On extremal cases of pop-stack sorting

Andrei Asinowski University of Klagenfurt

This talk is based on joint work with Cyril Banderier (University of Paris North) and Benjamin Hackl (University of Klagenfurt) [1].

Pop-stack sorting is a natural sorting procedure and a fascinating process to analyse. It finds its roots in the seminal work of Knuth on sorting algorithms and permutation patterns [**?**]. We present several results on permutations that need few (resp. many) iterations of this procedure to be sorted. In particular, we represent the "2-pop-stack sortable permutations" by lattice paths to prove conjectures raised by Pudwell and Smith, and we characterize some families of permutations related to the image of the pop-stack sorting and to its "worst case".

Introduction and definitions

Each permutation can be split uniquely into *runs* – the maximal ascending strings, and into *falls* – the maximal descending strings. For example, the permutation 413625 is split into runs as $4|136|25$, and into falls as $41|3|62|5$.

One iteration of pop-stack sorting is defined as the transformation T that reverses all the falls. For example, $T(41|3|62|5) = 143265$. If, given a permutation π of size n, one applies T successively sufficiently many times (thus obtaining $T(\pi)$, $T^2(\pi)$, etc.), one eventually reaches the identity permutation Id. Ungar proved [**?**] that each each permutation of size n needs at most $n-1$ iterations of T to be sorted by pop-stack [2]. Equivalently: for each permutation of size n we have $T^{n-1}(\pi) = \text{Id}$). This bound is tight: there are permutations that need n, but not fewer, iterations of T to be sorted. Thus, we refer to this situation as the "worst case".

A permutation is *k-pop-stack-sortable* (*kPS*) Avis and Newborn showed that if $T^k(\pi) = \text{Id}$. 1PS-permutations are precisely the layered permutations [**?**]. Pudwell and Smith [**?**] found a structural characterization of 2PS-permutations and showed that their generating function is rational. Claesson and Guðmundsson [**?**] generalized the latter result showing that for each fixed k, the generating function for kPS-permutations is rational. The pop-stack sorting process offers many fascinating open questions (the main one being the average cost analysis of the corresponding algorithm). In this article, we offer new links with other combinatorial objects to derive further results.

Results concerning 2-pop-stack sortable permutations

First, let us concentrate on permutations which need few iterations of T to be sorted. Specifically, we prove two conjectures on 2PS-permutations by Pudwell and Smith, and reprove one of their theorems in a more combinatorial way which allows us to keep track of additional parameters.

[1] Research of Andrei Asinowski and Benjamin Hackl was supported by the project *Analytic Combinatorics: Digits, Automata and Trees* (P 28466) funded by the Austrian Science Fund (FWF).

[2] Ungar proved this result as a lemma for solving a geometric problem concerning the number of directions determined by a planar set of points.

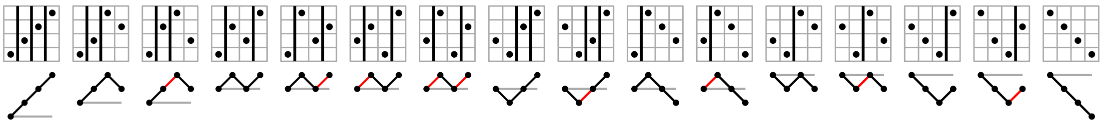
Theorem 1 ([?] Thm. 2)**.** *The generating function of 2-pop-stack-sortable permutations is* $A(x,y) = \sum a_{n,k}x^n y^k = x(1+x^2y)/(1-x-xy-x^2y-2x^3y^2)$, *where* $a_{n,k}$ *is the number of 2PS permutations of size* n *with exactly* k *ascents.*

Proof (sketch). We count 2PS-permutations taking descents rather than ascents as the second parameter: let $\mathcal{C}_{n,k}$ be the set of 2PS-permutations of size n with k descents, $c_{n,k} = |\mathcal{C}_{n,k}|$, $C(x,y) = \sum c_{n,k}x^n y^k$. We have $a_{n,k} = c_{n,n-1-k}$ and $A(x,y) = \frac{1}{y}C(xy, 1/y)$.

For fixed k, let F_k be the generating function for 2PS-permutations with k descents: $F_k(x) = \sum_{n \geq 0} c_{n,k}x^n$. We show $F_0(x) = \frac{x}{1-x}$ and, for $k \geq 1$, $F_k(x) = \frac{x^{k+1}(1+x)^2(1+x+2x^2)^{k-1}}{(1-x)^{k+1}}$. The cases of $F_0(x)$ and $F_1(x)$ are easily seen directly. We show that, for $k \geq 2$, we have $F_k(x)/(xF_{k-1}(x)) = (1+x+2x^2)/(1-x) = 1+2x+4x^2+4x^3+4x^4+\ldots$ We introduce a third parameter: let $\mathcal{C}_{n,k,d}$ be the set of those permutations in $\mathcal{C}_{n,k}$, in which the distance between the two rightmost descents is d. We construct a mapping which is a union of a 1:1 bijection between $\mathcal{C}_{n,k,1}$ and $\mathcal{C}_{n-1,k-1}$; a 2:1 bijection between $\mathcal{C}_{n,k,2}$ and $\mathcal{C}_{n-2,k-1}$; and a 4:1 bijection between $\mathcal{C}_{n,k,d\geq 3}$ and $\mathcal{C}_{n-d,k-1}$. This proves the result for $F_k(x)$. Now, $C(x,y)$ is obtained as the sum of geometric series $\sum_{k\geq 0} y^k F_k(x)$. □

Theorem 2 ([?] Conj. 2)**.** *The generating function for* $(a_{2n+1,n})_{n\geq 0}$ *is* $\sqrt{(1+x)/(1-7x)}$, *and the numbers are* $a_{2n+1,n} = \sum_{i=0}^{n-1}(-1)^i 2^{n-i}\binom{2(n-i)}{n-i}\binom{n-1}{i}$.

Proof [3] *(sketch).* As shown in [?], a 2PS-permutation is determined by positions of ascents / descents and indicating, for each ascent, whether the maximum of the run to its left is smaller (by 1) or larger (by 1) than the minimum of the run to its right. The second option is only possible when at least one of the adjacent runs has length > 1. Therefore, 2PS-permutations of size n are in bijection with Dyck walks ($\mathsf{U} = 1$ for ascents, $\mathsf{D} = -1$ for descents) of size $n-1$ where U's that have a D neighbor can be colored black or red, and U's that have no D neighbor can be colored only black.

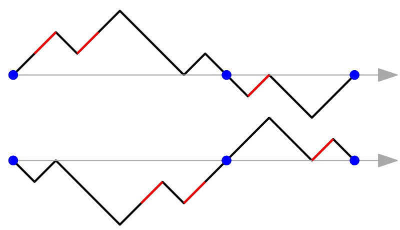


The paths that correspond to the diagonal values $a_{2n+1,n}$ are precisely the *bridges* – the walks that terminate at altitude 0. We first consider *excursions* – bridges that never go below the x-axis, and forget the colors for the time being, and get the generating function $E = E(x,y)$ for such excursions, where x is the variable for the semi-length, and y for the number of U's that have at least one adjacent D. We compose bridges from excursions and their reflections, and get their generating function $B(x,y) = E/(1-(E-1)) = \sqrt{(1-x+xy)/(1-x-3xy)}$. Since each non-colored bridge with k regular U's generates 2^k colored bridges, their generating function is $B(x,2) = \sqrt{(1+x)/(1-7x)} = 1/\sqrt{1-\ 8x/(1+x)} = (1/\sqrt{1-4t})|_{t=2x/(1+x)}$, and the coefficients of $1/\sqrt{1-4t}$ are well known to be central binomial coefficients. □

[3] It is possible to obtain the generating function by residue calculus, but we give a structural proof.

Additionally, we adjust the structure (Dyck walks with fixed final altitude) to get generating functions for any array of coefficients of $A(x,y)$ parallel to the diagonal, at distance m. Their shapes are $\sqrt{(1+x)/(1-7x)}((1-x-\sqrt{(1+x)(1-7x)})/(2x))^m$ or $\sqrt{(1+x)/(1-7x)}((1-x-\sqrt{(1+x)(1-7x)})/((2x)(1+2x)))^m$ (depending on the side). Another explicit formula for $a_{2n+1,n}$ is $\sum_{k\geq 0}\binom{n}{2k}\binom{2k}{k}2^{2k+1}3^{n-2k-1}(2-\frac{k}{n})$.

Theorem 3 ([?] Conj. 3)**.** *Let $\mathcal{B}_{n,k}$ be the set of permutations in $\mathcal{A}_{n,k}$ (2PS, size n, k ascents) whose last fall has size 1, and let $b_{n,k} = |\mathcal{B}_{n,k}|$. Then we have $a_{2n+1,n} = 2b_{2n+1,n}$.*



Proof (sketch). The last fall has size 1 if and only if we have an ascent at $2n$. Thus we need to prove that precisely one half of $\mathcal{A}_{2n+1,n}$ ends with an ascent. As above, we represent the permutations from $\mathcal{A}_{2n+1,n}$ by colored Dyck bridges. For such a bridge, split it into maximal excursions and anti-excursions and rotate each such string by 180°. Then the bridges with last step U are mapped bijectively to the bridges with last step D. This yields an autobijection in $\mathcal{A}_{2n+1,n}$ such that "ascent at $2n \leftrightarrow$ descent at $2n$". □

Results concerning the image of T

The image of the pop-stack sorting transformation has the following characterization.

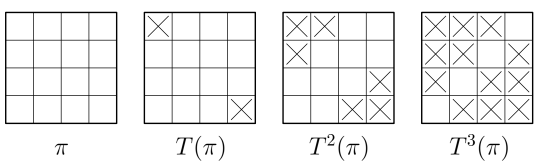
Theorem 4. *A permutation belongs to* $\mathrm{Im}(T)$ *if and only if its adjacent runs overlap.*

Enumerative aspects concerning the image of T can be found in our paper [?]. In the present work we study the structural and the enumerative aspects of $\mathrm{Im}(T^m)$.

Theorem 5. *If $\tau = a_1a_2\dots a_n \in \mathrm{Im}(T^m)$ (where $0 \leq m \leq n-1$), then for each i ($1 \leq i \leq n$), we have $|a_i - i| \leq n-m-1$.*

Proof (sketch). Our proof generalizes Ungar's argument [?]. It uses a poset structure associated to a "projection" of the successive images of the permutation π, and it is conveniently visualized by "forbidden corners" in the diagrams of these images. □

The "forbidden corners" for $n=4$:



The case $m = n-1$ of Theorem ?? gives Ungar's result: each permutation of size n is sorted by at most $n-1$ iterations of T. Our main result is the characterization and enumeration of $\mathrm{Im}(T^{n-2})$, the pre-image of Id in the longest chains $\pi \to T(\pi) \to T^2(\pi) \to \dots \to T^{n-1}(\pi) = \mathrm{Id}$.

Theorem 6. *A permutation $\tau = a_1a_2\dots a_n$ belongs to $\mathrm{Im}(T^{n-2})$ if and only if it is thin*[4] *and has no inner runs of odd size. This implies $|\mathrm{Im}(T^{n-2})| = 2^{n/2-1} + 2^{n/2} - 1$ for even n, $2^{(n+1)/2} - 1$ for odd n (OEIS A052955).*

[4] A permutation $a_1a_2\dots a_n$ is *thin* if $|a_i - i| \leq 1$ for each i.

Proof (sketch). Consider a thin permutation $\tau \neq \mathrm{Id}$ without odd inner runs. The runs of τ (listed left to right) are of lengths $(r_1, r_2, \ldots, r_s)$, where $s \geq 2$ and $r_2, \ldots, r_{s-1}$ are even. Now, let π be *the* skew layered permutation[5] with runs of lengths $(r_s, r_{s-1}, \ldots, r_1)$. It can then be checked that $T^{n-2}(\pi) = \tau$. Let us now prove the reciprocal. If one considers π and τ such that $T^{n-2}(\pi) = \tau$, then τ must be thin by Theorem **??**. If we assume that τ has an odd inner run, then analysing the successive images $T^m(\pi)$, $1 \leq m \leq n-2$, leads to a contradiction. Namely, it can be shown that in this case all the letters in τ before (or after) this odd run are already sorted. This contradicts the fact that an inner run starts and ends with a descent.
For the enumeration, as the run lengths determine a thin permutation uniquely, we just need to choose the even/odd positions for the borders between runs. □

In particular, this proof shows that each skew-layered permutation of size n without odd inner runs needs exactly $n-1$ iterations of T to be sorted. We conclude with the following conjectured (and supported by computer experiments) complete classification of skew-layered permutations with respect to the number of iterations of T needed to sort them. Denote by $\sigma(\pi)$ the smallest number m such that $T^m(\pi) = \mathrm{Id}$.

Conjecture 7. *Let π be a skew-layered permutation of size n such that π is neither the identity nor the anti-identity permutation. If n is even, then $\sigma(\pi) = n-1$. For odd n, $\sigma(\pi)$ depends on the structure of π around the central letter (that is, $(n+1)/2$) as follows: if this letter is the middle of a run/fall of size ≥ 3, then $\sigma(\pi) = n-2$; otherwise, $\sigma(\pi) = n-1$. From the enumerative point of view: apart for the identity and the anti-identity, for even n we have $2^{n-1} - 2$ skew-layered permutations with $\sigma(\pi) = n-1$; for odd n we have $(2^{n-2} - 2)/3$ (OEIS A020988) skew-layered permutations with $\sigma(\pi) = n-2$, and $(5 \cdot 2^{n-2} - 4)/3$ (OEIS A080675) skew-layered permutations with $\sigma(\pi) = n-1$.*

References

[1] Andrei Asinowski, Cyril Banderier, Sara Billey, Benjamin Hackl, and Svante Linusson. Pop-stack sorting and its image: Permutations with overlapping runs. (*Eurocomb 2019*).

[2] David Avis and Monroe Newborn. On pop-stacks in series. Utilitas Mathematica 19 (1981), 129–140.

[3] Donald E. Knuth. *The Art of Computer Programming. Vol. 3.* Addison-Wesley, Reading, MA, 1998. Sorting and searching, Second edition (1st ed: 1973).

[4] Lara Pudwell and Rebecca Smith. Two-stack-sorting with pop stacks. arXiv:1801.05005. Permutation Patterms 2017, Reykjavík, Iceland.

[5] Anders Claesson and Bjarki Ágúst Guðmundsson. Enumerating permutations sortable by k passes through a pop-stack. arXiv:1710.04978. *Séminaire Lotharingien de Combinatoire* 80B (Proc. FPSAC 2018).

[6] Peter Ungar. $2N$ noncollinear points determine at least $2N$ directions. Journal of Combinatorial Theory, Series A, 33:3 (1982), 343–347.

[5] A permutation is *skew layered* if it is the skew product of its runs.