(z)-\tau) + O((y(z)-\tau)^2)\right). \end{aligned}$$

Hence we find

$$(y(z)-\tau)^2 = \frac{2\Phi(\tau)}{\Phi''(\tau)}\left(1 - \frac{z}{r}\right)\left(1 - \tilde{d}_2(y(z)-\tau) + O((y(z)-\tau)^2)\right).$$

After applying the square root, we still have to decide for a branch of the square root—and by recalling $y(z)-\tau<0$ for $z\in(0,r)$ we choose the negative branch. Overall, this yields

$$y(z)=\tau-\sqrt{\frac{2\Phi(\tau)}{\Phi''(\tau)}}\sqrt{1-z/r}\left(1-\frac{\tilde{d}_2}{2}(y(z)-\tau)+O((y(z)-\tau)^2)\right). \tag{3.4}$$

Iteratively substituting the left-hand side of (3.4) into the right-hand side proves (3.2). Finally, (3.3) follows from applying Singularity Analysis to (3.2). □

With Lagrange inversion and this extension of Singularity Analysis at hand, the (exact and asymptotic) analysis of several basic classes of trees becomes quite mechanical. This is illustrated in the following section.

3.2.3 Applications

We begin with those trees that we already introduced in Example 3.2.1: binary trees and Motzkin trees.

Example 3.2.6 (Analysis of binary trees and Motzkin trees).
In Example 3.2.1 we already constructed the combinatorial classes related to binary trees and Motzkin trees. For binary trees we found the functional equations

$$\tilde{B}(z)=z(1+\tilde{B}(z))^2 \quad\text{and}\quad B_\bullet(z)=z(1+B_\bullet(z)^2)$$

where $B(z)=\tilde{B}(z)+1$ and $B_\bullet(z)$ enumerate the number of binary trees with respect to the number of inner nodes and overall nodes, respectively.

With Lagrange inversion, we find

$$[z^n]\tilde{B}(z)=\frac{1}{n}[u^{n-1}](1+u)^{2n}=\frac{1}{n}\binom{2n}{n-1}=\frac{1}{n+1}\binom{2n}{n}=C_n,$$

that is the number of binary trees with $n>0$ inner nodes is precisely the n-th Catalan number. We already discussed their asymptotic behavior in Example 2.3.3, namely $C_n\sim\frac{4^n}{\sqrt{\pi n^3}}$, and it is no surprise that applying Theorem 3.2.5 gives the same result.

For the sake of completeness, we also analyze $B_\bullet(z)$: observe that binary trees always have an odd number of nodes, meaning that we will only consider the coefficients of odd powers of $B_\bullet(z)$. For those, Lagrange inversion gives

$$[z^{2n+1}]B_\bullet(z)=\frac{1}{2n+1}[u^{2n}](1+u^2)^{2n+1}=\frac{1}{2n+1}\binom{2n+1}{n}=\frac{1}{n+1}\binom{2n}{n}=C_n.$$

Again, this is not very surprising considering that by induction it is easy to see that a tree with n inner nodes has $n+1$ leaves, and thus $2n+1$ nodes overall.

Motzkin trees can be analyzed analogously: the functional equation

$$M_\bullet(z)=z(1+M_\bullet(z)+M_\bullet(z)^2)$$

for the generating function enumerating the number of Motzkin trees with respect to the size of the tree (i.e. the number of all nodes) leads to

$$[z^n]M_\bullet(z) = \frac{1}{n}[u^{n-1}](1+u+u^2)^n =: M_{n-1}$$

for $n > 0$. The numbers M_n are called Motzkin numbers, and they are enumerated by sequence A001006 in [34].

Their asymptotic behavior can be found by means of Theorem 3.2.5: the function $\Phi(u) = 1+u+u^2$ is aperiodic (i.e. $p = 1$), and the equation $\frac{\tau\Phi'(\tau)}{\Phi(\tau)} = 1$ has the positive solution $\tau = 1$. Therefore, the radius of convergence of $M_\bullet(z)$ is given by $r = \frac{\tau}{\varphi(\tau)} = \frac{1}{3}$. Therefore, (3.3) gives the asymptotic behavior

$$[z^n]M_\bullet(z) \sim \sqrt{\frac{3}{4}}\frac{3^n}{\sqrt{\pi n^3}},$$

which translates into $M_n \sim \sqrt{\frac{27}{4}}\frac{3^n}{\sqrt{\pi n^3}}$ for the Motzkin numbers. $\triangle$

The next class of trees we want to investigate more closely is the class of rooted plane trees itself.

Example 3.2.7 (Analysis of rooted plane trees).
Recall that we already encountered rooted plane trees in Example 1.2.9: there, we already derived the combinatorial construction $\mathcal{T} = \{\bullet\} \times \mathcal{T}^*$ which translates into the functional equation $T(z) = z \cdot \frac{1}{1-T(z)}$. Thus, with $\Phi(u) = \frac{1}{1-u} = 1+u+u^2+\cdots$, we find (by means of Lagrange inversion)

$$\begin{aligned}[z^n]T(z) &= \frac{1}{n}[u^{n-1}](1-u)^{-n} = \frac{(-1)^{n-1}}{n}\binom{-n}{n-1} = \frac{(-1)^{n-1}}{n}\frac{(-n)(-n-1)\cdots(-2n+2)}{(n-1)!} \\ &= \frac{1}{n}\frac{(2n-2)!}{(n-1)!\,(n-1)!} = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1}.\end{aligned}$$

Hence, rooted plane trees can be enumerated by Catalan numbers: there are precisely C_{n-1} distinct rooted plane trees with size n. Asymptotically, Theorem 3.2.5 as well as the results on Catalan numbers from before yield the asymptotic behavior

$$[z^n]T(z) \sim \frac{4^n}{\sqrt{16\pi n^3}}.$$

Alternatively, following the approach from Example 1.2.9, we solve the quadratic equation for $T(z)$ that can be derived from the functional equation, namely $T(z)^2 - T(z) + z = 0$. This equation has the solutions

$$\frac{1+\sqrt{1-4z}}{2} \quad \text{and} \quad \frac{1-\sqrt{1-4z}}{2}.$$

However, only $T(z) = \frac{1-\sqrt{1-4z}}{2}$ is a valid choice: the other solution gives 1 tree of size 0 (when in fact there are no trees without nodes)—and additionally, the coefficients of z^n in the expansion of $\frac{1+\sqrt{1-4z}}{2}$ are negative for $n \geq 1$.

Applying Theorem 1.3.3 to this explicit formula also gives $[z^n]T(z) \sim \frac{4^n}{\sqrt{16\pi n^3}}$. △

Before turning to the next elementary class of trees, we want to shed some more light on the relation between Catalan paths (i.e. simple excursions, cf. Example 2.3.3) and rooted plane trees.

Consider some rooted plane tree T. A very common problem in computer science is searching for a record that is stored in some node of T. If T is not sorted in any way, this essentially reduces to the problem of *traversing* the tree, i.e. finding a "walk" through all nodes of the tree.

One approach for traversing such a tree is the so-called *depth-first traversal*, which can be described as follows:

(1) Start in the root node r of a rooted plane tree $T = (V, E, r)$.

(2) Visit the first unvisited successor of the current node. If the current node has no successor, or all successors are marked as visited, then mark the node itself as visited and go back to the predecessor of your current node (or stop if the current node is the root itself).

Consider this algorithm on a tree of size n: it is easy to see that every edge of the tree is visited exactly twice. As a tree of size n has $n-1$ edges, the algorithm thus terminates after visiting $2n-2$ nodes. Moreover, tracking the depth of the nodes we visit by traversing the tree with this strategy generates a Catalan path of length $2n-2$.

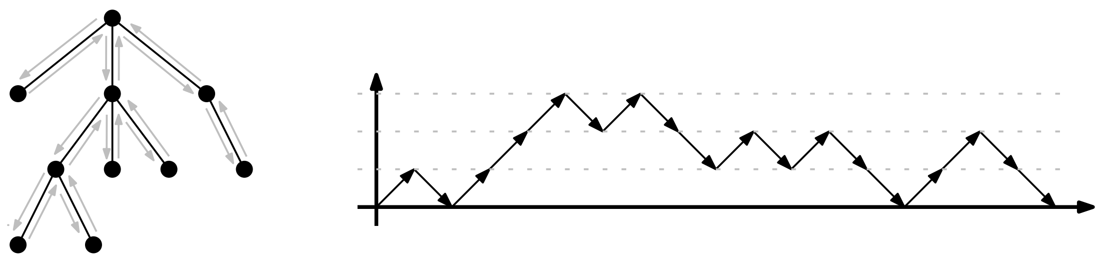

Figure 3.5: Bijection between rooted plane trees and Catalan paths

On the other hand, every Catalan path of length $2n-2$ can be interpreted as the result of the depth-first traversal of a rooted plane tree. This establishes a bijection between Catalan paths of length $2n-2$ and rooted plane trees with n nodes, which is illustrated in Figure 3.5. This bijection explains why C_{n-1}, the number of rooted plane trees with n nodes, is equal to the number of Catalan paths of length $2n-2$. Note that this also justifies that rooted plane trees are also often called *Catalan trees*.

Finally, to conclude this basic analysis, we turn to the analysis of labeled trees *without* differentiating between different embeddings in the plane. Note that these trees are also called *Cayley trees*.

Example 3.2.8 (Cayley trees).
For the sake of simplicity, we consider *rooted* Cayley trees: if $\tilde{L}_n$ and L_n denote the number of Cayley trees and rooted Cayley trees, respectively, then the simple relation $L_n = n\tilde{L}_n$ holds, because for every Cayley tree of size n there are n possible choices of the root.

Note that we will have to follow a slightly different path for the construction of this labeled structure than for the unlabeled structures we investigated so far. First of all, when enumerating labeled structures, we investigate the *exponential generating function* of the form

$$L(z) = \sum_{n\geq 0} \frac{L_n}{n!} z^n.$$

Analogue to the framework introduced in Section 1.2, similar construction rules hold for *labeled* combinatorial classes and exponential generating functions.

In particular, a Cayley tree is a root node with a *set* of Cayley trees attached. Because the trees are labeled, we do not have to consider a multiset; structurally equal trees already differ because they are labeled differently.

Note that when combining two different labeled structures, duplicate labels have to be avoided, and thus the combined object has to be relabeled in a way that does not destroy the underlying structure. We call this *consistent* relabeling (see [17, Chapter II] for a thorough introduction to labeled combinatorial classes). Fortunately, the corresponding generating function is simply the product of the exponential generating functions of the factors.

We want to translate the *set construction* of a combinatorial class $\mathcal{A}$ into an operation on the corresponding generating function. To do so, we note that the class $\mathrm{Set}(\mathcal{A})$ contains all consistent relabelings of an arbitrary number of objects from $\mathcal{A}$, where the order of the objects is not taken into account.

It is easy to see that the exponential generating function of all consistent relabelings of k objects is given by $A(z)^k$. Hence, dividing by $k!$ yields the exponential generating function $\frac{A(z)^k}{k!}$, which enumerates all sets consisting of k objects from $\mathcal{A}$.

By summing over k in order to obtain arbitrary set size we find

$$1 + A(z) + \frac{1}{2!}A(z)^2 + \frac{1}{3!}A(z)^3 + \cdots = \sum_{k\geq 0} \frac{1}{k!}A(z)^k = \exp(A(z)).$$

Finally, this allows us to construct the exponential generating function of the class of rooted Catalan trees $\mathcal{L}$: symbolically, we have $\mathcal{L} = \{\bullet\} * \mathrm{Set}(\mathcal{L})$, where "$*$" represents the combinatorial operation where two labeled classes are combined and relabeled consistently. For the generating function, this translates to the functional equation

$$L(z) = z\exp(L(z)).$$

By applying Lagrange inversion, we find

$$[z^n]L(z) = \frac{1}{n}[u^{n-1}]\exp(u)^n = \frac{1}{n}[u^{n-1}]\exp(un) = \frac{1}{n!}n^{n-1}.$$

Note that because of $[z^n]L(z) = \frac{1}{n!}L_n$, the number of rooted Catalan trees is $L_n = n^{n-1}$. Thus—as observed above—the number of Catalan trees is $\tilde{L}_n = n^{n-2}$, which is Cayley's famous formula.

In order to apply Theorem 3.2.5 we observe that for $\Phi(u) = \exp(u)$ we find $\tau = 1$, $p = 1$, as well as $r = \frac{\tau}{\Phi(\tau)} = e^{-1}$. This yields

$$[z^n]L(z) \sim \sqrt{\frac{e}{2e}}\frac{e^n}{\sqrt{\pi n^3}} = \frac{e^n}{\sqrt{2\pi n^3}}.$$

By comparing this with the exact result from Lagrange inversion, we find

$$\frac{n^{n-1}}{n!} \sim \frac{e^n}{\sqrt{2\pi n^3}} \qquad \Longleftrightarrow \qquad n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n},$$

which is a proof for the main term of Stirling's approximation for the factorial. △

3.3 Average Number of Deepest Nodes

In this section we investigate the number of *deepest nodes* in rooted plane trees. Consider some data structure where the objects are stored in the leaves of a rooted plane tree. When some recursive algorithm processes this tree, these deepest nodes are usually handled in the deepest recursion step—so a precise description of the number of deepest nodes (on average) is desirable. Essentially, this section is based on [28].

Like in Example 3.2.7, let $\mathcal{T}$ denote the combinatorial class of rooted plane trees. The number of *deepest nodes* of a tree $T \in \mathcal{T}$ is the number of nodes that share the same (maximal) distance from the root, i.e. the nodes on the maximal level of T.

Before thinking about asymptotic expressions, we derive an interesting relation between the number of rooted plane trees of size n, height at most k, and root degree r (which we will denote by $B_{n,k,r}$)—and the number of rooted plane trees of size n, height k, and r deepest nodes (denoted by $Q_{n,k,r}$). This requires us to observe some basic properties of these trees.

The size-generating function $B_{k,r}(z) := \sum_{n\geq 0} B_{n,k,r} z^n$ of the number of trees with height at most k and root degree r satisfies a recurrence relation of the form

$$B_{k+1,r}(z) = zA_k(z)^r,$$

where $A_k(z) = \sum_{n\geq 0} A_{n,k} z^n$ is the generating function of all rooted plane trees with height at most k. This follows directly from the symbolic method: a tree enumerated by $B_{n,k+1,r}$ is a root node with r trees of height at most k attached. Furthermore, the generating functions $A_k(z)$ can themselves be described via the recurrence relations

$$A_{-1}(z) = 0, \qquad A_{k+1}(z) = \frac{z}{1 - A_k(z)}. \tag{3.5}$$

Here we have $A_0(z) = z$, as there is exactly one tree of size 1 with height 0 (the tree consisting only of the root), and the recurrence comes from the fact that a rooted plane tree of height at most $k+1$ can be seen as a sequence of trees of height at most k attached to a new root.

Following the discussions in [10], the solution of this recurrence is given by

$$A_k(z) = 2z \frac{(1+\sqrt{1-4z})^{k+1} - (1-\sqrt{1-4z})^{k+1}}{(1+\sqrt{1-4z})^{k+2} - (1-\sqrt{1-4z})^{k+2}}.$$

Next, we want to find a similar recurrence for the bivariate generating function $Q_k(z,u) := \sum_{n,r\geq 0} Q_{n,k,r} z^n u^r$, where z and u mark the size of the tree and the number of deepest nodes, respectively.

Lemma 3.3.1.
The bivariate generating function $Q_k(z,u)$ associated to $Q_{n,k,r}$, the number of rooted plane trees with n nodes, r deepest nodes, and (fixed) height k fulfills the recurrence relation

$$Q_0(z,u) = zu, \qquad Q_{k+1}(z,u) = \frac{A_k(z) Q_k(z,u)}{1 - A_{k-1}(z) - Q_k(z,u)}$$

for $k \geq 0$.

Proof. It is easy to see that $Q_0(z,u) = zu$ has to hold: the tree with the root as its only vertex is the only tree of height 0, and this tree has exactly one deepest node.

The recurrence relation itself again follows by the symbolic method: consider a non-empty sequence of trees enumerated by $Q_k(z,u)$ (i.e. trees with height k) and rooted plane trees whose height is at most $k-1$. However, at least one tree of height k has to occur. Attaching such a sequence of trees to a new node yields a tree associated to $Q_{k+1}(z,u)$. Thus, we obtain

$$\begin{aligned} Q_{k+1}(z,u) &= z\left[\frac{Q_k(z,u) + A_{k-1}(z)}{1 - (Q_k(z,u) + A_{k-1}(z))} - \frac{A_{k-1}(z)}{1 - A_{k-1}(z)}\right] \\ &= \frac{zQ_k(z,u)}{(1 - Q_k(z,u) - A_{k-1}(z))(1 - A_{k-1}(z))}, \end{aligned}$$

which, together with (3.5), proves the lemma. □

Fortunately, we are able to solve this recurrence—the solution is given in the following lemma.

Lemma 3.3.2.
For $k \geq 0$, the bivariate generating functions $Q_k(z,u)$ are given by

$$Q_k(z,u) = \frac{u(A_{k-1}(z)^2 - A_{k-1}(z) + z)}{1 - uA_{k-1}(z)} \tag{3.6}$$

Proof. By means of induction, it is easy to verify that $Q_k(z,u)$ actually solves the recurrence relation from Lemma 3.3.1. □

At this point we are able to prove the relation between the numbers $B_{n,k,r}$ and $Q_{n,k,r}$ mentioned above.

Theorem 3.3.3.
For the number of rooted plane trees of size n, root degree r, and height at most k (enumerated by $B_{n,k,r}$), and the number of rooted plane trees of size n, height k, and r deepest nodes (enumerated by $Q_{n,k,r}$), the recurrence relation

$$Q_{n,k,r} = B_{n+1,k,r+1} - B_{n+1,k,r} + B_{n,k,r-1} \tag{3.7}$$

holds for all n, k, $r \geq 0$.

Proof. The proof is a consequence of Lemma 3.3.2: by developing the denominator of (3.6) into a geometric series, we find

$$Q_k(z,u) = u(A_{k-1}(z)^2 - A_{k-1}(z) + z) \sum_{j \geq 0} u^j A_{k-1}(z)^j.$$

Comparing the coefficients of u^r on the left- and right-hand side, we find

$$\sum_{n \geq 0} Q_{n,k,r} z^n = A_{k-1}(z)^{r+1} - A_{k-1}(z)^r + zA_{k-1}(z)^{r-1}.$$

Recall that for the generating functions $B_{k,r}(z)$ of the numbers $B_{n,k,r}$ we had the relation $B_{k+1,r}(z) = zA_k(z)^r$. Using this to compare the coefficients on the left- and right-hand side again, we find

$$Q_{n,k,r} = B_{n+1,k,r+1} - B_{n+1,k,r} + B_{n,k,r-1}. \qquad \square$$

Note that with the help of (3.7), we are able to obtain the asymptotic behavior of $Q_{n,k,r}$ by analyzing $B_{n,k,r}$. This is done in the following lemma.

Lemma 3.3.4.
For fixed k, $r \in \mathbb{N}$ we have

$$B_{n,k,r} \sim (k+1)^{-r} \tan^{2r}\left(\frac{\pi}{k+1}\right)\left(4\cos^2\left(\frac{\pi}{k+1}\right)\right)^{n-1} \frac{n^{r-1}}{(r-1)!}.$$

Proof. We want to apply Singularity Analysis in order to determine the growth of the counting sequence. Recall that

$$A_{k-1}(z) = 2z\frac{(1+\sqrt{1-4z})^k - (1-\sqrt{1-4z})^k}{(1+\sqrt{1-4z})^{k+1} - (1-\sqrt{1-4z})^{k+1}}.$$

Following the idea of [10], we can identify $(4\cos^2(\frac{j\pi}{k+1}))^{-1}$ for $1 \le j \le k/2$ as the poles of $A_{k-1}(z)$. This enables us to determine a partial fraction decomposition of the form

$$A_{k-1}(z) = \sum_{1\le j\le k/2} \frac{\tan^2\frac{j\pi}{k+1}}{(k+1)\big(1-\big(4\cos^2\frac{j\pi}{k+1}\big)z\big)} + a_k + b_k z,$$

where a_k and b_k are some factors that depend on the parity of k—however, we are only interested in the summand belonging to the dominant singularity. It is easy to see that this is exactly $(4\cos^2(\frac{\pi}{k+1}))^{-1}$. The statement of the lemma now follows by applying Singularity Analysis (Theorem 1.3.3) to $B_{k,r}(z) = zA_{k-1}(z)^r$. □

As a simple consequence, we now also know the asymptotic behavior of $Q_{n,k,r}$.

Corollary 3.3.5.
For fixed k, $r \in \mathbb{N}$ we obtain

$$Q_{n,k,r} \sim (k+1)^{-r-1}\tan^{2r+2}\Big(\frac{\pi}{k+1}\Big)\Big(4\cos^2\Big(\frac{\pi}{k+1}\Big)\Big)^n \frac{n^r}{r!}.$$

Essentially, this result characterizes the behavior of the number of trees with a fixed height and a fixed number of deepest nodes. However, we are also interested in the stochastic behavior of the number of deepest nodes (under the assumption that all trees of size n are equally likely).

Let $\mathcal{T}_n$ denote the combinatorial class of rooted plane trees of size n, and let $X_n\colon \mathcal{T}_n \to \mathbb{N}$ be the random variable associated to the number of deepest nodes over the probability space where every $T \in \mathcal{T}_n$ is equally likely. The following theorem characterizes this random variable.

Theorem 3.3.6 ([28, Theorem 2]).
The probability that a rooted plane tree of size n has $r \in \mathbb{N}$ deepest nodes is given by

$$\mathbb{P}(X_n = r) = 2^{-r} + O\Big(\frac{\log n}{n^{1/2-\varepsilon}}\Big)$$

with some fixed $\varepsilon > 0$.

Additionally, the ℓ-th central moment of this random variable is

$$\mathbb{E}X_n^\ell = E_\ell(2) + O\Big(\frac{\log n}{n^{1/2-\varepsilon}}\Big),$$

where $E_\ell(2)$ denotes the ℓ-th Euler polynomial (cf. [11, 24.2.10]) evaluated at 2 and $\varepsilon > 0$.

Proof. By definition of the random variable X_n, we have

$$\mathbb{P}(X_n = r) = \frac{\sum_{k\geq 0} Q_{n,k,r}}{T_n},$$

where $T_n = C_{n-1}$ is the number of rooted plane trees of size n. Note that in particular, $Q_{n,k,r} = 0$ for $k > n - r$.

Applying Theorem 3.3.3 thus gives

$$\mathbb{P}(X_n = r) = \frac{1}{T_n} \sum_{k\geq 0} \big(B_{n+1,k,r+1} - B_{n+1,k,r} + B_{n,k,r-1} \big).$$

For the next step, we require a result from [26]: the average height $\bar{h}_r(n)$ of a rooted plane tree of size n and with root degree r is given by[2]

$$\bar{h}_r(n) = n - r - \frac{1}{T_{n,r}} \sum_{0\leq k<n-r} B_{n,k,r} = \sqrt{n\pi} - r/2 + O\Big(\frac{\log n}{n^{1/2-\varepsilon}} \Big), \tag{3.8}$$

where $T_{n,r}$ is the number of rooted plane trees of size n and root degree r, and $\varepsilon > 0$ is fixed. Furthermore, it is shown that the numbers $T_{n,r}$ are precisely given by

$$T_{n,r} = \frac{r}{n-1} \binom{2n-r-3}{n-2}.$$

By using the relation given in (3.8), the expression for $\mathbb{P}(X_n = r)$ can be rewritten into

$$\begin{aligned}\mathbb{P}(X_n = r) &= (n+1)\frac{1}{T_n}(T_{n+1,r+1} - T_{n+1,r} + T_{n,r-1}) \\ &\quad + \frac{1}{T_n}(T_{n+1,r}\bar{h}_r(n+1) - T_{n+1,r+1}\bar{h}_{r+1}(n+1) - T_{n,r-1}\bar{h}_{r-1}(n)).\end{aligned}$$

By substituting the exact values of $T_{n,r}$ into the expression on the first line, we find that this gives 0 and only the expression on the second line remains. Finally, by applying Stirling's approximation to the quotients $T_{n+a,r}/T_n$, we find

$$\frac{T_{n+a,r}}{T_n} = \frac{r}{2^{r+1-2a}} + O\Big(\frac{1}{n}\Big),$$

which results in

$$\mathbb{P}(X_n = r) = 2^{-r} + O\Big(\frac{\log n}{n^{1/2-\varepsilon}} \Big).$$

This proves the first part of the theorem. For the second part of the proof we refer to [28, Theorem 2]. □

[2] We slightly adapted this result in order to fit our definition of height. In [10], [26], as well as [28] the height of a tree is defined such that the root has height 1 etc.

This result on the stochastic behavior of the random variable X_n is particularly interesting: for large n, the probability that a random rooted plane tree has exactly one deepest node is approximately $1/2$. Moreover, the probability that such a random tree has at most 7 deepest nodes is more than 0.99.

Assume now, that we do not fix the number of deepest nodes—but the height instead. Then we are able to carry out a similar analysis for the average number of deepest nodes. Let $Y_{n,k} \colon \mathcal{T}_{n,k} \to \mathbb{N}$ be the associated random variable, where $\mathcal{T}_{n,k}$ is the combinatorial class of rooted plane trees of size n and height k. Then the following theorem holds.

Theorem 3.3.7 ([26, Theorem 3]).
Under the assumption that all trees in $\mathcal{T}_{n,k}$ are equally likely, the expected number of deepest nodes is given by

$$\mathbb{E}Y_{n,k} = 4\frac{n+1}{k+2}\sin^2\Big(\frac{\pi}{k+2}\Big) - \frac{6}{k+2} + O\Big(\frac{1}{n}\Big).$$

Sketch of the proof. The proof relies on a relation derived in the second part of the proof of Theorem 2 in [28]. This relation lets us write $\mathbb{E}Y_{n,k} = \frac{Q_{n+1,k+1,1}}{A_{n,k}-A_{n,k-1}}$. The result then follows from a “lengthy, but elementary” computation where Theorem 3.3.3 and the asymptotic expansion of $B_{n,k,r}$ from Lemma 3.3.4 are used. □

This concludes our chapter on the asymptotic analysis of special classes of trees.

A SageMath Implementations

Admissible Lattice Paths

The following listings contain SageMath code associated to Section 2.4. In addition to this code, there are also two IPython notebooks available. A notebook containing all calculations related to admissible lattice paths on $\mathbb{N}_0$ and $\mathbb{Z}$ can be found at `http://arxiv.org/src/1503.08790/anc/random-walk_NN.ipynb` and `http://arxiv.org/src/1503.08790/anc/random-walk_ZZ.ipynb`, respectively.

Admissible Lattice Paths on $\mathbb{N}_0$

```
sage: def tau(h, k):
....:     return (h + 1)*(2*k + 1)/2
sage: def p(m, h=None):
....:     if h is None:
....:         return sum(p(m, h) for h in range(2*m+1))
....:     else:
....:         return 4^(1-m) * sum((-1)^k * tau(h,k)/m * binomial(2*m, m-tau(h,k))
....:                                 if mod(h+1-2*m, 2) == 0 else 0
....:                                 for k in range(2*m + 1))

sage: [p(m/2) for m in range(2, 10)]
[1, 1/2, 3/4, 1/2, 9/16, 15/32, 29/64, 55/128]
```

Listing A.1: Explicit Formula for p_m (Theorem 2.4.5)

```
sage: R.<M> = LaurentSeriesRing(QQ, default_prec=20)
sage: R2.<a> = LaurentSeriesRing(R, default_prec=20)
sage: def stirling_coef(k): # due to G. Nemes, http://arxiv.org/abs/1003.2907
....:     return factorial(2*k)/(2^k * factorial(k)) \
....:             * sum(binomial(k + i - 1/2, i) * binomial(3*k + 1/2, 2*k - i) *
....:             2^i * sum(binomial(i,j) * (-1)^j * factorial(j) *
....:                     stirling_number2(2*k + i + j,j)/factorial(2*k + i + j)
....:                     for j in range(i + 1)) for i in range(2*k + 1))
....:

sage: def truncate_inner(expr, r):
....:     coefs = expr.coefficients()
....:     expos = expr.exponents()
....:     for j in range(len(coefs)):
....:         coefs[j] = coefs[j].truncate_neg(-expos[j] - r)
....:     return sum(a^expos[j] * coefs[j] for j in range(len(coefs)))
```

```
sage: def S(a, M, prec=20):
....:     return (sum(stirling_coef(r) * (2*M)^-r for r in range(prec)) *
....:             sum((-1)^r * stirling_coef(r) * (M + a)^-r
....:                 for r in range(prec)).truncate(prec) *
....:             sum((-1)^r * stirling_coef(r) * (M - a)^-r
....:                 for r in range(prec)).truncate(prec) *
....:             sum((-1)^r * binomial(-1/2, r) * (a/M)^(2*r)
....:                 for r in range(prec)).truncate(prec) *
....:             sum(1/factorial(r) * a^(4*r) / M^(3*r) *
....:                 (sum(-1/((t + 2) * (2*t + 3)) * (a/M)^(2*t)
....:                      for t in range(prec))^r).truncate(prec)
....:                 for r in range(prec))).truncate(prec)
```

Listing A.2: Shifted central binomial coefficient (series $S(\alpha, n)$, Lemma 2.4.6)

```
sage: var('m')
m
sage: dirichlet_beta = {1:pi/4, 2:catalan, 3:pi^3/32} # particular values of the
    Dirichlet beta function
sage: def mellin_translation(expr, r):  # compute the factors for the r-th moment
....:     coefs = expr.coefficients()
....:     expos = expr.exponents()
....:     erg = 0
....:     for k in range(len(coefs)):
....:         j = (expos[k] - 1)/2
....:         erg = erg + (2^(r-1) * gamma(j+1+r/2) * dirichlet_beta[r+1] * m^(j+1)
....:                      * sqrt(m)^r) * coefs[k].subs(M=m)
....:     return expand(erg)
```

Listing A.3: Mellin translation (Lemma 2.4.7)

```
sage: var('m n')
(m, n)
sage: asy_prob = (4/(sqrt(pi)*sqrt(m)) \
....:             * mellin_translation(truncate_inner(S(a,M,15), 15) * a/M,
....:             0)).subs(m = (n+1)/2)

sage: asy_prob = expand(asy_prob).taylor(n, oo, 10); asy_prob
1/2*sqrt(2)*sqrt(pi)/sqrt(n) - 5/24*sqrt(2)*sqrt(pi)/n^(3/2) + 127/960*sqrt(2)*sqrt
    (pi)/n^(5/2) - 1571/16128*sqrt(2)*sqrt(pi)/n^(7/2) - 1896913/184320*sqrt(2)*
    sqrt(pi)/n^(9/2) + 3716111711/4866048*sqrt(2)*sqrt(pi)/n^(11/2) -
    456593290865603/29520691200*sqrt(2)*sqrt(pi)/n^(13/2) +
    184340777593171739/1062744883200*sqrt(2)*sqrt(pi)/n^(15/2) -
    4393508939792266767 7/34007836262400*sqrt(2)*sqrt(pi)/n^(17/2) +
    127999367899555774 1521/190443883069440*sqrt(2)*sqrt(pi)/n^(19/2)

sage: asy_exp = (4/(sqrt(pi)*sqrt(m)) \
....:            * mellin_translation(truncate_inner(S(a,M,15), 15) * a/M,
....:            1)).subs(m=(n + 1)/2)/asy_prob - 1

sage: (asy_exp - 1).taylor(n, oo, 10)
2*sqrt(2)*catalan*sqrt(n)/sqrt(pi) + 5/6*sqrt(2)*catalan/(sqrt(pi)*sqrt(n)) -
    131/720*sqrt(2)*catalan/(sqrt(pi)*n^(3/2)) + 1129/12096*sqrt(2)*catalan/(sqrt(
    pi)*n^(5/2)) - 88061611/907200*sqrt(2)*catalan/(sqrt(pi)*n^(7/2)) +
    65631622327/9580032*sqrt(2)*catalan/(sqrt(pi)*n^(9/2)) -
    172247261860077449/1307674368000*sqrt(2)*catalan/(sqrt(pi)*n^(11/2)) +
    4399396764901604611/3138418483200*sqrt(2)*catalan/(sqrt(pi)*n^(13/2)) -
```

```
    94737237358744207421/9607403520000*sqrt(2)*catalan/(sqrt(pi)*n^(15/2)) +
    182335851895636802413З/38023147008000*sqrt(2)*catalan/(sqrt(pi)*n^(17/2)) -
    350317121390781200174682747240 7/13362733110421094400000*sqrt(2)*catalan/(sqrt(
    pi)*n^(19/2)) - 2

sage: asy_var = (4/(sqrt(pi)*sqrt(m)) \
....:             * mellin_translation(truncate_inner(S(a,M,15), 15) * a/M,
....:             2)).subs(m=(n + 1)/2)/asy_prob - asy_exp^2

sage: asy_var.taylor(n, oo, 10)
4*sqrt(2)*catalan*sqrt(n)/sqrt(pi) + 1/4*(pi^3 - 32*catalan^2)*n/pi + 5/3*sqrt(2)*
    catalan/(sqrt(pi)*sqrt(n)) - 1/6*(6*pi - pi^3 + 40*catalan^2)/pi - 131/360*sqrt
    (2)*catalan/(sqrt(pi)*n^(3/2)) - 1/180*(pi^3 - 12*catalan^2)/(pi*n) +
    1129/6048*sqrt(2)*catalan/(sqrt(pi)*n^(5/2)) + 1/1890*(11*pi^3 - 265*catalan^2)
    /(pi*n^2) - 88061611/453600*sqrt(2)*catalan/(sqrt(pi)*n^(7/2)) -
    1/1814400*(75273088*pi^3 - 1408301149*catalan^2)/(pi*n^3) +
    65631622327/4790016*sqrt(2)*catalan/(sqrt(pi)*n^(9/2)) +
    1/119750400*(342572847616*pi^3 - 6524406989415*catalan^2)/(pi*n^4) -
    17224726186007744 9/653837184000*sqrt(2)*catalan/(sqrt(pi)*n^(11/2)) -
    1/1307674368000*(6989796911856025 6*pi^3 - 134802330308073038 9*catalan^2)/(pi*n
    ^5) + 4399396764901604611/1569209241600*sqrt(2)*catalan/(sqrt(pi)*n^(13/2)) +
    1/392302310400 0*(21684164310700687 36*pi^3 - 42251792993342156275*catalan^2)/(pi
    *n^6) - 94737237358744207421/4803701760000*sqrt(2)*catalan/(sqrt(pi)*n^(15/2))
    - 1/58845346560000*(221638055423379894272*pi^3 - 4360261780008844857615*catalan
    ^2)/(pi*n^7) + 1823358518956368024133/19011573504000*sqrt(2)*catalan/(sqrt(pi)*
    n^(17/2)) + 1/2471504555520000*(4342967432561763851468 8*pi^3 -
    85769197755322913813643 5*catalan^2)/(pi*n^8) -
    3503171213907812001746827472407/6681366555210547200000*sqrt(2)*catalan/(sqrt(pi
    )*n^(19/2)) - 1/10690186488336875520000 0*(746516193554990299586534423323 9*pi^3
    - 19078976940820699705734918771654 4*catalan^2)/(pi*n^9) +
    23/641411189300212531200000*(7874503516501983226685399503801*pi^3 -
    5335685721836737999056982813440 0*catalan^2)/(pi*n^10)
```

Listing A.4: Computing the asymptotic expansions (Theorem 2.4.8)

Admissible Lattice Paths on $\mathbb{Z}$

```
sage: def upsilon(h, k):
....:     return (h + 2)*(2*k + 1)/2
sage: def q(m, h=None):
....:     if h is None:
....:         return sum(q(m, h) for h in range(2*m + 1))
....:     else:
....:         return 4^(1 - m) * sum((2*upsilon(h,k)^2 - m)/(m * (2*m - 1)) *
....:                                binomial(2*m, m - upsilon(h,k))
....:                                if mod(h - 2*m, 2) == 0 else 0
....:                                for k in range(2*m + 1))

sage: [q(m/2) for m in range(2, 10)]
[1, 1/2, 1/4, 1/4, 3/16, 5/32, 9/64, 15/128]
```

Listing A.5: Explicit formula for q_m (Theorem 2.4.5)

```
sage: var('m')
m
sage: def mellin_translation(expr, r, parity="even"): # specify the factor (h+2)^r
....:     coefs = expr.coefficients()
```

```
....:     expos = expr.exponents()
....:     erg = 0
....:     if r == 0:
....:         for k in range(len(coefs)):
....:             j = expos[k]/2
....:             if j == 0:
....:                 erg = erg + sqrt(m * pi)/4 * coefs[k].subs(M=m)
....:             else:
....:                 if parity == "even":
....:                     erg = erg + j/2 * gamma(j + 1/2) \
....:                           * (log(m)/2 + (2*euler_gamma + 1/(2*j)
....:                                         + psi(j + 1/2)/2 + log(2))) \
....:                           * m^j * sqrt(m) * coefs[k].subs(M=m)
....:                 elif parity == "odd":
....:                     erg = erg + j/2 * gamma(j + 1/2) \
....:                           * (log(m)/2 + 2*euler_gamma + 3*log(2)
....:                              + psi(j + 1/2)/2 - 2 + 1/(2*j)) \
....:                           * m^j * sqrt(m) * coefs[k].subs(M=m)
....:                 else:
....:                     raise ValueError("parity has to be even or odd")
....:     else:
....:         for k in range(len(coefs)):
....:             j = expos[k]/2
....:             if parity == "even":
....:                 erg = erg + (j * gamma(j+1/2) * 2^(r-1) * zeta(1-r) * m^j
....:                              * sqrt(m) + (j + r/2)/2 * gamma(j + (r+1)/2)
....:                              * (2^(r+1) -1) * zeta(r+1) * m^j
....:                              * sqrt(m)^(r+1)) * coefs[k].subs(M=m)
....:             elif parity == "odd":
....:                 erg = erg + (j * gamma(j+1/2) * ((1-2^(r-1)) * zeta(1-r) - 1)
....:                              * m^j * sqrt(m) + (j + r/2)/2 * gamma(j+(r+1)/2)
....:                              * (2^(r+1) -1) * zeta(r+1) * m^j
....:                              * sqrt(m)^(r+1)) * coefs[k].subs(M=m)
....:             else:
....:                 raise ValueError("parity has to be even or odd")
....:     return expand(erg)
```

Listing A.6: Mellin translation (Lemma 2.4.9)

```
sage: var('m n')
(m, n)
sage: asy_prob_even = (4/(sqrt(pi)*sqrt(m)*(2*m - 1))
....:                  * mellin_translation(truncate_inner(S(a,M,30), 30),
....:                                       0, parity="even")).subs(m = (n + 2)/2)
sage: asy_prob_odd = (4/(sqrt(pi)*sqrt(m)*(2*m - 1))
....:                 * mellin_translation(truncate_inner(S(a,M,30), 30),
....:                                      0, parity="odd")).subs(m = (n + 2)/2)

sage: asy_prob_even.taylor(n,oo,8)
1/n - 4/3/n^2 + 88/45/n^3 - 976/315/n^4 + 3488/675/n^5 - 276928/31185/n^6 +
    220605568/14189175/n^7 - 6724864/243243/n^8
sage: asy_prob_odd.taylor(n,oo,8)
1/n - 4/3/n^2 + 88/45/n^3 - 976/315/n^4 + 3488/675/n^5 - 276928/31185/n^6 +
    220605568/14189175/n^7 - 6724864/243243/n^8
sage: bool(asy_prob_even.taylor(n, oo, 8) == asy_prob_odd.taylor(n, oo, 8))
True
sage: asy_prob = asy_prob_even.taylor(n, oo, 8); asy_prob
```

```
1/n - 4/3/n^2 + 88/45/n^3 - 976/315/n^4 + 3488/675/n^5 - 276928/31185/n^6 +
    220605568/14189175/n^7 - 6724864/243243/n^8

sage: asy_exp_even = (4/(sqrt(pi)*sqrt(m) * (2*m - 1))
....:                 * mellin_translation(truncate_inner(S(a,M,15), 15), 1,
....:                                           parity="even")).subs(m = (n + 2)/2)
sage: asy_exp_even /= asy_prob_even
sage: asy_exp_odd = (4/(sqrt(pi)*sqrt(m) * (2*m - 1))
....:                * mellin_translation(truncate_inner(S(a,M,15), 15), 1,
....:                                          parity="odd")).subs(m = (n + 2)/2)
sage: asy_exp_odd /= asy_prob_odd

sage: asy_exp_even.taylor(n, oo, 7) - 2
1/4*sqrt(2)*pi^(3/2)*sqrt(n) + 3/16*sqrt(2)*pi^(3/2)/sqrt(n) - 539/5760*sqrt(2)*pi
    ^(3/2)/n^(3/2) + 50713/483840*sqrt(2)*pi^(3/2)/n^(5/2) - 16671323/116121600*
    sqrt(2)*pi^(3/2)/n^(7/2) + 13114961/63078400*sqrt(2)*pi^(3/2)/n^(9/2) -
    52266077173201/167382319104000*sqrt(2)*pi^(3/2)/n^(11/2) +
    1001945317462289/2008587829248000*sqrt(2)*pi^(3/2)/n^(13/2) - 2
sage: asy_exp_odd.taylor(n, oo, 7) - 2
1/4*sqrt(2)*pi^(3/2)*sqrt(n) + 3/16*sqrt(2)*pi^(3/2)/sqrt(n) - 539/5760*sqrt(2)*pi
    ^(3/2)/n^(3/2) + 50713/483840*sqrt(2)*pi^(3/2)/n^(5/2) - 16671323/116121600*
    sqrt(2)*pi^(3/2)/n^(7/2) + 13114961/63078400*sqrt(2)*pi^(3/2)/n^(9/2) -
    52266077173201/167382319104000*sqrt(2)*pi^(3/2)/n^(11/2) +
    1001945317462289/2008587829248000*sqrt(2)*pi^(3/2)/n^(13/2) - 2
sage: bool(asy_exp_even.taylor(n, oo, 7) - 2 == asy_exp_odd.taylor(n, oo, 7) - 2)
True
sage: asy_exp = asy_exp_even.taylor(n, oo, 7) - 2; asy_exp
1/4*sqrt(2)*pi^(3/2)*sqrt(n) + 3/16*sqrt(2)*pi^(3/2)/sqrt(n) - 539/5760*sqrt(2)*pi
    ^(3/2)/n^(3/2) + 50713/483840*sqrt(2)*pi^(3/2)/n^(5/2) - 16671323/116121600*
    sqrt(2)*pi^(3/2)/n^(7/2) + 13114961/63078400*sqrt(2)*pi^(3/2)/n^(9/2) -
    52266077173201/167382319104000*sqrt(2)*pi^(3/2)/n^(11/2) +
    1001945317462289/2008587829248000*sqrt(2)*pi^(3/2)/n^(13/2) - 2

sage: asy_var_even = (4/(sqrt(pi)*sqrt(m) * (2*m - 1))
....:                 * mellin_translation(truncate_inner(S(a,M,15), 15), 2,
....:                                           parity="even")).subs(m = (n + 2)/2)
sage: asy_var_even = asy_var_even / asy_prob_even - asy_exp_even^2
sage: asy_var_odd = (4/(sqrt(pi)*sqrt(m) * (2*m - 1))
....:                * mellin_translation(truncate_inner(S(a,M,15), 15), 2,
....:                                          parity="odd")).subs(m = (n + 2)/2)
sage: asy_var_odd = asy_var_odd / asy_prob_odd - asy_exp_odd^2

sage: asy_var_even.taylor(n, oo, 4)
-3/16*pi^3 - 1/8*(pi^3 - 28*zeta(3))*n + 1/2880*(67*pi^3 - 1792*zeta(3))/n -
    1/120960*(4189*pi^3 - 107520*zeta(3))/n^2 + 1/2073600*(98381*pi^3 - 2539520*
    zeta(3))/n^3 - 59/958003200*(990593*pi^3 - 27066368*zeta(3))/n^4 +
    1/5230697472000*(421642510377*pi^3 - 12492906954752*zeta(3))/n^5 -
    1/31384184832000*(3973982368189*pi^3 - 120862713839616*zeta(3))/n^6 + 14/3*zeta
    (3)
sage: asy_var_odd.taylor(n, oo, 4)
-3/16*pi^3 - 1/8*(pi^3 - 28*zeta(3))*n + 1/2880*(67*pi^3 - 1792*zeta(3))/n -
    1/120960*(4189*pi^3 - 107520*zeta(3))/n^2 + 1/2073600*(98381*pi^3 - 2539520*
    zeta(3))/n^3 - 59/958003200*(990593*pi^3 - 27066368*zeta(3))/n^4 +
    1/5230697472000*(421642510377*pi^3 - 12492906954752*zeta(3))/n^5 -
    1/31384184832000*(3973982368189*pi^3 - 120862713839616*zeta(3))/n^6 + 14/3*zeta
    (3)
sage: bool(asy_var_odd.taylor(n, oo, 2) == asy_var_even.taylor(n, oo, 2))
True
```

```
sage: asy_var = asy_var_even.taylor(n, oo, 2); asy_var
-3/16*pi^3 - 1/8*(pi^3 - 28*zeta(3))*n + 1/2880*(67*pi^3 - 1792*zeta(3))/n -
    1/120960*(4189*pi^3 - 107520*zeta(3))/n^2 + 1/2073600*(98381*pi^3 - 2539520*
    zeta(3))/n^3 - 59/958003200*(990593*pi^3 - 27066368*zeta(3))/n^4 +
    1/5230697472000*(421642510377*pi^3 - 12492906954752*zeta(3))/n^5 -
    1/31384184832000*(3973982368189*pi^3 - 120862713839616*zeta(3))/n^6 + 14/3*zeta
    (3)
```

Listing A.7: Computing the asymptotic expansions (Theorem 2.4.10)

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